Nonlinear Ordinary Differential Equations: A discussion on Symmetries and Singularities

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Abstract

Two essential methods, the symmetry analysis and of the singularity analysis, for the study of the integrability of nonlinear ordinary differential equations are discussed. The main similarities and differences of these two different methods are given.

Keywords: Lie symmetries; Singularity analysis; Integrability

1   Introduction

The systematic analysis of the symmetries and singularities of ordinary differential equations began in the last quarter of the nineteenth century. In a series of papers and books [1, 2, 3, 4, 5, 6],

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seeking to do for ordinary differential equations what Galois had done for algebraic equations, the Norwegian mathematician, Sophus Lie, wrought a much greater achievement which even to this day influences every area in which differential equations, indeed difference equations, arise. The genius of Lie's work was to take the infinitesimal representations of the finite transformations of continuous groups, thereby moving from the group to a local algebraic representation, and to study the invariance properties under them. This resulted in linearization of all equations and/or functions under consideration. The infinitesimal transformation

$$\bar{x} = x + \varepsilon\xi, \qquad \bar{y} = y + \varepsilon\eta, \tag{1}$$

where ε is the infinitesimal parameter of the transformation, could be represented in terms of the differential operator G given by

$$G = \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} \tag{2}$$

as the deformation from the identity

$$\bar{x} = (1 + \varepsilon G)\, x, \qquad \bar{y} = (1 + \varepsilon G)\, y. \tag{3}$$

The effect of the infinitesimal transformation on functions or equations involving derivatives could be determined by the extension of G to deal with derivatives. For the first derivative we have

$$
\begin{aligned}
\frac{\bar{y}}{\bar{x}} &= \frac{y + \varepsilon\eta}{x + \varepsilon\xi} \\
&= \frac{y' + \varepsilon\eta'}{1 + \varepsilon\xi'} \\
&= y' + \varepsilon(\eta' - y'\xi')
\end{aligned}
\tag{4}
$$

and we write the first extension of G as

$$G^{[1]} = G + (\eta' - y'\xi')\partial_{y'}. \tag{5}$$

The extension for higher derivatives is determined in the same fashion and we can write the nth extension in a recursive form as [7]

$$G^{[n]} = G^{[n-1]} + \left\{\eta^{(n)} - \sum_{i=0}^{n-1}\binom{n}{i+1} y^{(n-i)}\xi^{(i+1)}\right\}\partial_{y^{(n)}}. \tag{6}$$

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A function or differential equation containing up to the nth derivative invariant under the action of $G^{[n]}$ is said to possess the symmetry G. This is expressed as

$$G^{[n]} f(x, y, y'^{(n)}) = 0 \tag{7}$$

in the case of the function $f(x, y, y'^{(n)})$ and

$$G^{[n]} f(x, y, y'^{(n)})\big|_{f=0} = 0 \tag{8}$$

in the case of the nth-order differential equation, $f(x, y, y'^{(n)}) = 0$. The set of all such symmetries constitutes a Lie algebra under the operation of taking the Lie Bracket, namely

$$[G_i, G_j]_{LB} = G_i G_j - G_j G_i. \tag{9}$$

In the case of point symmetries, ie, the coefficient functions ξ and η are functions of x and y only, the algebraic properties of the set of symmetries $G_i, i = 1, m$, are invariant under extension. In the case of contact symmetries, for which the coefficient functions can also contain y' in such a way that

$$\frac{\partial \eta}{\partial y'} = y' \frac{\partial \xi}{\partial y'}, \tag{10}$$

the same applies. When one has ensured that the first extension has a coefficient depending upon x, y and y' only, the algebraic properties must be established using the set $G_i^{[1]}, i = 1, m$. In the case of generalized symmetries, for which the dependence upon derivatives is limited only by the order of the differential equation in the case that a differential equation is being considered, and in the case of nonlocal symmetries, in which the coefficient functions can depend upon integrals in a nontrivial fashion, the calculation of the algebraic properties is a somewhat more complicated affair. Although generalised and nonlocal symmetries play important roles in certain problems, fortunately the algebraic difficulties have, to date, not been a great problem in the study of these types of symmetries of differential equations.

The original work of Lie was motivated by geometric considerations and he commenced with point transformations, and so point symmetries, and then extended his work to include contact symmetries so that the transformations were from (x, y) space to $(\bar{x}, \bar{y})$ space or from (x, y, y') space to $(\bar{x}, \bar{y}, \bar{y}')$ space. The use of generalized transformations was firmly established by the work of Noether on the invariance of the Action Integral of the Calculus of Variations under infinitesimal transformations [8]. The use of nonlocal symmetries arose in the last part of the twentieth century.

The major thrust of the singularity analysis of differential equations is associated with the French school led by Painlevé in the last years of the nineteenth century and the early years of the twentieth century [9, 10, 11, 12] following its successful application to the determination of the third integrable case of Euler's equations for a spinning top by Kowalevskaya [13]. Since then considerable work has been done on the classification problem of higher-order and higher-degree ordinary differential equations by Bureau [14, 15, 16] and Cosgrove *et al* [18, 19, 20]. Application to partial differential equations became widespread in the second half of the twentieth century with significant contributions being made by Kruskal [76]. The development of the Painlevé Test for the determination of integrability of a given equation or system of equations and its systematization in the ARS algorithm [22, 23, 24] has made the singularity analysis a routine tool for the practising applied mathematician. Popular expositions such as that found in the review by Ramani, Grammaticos and Bountis [25] and the monograph of Tabor [26] provide clear guides for the implementation of the algorithm. More precise prescriptions are found in the somewhat more technical works of Conte [27, 28].

The basic purpose of both forms of analysis is to facilitate the solution of differential equations. The existence for a differential equation of a sufficient number of Lie symmetries of the right type enables one to solve the differential equation by means of repeated reduction of order and a reverse series of quadratures or by means of the determination of a sufficient number of first integrals. The latter is the route taken in Noether's Theorem. In singularity analysis the differential equation (or system of differential equations; this should be implied unless specifically excluded) is deemed integrable if it possesses the Painlevé Property. The Painlevé Property in brief is that the differential equation possesses a Laurent expansion about a movable polelike singularity in the complex plane of the independent variable with the requisite number of arbitrary constants to provide the general solution of differential equation. The Laurent expansion implies that the solution of the differential equation is analytic except at the singularities. This is somewhat stronger than that which the Lie approach gives and there still exists the question of a complete reconciliation between the two approaches. For a differential equation of more than simple complexity there is the possibility of different patterns of singular behaviour and it is the conventional wisdom [26] [p 300] that for each pattern of singular behaviour there must exist a solution in terms of an analytic function, expressed in terms of a Laurent expansion, with the requisite number of arbitrary constants for the equation to possess the Painlevé Property and so be integrable. (In all of these considerations we must allow for the possibility of the so-called 'weak Painlevé Property' in which the polelike singularity is replaced by an algebraic

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branch point and the Laurent expansion is in terms of fractional powers of the complex variable. The discussion is *mutatis mutandis* identical.) However, this very convenient criterion has been shown, by way of counterexample, to be too strong [29].

In the context of the Lie analysis the concept of integrability is not as strong as it is in the Painlevé analysis. For the latter the solution of the differential equation must be analytic apart from isolated movable polelike singularities or have branch point singularities so that the complex plane of the independent variable can be divided into sections and in each section the solution is analytic. One can say that a differential equation is integrable in the sense of Lie if it possesses a sufficient number of symmetries for it to be reducible to an algebraic equation, although generally one would not bother with the ultimate step and be content with reduction to a separable first-order ordinary differential equation, *ie* to a quadrature. To the sufficiency of the number of symmetries one must sound a note of caution. If an nth-order system has n Lie point symmetries with a solvable algebra, one knows that the system is reducible to quadratures. The absence of this property does not immediately obviate the possibility of integrability. In the process of reduction of order of a system using the known symmetries additional point symmetries may arise. Such symmetries are known as Type II hidden symmetries [30, 31, 32] and originate from nonlocal symmetries in the main. A similar phenomenon can be observed when the order of a differential equation is increased, a technique occasionally of use in the solution of certain equations [33, 34]. A Lie point symmetry which arises on the increase of order of a differential equation is called a Type I hidden symmetry and originates in a nonlocal symmetry of the original equation.

The possession of the Painlevé Property is representation dependent and its preservation under transformation is guaranteed only in the case of a Möbius transformation. A Lie symmetry exists independently of the representation. In fact one could say that symmetries can be neither created nor destroyed. The particular nature of a symmetry can change with the representation. Thus the origin of hidden symmetries, *ie* Lie point symmetries which appear, as it were, from nowhere on a change of order of the differential equation, is found in nonlocal symmetries[1]. The symmetry was there all the time, but was hidden from view due to the restriction to the viewing of point symmetries only. Preservation of the type of Lie symmetry is guaranteed only by a transformation of the same quality as that of the symmetry. Point symmetries are preserved under point transformations, contact symmetries are preserved under contact transformations, generalized symmetries are preserved under generalized transformations and nonlocal symme-

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[1] Occasionally contact or generalise symmetries, but generally nonlocal symmetries.

tries under nonlocal transformations.

In addition to their use in the solution of differential equations Lie symmetries are used for the classification of equations and the establishment of equivalence classes of equations, *ie* those equations obtainable by means of transformations of the same nature as the symmetries used in the classification. Usually the classification is in terms of Lie point symmetries. In the case of third-order equations contact symmetries are also used. Because the Lie point symmetries constitute an algebra under the operation of taking the Lie Bracket, the classification is conveniently commenced from the algebra. The task of establishing all representations of Lie algebras which admit a differential equation of given order is a tediously formidable task. The reader is referred to the works of Mubarakzyanov [35, 36, 37] for the classification scheme of Lie algebras and of the Montreal School for the representations [38, 39, 40]. Additional to these systematic investigations there are results for specific types of equations. There have also been studies of the algebraic properties of first integrals of differential equations [41, 42, 43]. More recently the concept of a complete symmetry group, *ie* the number of symmetries required to specify completely a differential equation or its first integrals, has attracted attention [44, 45, 46] and has been used to show the identity of a number of nonlinear systems of somewhat different properties [47].

Before we commence our treatment of nonlinear ordinary differential equations we present a short summary of the results for linear equations. The Painlevé analysis is not really relevant to linear systems as there is no question of their possessing movable singularities. However, linear systems are known to be integrable. A scalar $n \geq 3$th-order linear equation has either $n + 1$, $n + 2$ or $n + 4$ Lie point symmetries. In the case of a linear second-order equation the number of Lie point symmetries is always eight. Apart from the symmetry related to the very linearity of the equation all other symmetries require a knowledge of the solution of the original equation and so the theoretical plethora of Lie point symmetries is of no great practical value. Systems of linear equations have been little studied. In the case of systems of two second-order linear equations the number of symmetries has been shown to range from 5 to 15 [48, 7] and in the case of n autonomous second-order linear systems to range from $n + 1$ to $(n + 2)^2 - 1$ with the latter being a representation of the algebra $sl(n + 2, R)$ [49]. In the case of n equations of the mth order the maximum number of Lie point symmetries is $n^2 + nm + 3$ [50].

2 Applications of Lie symmetries to nonlinear ordinary differential equations

The calculation of the Lie symmetries of a function or differential equation, be it ordinary or partial, linear or nonlinear, is a tedious business except for the simplest of expressions. One is advised to make use of the codes written in one of the symbolic manipulation packages, such as Nucci's interactive code [51, 52], the more automated code of Head [53, 54] or the Mathematica Add-on, Sym, developed by Dimas [56, 57, 58, 55]. The codes of Head, Nucci and Dimas are readily available. These codes are designed primarily for the computation of Lie point symmetries, but they can be used for the computation of contact and generalized symmetries and some specialised forms of symmetry such as approximate [59]. There is an application for which these codes are not so satisfactory and that is the calculation of possibly nonlocal symmetries to determine the existence of integrating factors [60].

It is possible that a nonlinear ordinary differential equation is in fact a linear ordinary differential equation written using inappropriate variables. This in fact is the case with many of the nonlinear equations listed in the classic compendium of Kamke [61]. We select one of them, [61] [6.51, p 554], namely

$$y'' + f(y)y'^2 + g(x)y' = 0. \tag{11}$$

A second-order equation requires two Lie point symmetries to be reducible to quadratures. Given that an equation with which we are working may not be written in the ideal variables, we must allow for the possibility that one of the symmetries could be nonlocal. For a second-order equation the determining equation, (8), becomes

$$\frac{\partial f}{\partial y} + (\eta' - y'\xi')\frac{\partial f}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial f}{\partial y''} \tag{12}$$

For the calculation of nonlocal symmetries (12) is extremely awkward. One can regard it as a second-order linear equation in either ξ or η with the other function being at one's disposal. It is usually convenient to be regarded as a linear equation in η and to set $\xi = 0$. Provided that one can solve the equation for η, there are two symmetries, as required. However, there are four possible two-dimensional Lie algebras with a standard representation and it is necessary to find the equivalent form in the representation we have adopted to solve (12). We list these in Table 13 (adapted from [60] [Table 1].

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Type	$\{G_1, G_2\}$	Canonical forms of G_1 and G_2	Form of equation	Present forms of G_1 and G_2
I	0	$G_1 = \frac{\partial u}{\partial x}$ $G_2 = \frac{\partial u}{\partial y}$	$y'' = f(y')$	$G_1 = y'\frac{\partial u}{\partial y}$ $G_2 = \frac{\partial u}{\partial y}$
II	0	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = x\frac{\partial u}{\partial y}$	$y'' = f(x)$	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = x\frac{\partial u}{\partial y}$
III	G_1	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$	$xy'' = f(y')$	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = (xy' - y)\frac{\partial u}{\partial y}$
IV	G_1	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = y\frac{\partial u}{\partial y}$	$y'' = y'f(x)$	$G_1 = \frac{\partial u}{\partial y}$ $G_2 = y\frac{\partial u}{\partial y}$

$$(13)$$

We note that in cases Type II and Type IV the normal form of the equation is of the form we desire to use.

When we apply this procedure to (11), we obtain the equation

$$\left(\frac{\eta'}{y'}\right)' + (\eta f)' = 0 \tag{14}$$

which has a fairly obvious solution and so we have the two symmetries and Lie Bracket

$$G_1 = \exp\left[-\int f(y)y'\mathrm{d}x\right]\frac{\partial u}{\partial y}, \tag{15}$$

$$G_2 = \exp\left[-\int f(y)y'\mathrm{d}x\right]\int y'\exp\left[\int f(y)y'\mathrm{d}x\right]\mathrm{d}x\frac{\partial u}{\partial y}\frac{\partial u}{\partial y}, \tag{16}$$

$$[G_1, G_2]_{LB} = G_1 \tag{17}$$

so that our second-order equation, (11), is of Lie's Type IV and is transparently linear when written in the variables

$$x = x, \quad w = \exp\left[\int f(y)y'\mathrm{d}x\right] \quad \text{as} \quad w'' + g(x)w' = 0 \tag{18}$$

the integration of which is theoretically quite trivial.

In general we may apply one of the codes to determine the Lie point symmetries of a given differential equation. For example the well-known equation [62, 63, 64, 65]

$$y'' + 3yy'^3 = 0, \tag{19}$$

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often called the Painlevé-Ince Equation, has the eight Lie point symmetries [66]

$$G_1 = \tfrac{1}{2}x^2 y \frac{\partial u}{\partial x} + \left(xy^2 - \tfrac{1}{2}x^2 y^3 - y\right)\frac{\partial u}{\partial y}$$

$$G_2 = y\frac{\partial u}{\partial x} - y^3 \frac{\partial u}{\partial y}$$

$$G_3 = xy\frac{\partial u}{\partial x} + \left(y^2 - xy^3\right)\frac{\partial u}{\partial y}$$

$$G_4 = \left(-\tfrac{1}{2}x^2 y + x\right)\frac{\partial u}{\partial x} + \left(\tfrac{1}{2}x^2 y^3 - xy^2\right)\frac{\partial u}{\partial y}$$

$$G_5 = \left(\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 y\right)\frac{\partial u}{\partial x} + \left(-x - x^3 y^2 + \tfrac{1}{4}x^4 y^3 + \tfrac{3}{4}x^2 y\right)\frac{\partial u}{\partial y}$$

$$G_6 = \left(-\tfrac{1}{2}x^3 y + x^2\right)\frac{\partial u}{\partial x} + \left(xy + \tfrac{1}{2}x^3 y^3 - \tfrac{3}{2}x^2 y^2\right)\frac{\partial u}{\partial y}$$

$$G_7 = \left(-\tfrac{1}{2}x^3 y + \tfrac{3}{2}x^2\right)\frac{\partial u}{\partial x} + \left(1 + \tfrac{1}{2}x^3 y^3 - \tfrac{3}{2}x^2 y^2\right)\frac{\partial u}{\partial y}$$

$$G_8 = \frac{\partial u}{\partial x} \tag{20}$$

and so is equivalent under a point transformation to the equation

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2} = 0 \Rightarrow Y = AX + B, \quad A, B \quad \text{constants.} \tag{21}$$

The point transformation is

$$Y = -\tfrac{1}{2}x^2 + \frac{x}{y}, \quad X = x - \frac{1}{y} \tag{22}$$

and so the solution of (19) is obviously

$$y = \frac{2(1 + Ax)}{Ax^2 + 2x + C}. \tag{23}$$

In the more general form,

$$y'' + kyy'^3 = 0, \tag{24}$$

the equation has only the two obvious symmetries of invariance under translation in x and rescaling, namely

$$G_1 = \frac{\partial}{\partial x} \quad \text{and} \quad G_2 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{25}$$

and the solution is given by following expression

$$\begin{aligned}
y(x) &= k^{-\frac{1}{3}}\left(3(x + y_1) + k^{-\frac{1}{2}}\left(8y_0^3 + 9(x + y_1)^2 k\right)\right)^{\frac{1}{3}} + \\
&\quad -2y_0\left(3(x + y_1) + k^{-\frac{1}{2}}\left(8y_0^3 + 9(x + y_1)^2 k\right)\right)^{-\frac{1}{3}}
\end{aligned} \tag{26}$$

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where y_0, y_1 are the constant of integration. The performance of the quadrature in closed form is generally not possible and the inversion to obtain $y(x)$ even less so [67]. We meet (19) and (24) below in the discussion of the Painlevé Property.

In the case of third-order equations there is the possibility of linearization by means of a point transformation or by means of a contact transformation. The need to distinguish between the two possibilities arises from some differences in algebraic properties between second-order and third-order linear equations. In general second-order equations can have 0, 1,2,3 or 8 Lie point symmetries. Third-order linear equations can have 4, 5 or 7 Lie point symmetries and in the last case there are 10 Lie contact symmetries. Although a third-order linear equation cannot have six Lie point symmetries, there is no such restriction on a third-order nonlinear equation. For example the Kummer-Schwarz Equation

$$2y'y''' - 3y''^2 = 0 \tag{27}$$

has the 6 Lie point symmetries

$$\begin{aligned} G_1 &= \frac{\partial u}{\partial x} & G_4 &= \frac{\partial u}{\partial y} \\ G_2 &= x\frac{\partial u}{\partial x} & G_5 &= y\frac{\partial u}{\partial y} \\ G_3 &= x^2\frac{\partial u}{\partial x} & G_6 &= y^2\frac{\partial u}{\partial y} \end{aligned} \tag{28}$$

with the Lie algebra $sl(2, R) \oplus sl(2, R)$. It does, moreover, have 10 Lie contact symmetries and so can be transformed to the third-order equation of maximal symmetry by means of a contact transformation

$$\frac{\mathrm{d}^3 Y}{\mathrm{d}X^3} = 0. \tag{29}$$

By way of interest the generalized Kummer-Schwarz Equation

$$y'y''' + ny''^2 = 0, \tag{30}$$

which possesses only the four Lie point symmetries G_1, G_2, G_4 and G_5 of the six listed in (28), is also equivalent to the equation of (29) but by means of the nonlocal transformation,

$$X = x, \quad Y = \int y'^{n+1}\mathrm{d}x, \tag{31}$$

corresponding to the local symmetry G_1 of (30) rather than the contact transformation of (29).

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3 Singularity analysis of nonlinear ordinary differential equations

In the spirit of Sophie Kowalevskaya [13], we seek to determine whether or not a given differential equation possesses movable singularities. To take an example consider the Painleve-Ince Equation,

$$y'' + 3yy' + y^3 = 0, \tag{32}$$

which has attracted a certain amount of attention in recent decades [30, 68, 64, 66, 69, 67]. If a movable singularity exists then the solution of the latter equation will described by the power-law function $y(x) \simeq (x - x_0)^p$, where p is a negative number and x_0 indicates the position of the singularity. Of a movable singularity because the value of x_0, the position, depends on the initial conditions, that is, different initial conditions provide us with different positions for the singular point.

We substitute $y(x) = a_0 (x - x_0)^p = a_0 \chi^p$ in (32) and obtain the expression

$$a_0 p (p - 1) \chi^{p-2} + 3p (a_0)^2 \chi^{2p-1} + a_0 \chi^{3p}, \tag{33}$$

for which balance occurs if $p = -1$ and consequently $a_0 = 1$ or $a_0 = 2$. This means that the movable singularity is a simple pole and there are two possibilities which follow from the leading-order behaviour. The arbitrary location of the movable singularity gives one of the constants of integrations. As (32) is a second-order equation, the other constant of integration has to be determined from a series developed about the singularity.

We select the leading order $a_0 = 1$ write the solution as a Laurent expansion of the form

$$y(x) = \chi^{-1} + Y(\chi) \tag{34}$$

where $Y(\chi)$ represents the remained of the Laurent expansion. To avoid the tedium of coping with an infinite series we simply replace $Y(\chi)$ with $\mu \chi^{-1+s}$ in (32) and obtain

$$\mu \chi^{-s} (s - 1)(s + 1) + 3\mu^2 s \chi^{2s} + \mu^3 \chi^{3s} = 0. \tag{35}$$

We take the term linear in μ and equate it to zero so that $s = -1$, $s = 1$. The former value $s = -1$ is to be expected as it is associated with the movable singularity. The second value

$s = 1$, indicates the term in the series at which the second constant of integration occurs. We infer that the series increments by integral powers and write the Laurent expansion as

$$y(x) = \chi^{-1} + \sum_{I=1}^{\infty} a_I \chi^{-1+I}. \tag{36}$$

When we substitute (36) into (32), we obtain a recurrence relation for the remaining coefficients. The first terms are $a_2 = -a_1^2$, $a_3 = a_1^3$, $a_4 = -a_1^4$, $a_5 = a_1^5$..., that is,

$$a_I = (-1)^{-1+I} a_1^I. \tag{37}$$

In the case that we consider the second leading order $a_0 = 2$, the Laurent expansion that we find that it is a decreasing series. This is known as a Left Painlevé Series in contrast to the former results which was a Right Painlevé Series.

The explanation for the two distinct types of solutions is simple. We are in the complex plane and we integrate around the singularity. An increasing expansion means that we integrate from the singularity until a border. The decreasing series means that we integrate from the border to infinity. However, there exists a possibility that the Laurent expansion admits increasing and decreasing terms. The explanation of latter is that we integrate over annulus around the singularity which has two borders [70].

This approach to the singularity analysis has been succinctly summarised in the papers of Ablowitz, Ramani and Segur [22, 23, 24] and it is called the ARS algorithm from the initials of the authors. That algorithm can be briefly described as follows:

- Determine the leading-order behaviour, at least in terms of dominated exponent. The coefficient of the leading-order term may or may not be explicit.

- Determine the exponents at which the arbitrary constants of integration enter.

- Substitute an expansion up to the maximum resonance into the full equation to check for consistency.

For the singularity analysis to work the exponents of the leading-order term needs to be a negative integer or a nonintegral rational number. Equally the resonances have to be rational numbers. Excluding the generic resonance $s = -1$, for a Right Painlevé Series the resonances

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must be nonnegative, for a Left Painlevé Series the resonances must be nonpositive while for a full Laurent expansion the resonances they have to be mixed. Clearly for a second-order ordinary equation the possible Laurent expansions are Left or Right Painlevé Series.

4   Symmetries and Singularities

There are similarities and dissimilarities between the two methods, approaches for determining the integrability of differential equations. Singularity analysis indicates potential integrability. Symmetry analysis can give stronger results in that it can provide a route to the explicit solution of the equation in closed form. In the route of the latter there is the question the choice of the group invariant transformation which leaves invariant the differential equation. There is a choice of the type of symmetry, such as point symmetries, contact symmetries, Lie-Bäcklund symmetries, nonlocal symmetries and many others. The ease of applicability deteriorates with the increasing complexity of the functional forms permitted in the coefficient functions.

Singularity analysis is straighforward in principle as it does not offer so many choices. However, singularity analysis is coordinate dependent which is not true for symmetry analysis. To demonstrate that we give two well-known elementary equations, the free particle and the "hyperbolic" oscillator. Both of these equations are invariant under an eight-dimensional Lie algebra, $sl(3, R)$, i.e., they are maximally symmetric. That means that there exists a transformation which transform the one equation to the other one and vice versa.

From the singularity point of view, equations $y'' = 0$ and $y'' - y = 0$ do not possess any movable singularity and the ARS algorithm that we discussed above fails. However, that does not mean that there does not exist a coordinate system in which these two equations pass the singularity test. For the equation of motion of the free particle that is simple by selecting the new variable $w = y^{-1}$. Then the new equation is

$$ww'' - 2\left(w'\right)^2 = 0 \tag{38}$$

which admits the leading term with exponent $p = -1$ and arbitrary a_0. Easily we have that the resonances are $s = -1$, and $s = 0$. The second value is expected because the leading term has arbitrary constant, a_0, and a zero resonance provides us with that property.

As far as concerns the linear equation $y'' = y$, which does not pass the singularity test we perform the change of variables

$$x = -\ln\left(u\left(v\right)\right) \ , \ y = \frac{du\left(v\right)}{dv}, \tag{39}$$

and we have the third-order nonlinear equation

$$u^2 u_{,v} u_{,vvv} + \left(u \ (u_v)^2 - u^2 \right) u_{,vv} - 4 \left(u_{,v} \right)^4 = 0 \qquad (40)$$

which passes the singularity test hence the integrability is expressed also in terms of the singularity analysis.

One of the main differences between the two methods is that symmetries provide for the conservation laws of a dynamical system, *ie* functions which are invariant in time. This is generally not necessarily true in the case of singularity analysis. The conservation laws which follow from the symmetry analysis are applied for the analysis of the dynamical system as they provide surfaces in the phase space in which the solution evolves. On the other hand the solution which follows from the singularity analysis admits the correct number of constants of integration but in general information about the nature of conservation laws cannot be extracted. For instance for the free particle (38) it is easy to extract the conservation law $I = \frac{1}{w^2} \dot{w}$ from the symmetry vector $w\partial_w$ which is nothing else than the law of conservation of momentum. However, from the singularity analysis the solution of (38) is given as a Laurent expansion and one needs to calculate all of the coefficients to determine the solution. The conservation law cannot be determined, which does not mean that the conservation law does not exist.

The two forms of analysis can be regarded as complementary. Symmetry analysis is very effective when it works, singularity analysis is also very effective when it works. Neither method is a complete answer to question of integrability for the simple reason that there exist equations for which neither method provides a result, but which are trivially integrable.

In this paper we have concentrated upon nonlinear differential equations for the purposes of clarity of presentation. The considerations here can be extended *mutatis mutandus* to systems of ordinary differential equations and to partial differential equations.

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