

Opportunities to develop mathematical proficiency in Grade 6 mathematics classrooms in KwaZulu-Natal

Noor Ally & Iben Maj Christiansen

In this article, we propose a rubric for assessing the teacher's provision of opportunities to develop mathematical proficiency in classrooms. The rubric is applied to data from 30 video recordings of mathematics lessons taught in Grade 6 in one district of KwaZulu-Natal. The results suggest that opportunities to develop procedural fluency are common, but generally of a low quality; that opportunities to develop conceptual understanding are present in about half the lessons, but also are not of a high quality; and that overall opportunities to develop mathematical proficiency are limited, because learners are not engaging in adaptive reasoning, hardly have any opportunities to develop a productive disposition, and seldom are given the opportunity to engage in problem-solving which could develop their strategic competence.

Keywords: mathematical proficiency, opportunities to learn, mathematics teaching, South Africa, conceptual understanding, procedural fluency

Introduction: Mathematical proficiency in learning and teaching

Kilpatrick, Swafford & Findell (2001: 116) contend that the term mathematical proficiency (MP) was chosen to “capture what we believe is necessary for anyone to learn mathematics successfully”. Proficiency in school mathematics was characterised in terms of five strands:

- Conceptual understanding
- Procedural fluency
- Adaptive reasoning
- Productive disposition
- Strategic competency

Noor Ally
Durban University of Technology
E-mail: noora@dut.ac.za
Telephone: 031 3732281

Iben Maj Christiansen
University of KwaZulu-Natal
E-mail: christianseni@ukzn.ac.za
Telephone: 033 2606092

The expectation is that a successful mathematics learner is proficient in mathematics if s/he ‘possesses’ the five component strands in such a way that they can be brought to bear on different situations. These strands are interwoven and interdependent. While many teacher educators may regard conceptual understanding as superior and in contrast to procedural fluency (Bossé & Bahr, 2008), we stress Kilpatrick et al.’s point that the strands support rather than limit each other (2001).

This particular explication of MP was a result of the National Research Council of the USA convening a group of experts to review research on effective mathematics learning, in 1999. Since then, questions have been asked about how to assess for MP, and the strands of MP have been used as a framework for assessing the impact of teaching approaches (Langa & Setati, 2007; Moodley, 2008; Pillay, 2006; Samuelsson, 2010; Suh, 2007; Takahashi, Watanabe & Yoshida, 2006), or have informed classroom practice (Suh, 2007). In South Africa, the MP notion has been used to analyse the curriculum (Sanni, 2009) and inform initiatives for teachers (government gazette of 14th March 2008).

While “designing classroom environments and teaching pedagogies that effectively promote this vision, has proven more elusive” (Pape, Bell & Yetkin, 2003: 180), the five strands of proficiency are not useful descriptors of teaching or instructional situations. If the aforementioned attributes of proficient learners are to be developed over time, classroom teaching needs to be constructed so that each strand is promoted. This is in line with the notion of mathematical knowledge for teaching (MKT) as defined by Ball, Hill and Bass (2005), who regard MKT to be about disciplinary knowledge with a view to assisting learners and students in their development of MP.

When Kilpatrick and his team wrote about developing proficiency in teaching mathematics (Kilpatrick et al., 2001: Chapter 10), they did not engage indicators of teaching for each strand of MP. Instead, they discussed the knowledge base for teaching mathematics including three types, namely mathematical knowledge, knowledge of students, and knowledge of instructional practice. (This is in line with the typologies for mathematics teacher knowledge proposed by others. See, for instance, Adler & Patahuddin, 2012; Ball, Thames & Phelps, 2008; Christiansen & Bertram, 2012; Krauss & Blum, 2012; Shulman, 1986). Kilpatrick et al. (2001: 380) then went on to list five components of proficient teaching of mathematics, which are, however, not linked directly to the strands of MP. Thus, they do not serve as indicators of the extent to which teaching promotes MP.

Some years later, Schoenfeld and Kilpatrick (2008: 322) produced a different list of requirements for proficiency in teaching mathematics, but also not directly linked to the strands of MP. It included general competencies such as crafting and managing learning environments, developing classroom norms, and supporting classroom discourse as part of “teaching for understanding”, and so on.

They also mentioned that proficient mathematics teachers have both deep and broad knowledge of school mathematics, including representations (cf. Suh, 2007) and conceptual connections (cf. Hattie, 2003). A recent case study in South Africa took up the latter, finding that high school teachers displayed faulty or superficial mathematical connections (Mhlolo, Venkat & Schäfer, 2012). The notion of deep and broad knowledge has several similarities with the notion of profound understanding of fundamental mathematics proposed by Ma (1999). However, this notion was not explicitly connected to the specifics of the strands of mathematical proficiency.

A recent large-scale study in Southern Africa (to which the study reported on, in this article, was linked) used a range of different frameworks to code the teaching, under the notion of strands of mathematical proficiency (Carnoy, Chisholm & Chilisa, 2012). However, video recordings of lessons were coded for the presence of the strands of mathematical proficiency without making an analytical or methodological distinction between the strands as learning outcomes and the opportunities to develop these strands as reflected in the instructional situation.

It is within this frame of reference that we propose the notion of opportunity to develop mathematical proficiency (OTDMP) as ‘the existence of an opportunity to develop, promote or advance mathematical proficiency via one or a combination of its component strands’. The notion of OTDMP is composed of five categories matching the five strands of MP on a one-to-one basis: Opportunity To Develop Conceptual Understanding (OTDCU), Opportunity To Develop Procedural Fluency (OTDPF), Opportunity To Develop Productive Disposition (OTDPD), Opportunity To Develop Adaptive Reasoning (OTDAR), and Opportunity To Develop Strategic Competence (OTDSC).

The presence of an opportunity does not imply that the opportunity is realised in learning; it simply means that there is the prospect of learners engaging mathematically in such a way that one or more strands of mathematical proficiency could be furthered. Obviously, there may be circumstances or learner attributes which act as barriers to learning (Christiansen & Aungamuthu, 2012; Hattie, 2003; Moloï & Strauss, 2005; SACMEQ II, 2010; Spaul, 2011; Van der Berg et al., 2011). Yet making a distinction between MP and OTDMP allows us to distinguish between the opportunities presented through teaching and the extent to which such opportunities are realised in learning, thus expanding our possibilities of researching links between teaching and learning.

Proposing the OTDMP notion is based on the assumption that opportunities to develop the strands must be present in a mathematics lesson for learners to become mathematically proficient. While the presence of instructional materials and so on may contribute to opportunities to develop mathematical proficiency, teaching is still the most important in-school factor in promoting learning (Hattie, 2003).

The five strands of OTDMP provide a conceptual language in which to discuss learning and teaching, and as such they form an internal language of description,

in Bernstein's sense (Bernstein, 1996). However, in order to utilise this in describing classroom practices, an external language of description – or analytical framework – must be created. We postulate three necessary conditions of OTDMP that inform such a framework: the opportunities must exist, occur regularly, and occur with a degree of strength – of course, influenced by the learners' personal attributes, circumstances and their current understanding of the topic being dealt with (cf. Christiansen & Aungamuthu, 2012).

Five strands of OTDMP

Opportunity To Develop Conceptual Understanding

According to Kilpatrick et al. (2001: 116), conceptual understanding refers to the “grasp of mathematical ideas, its comprehension of mathematical concepts, operations and relations”. Without it, many of the other strands are significantly weakened – reasoning relies on understanding (Gilbert, 2008), and some studies indicate that the procedurally focused instruction without conceptual support plays a role in the development of mathematics anxiety (Rayner, Pitsolantis & Osana, 2009: 63).

Zandieh (1997: 105), referring to Thompson (1994), makes the point that it is our subjective sense of invariance that makes us perceive one underlying structure across a number of contexts, which we then perceive to be representations of that structure. Thus, conceptual understanding is intrinsically linked to representations (graphical, tabular, algebraic, narrative, manipulative, and so on). Lesh, Post and Behr (1987) even equated understanding to the ability to recognise, manipulate and translate an idea/concept in and between different representations, thus also stressing the point of connections. In his general discussion, Hugo (2013) emphasises that disciplinary knowledge means both understanding the ideas and how they connect.

This is reflected in discussions about teaching for understanding. ‘Using connected representations’ is one of three characteristics of excellent teachers, if the work of Hattie and colleagues is to be accepted (Hattie, 2003). Being able to make links between concepts and representations is a key skill of competent teachers (Ma, 1999). Suh (2007) stressed the use of representations to foster conceptual understanding, and Mhlolo et al. (2012) drew on Barmby, Harries, Higgins and Suggate's (2009) perspectives to characterise deep understanding of mathematics through connections between mathematical ideas, representations, and reasoning between mathematical ideas. The latter is supported by the finding from the USA that conceptual understanding develops when connected pieces of ideas are merged by reasoning and justifying (Richland, Zur & Holyoak, 2007). (See the discussion of OTDAR.) The authors of a well-known professional development book (Stein, Smith, Henningsen & Silver, 2000) also use the notion of connections to distinguish between procedures taught with or without conceptual understanding. Thus, we are in good company when focusing on connections as indicators of opportunities to develop conceptual understanding.

Opportunity To Develop Productive Disposition

Productive disposition refers to the “habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (Kilpatrick et al., 2001: 131). Thus, one aspect refers to the perceptions of the discipline, and one refers to perceptions of self, each of which has additional aspects. All of these aspects have been engaged by Siegfried (2012), who also presents a rubric for evaluating the productive disposition of learners.

Perceiving ‘mathematics as sensible’ could be related to personal sense-making, which some have found increases when engaged in structured inquiry-based learning (see, for instance, Smith & Klein, 2010). It links to the connectedness of mathematical ideas discussed earlier.

In a Realistic Mathematics Education context, Middleton, Leavy, Leader and Valdosta (2013: 1) found that “with curriculum designed to emphasize utility and interest, students forged a high degree of motivation”. The issue of ‘usefulness’, referring out of mathematics to its applications, is a complex and loaded one, and requires discussion of the extent to which the applications are mere sugar coating, distractors actually increasing extrinsic cognitive load (Artino, 2008), social positioning strategies limiting access to disciplinary knowledge (Dowling, 1998), or motivation for learning mathematics. For that reason, our analysis of the empirical data clearly separated out this aspect.

Without going into detail concerning self-efficacy theories (see, for instance, Bobis, Anderson, Martin & Way, 2011), these are concerned with existing levels of anxiety, past performance experiences on similar tasks, experiences of peers’ performance, experiences of fruitful persistence, experiences of learning from difficulties/failures, and suggestions by others on how one may perform (Bandura, 1977, in this instance from Siegfried, 2012). Thus, in the classroom situation it relates to the general atmosphere of the classroom, including encouraging the learning and doing of mathematics (Boaler, 2002), conducive socio-mathematical norms (Yackel & Cobb, 1996), and perseverance or a ‘growth mindset’ (Dweck, 2006). In our empirical analysis, we sought signs that the teacher showed sensitivity towards learners’ previous difficulties, encouraged persistence, and accepting mistakes as part of learning.

Opportunity To Develop Procedural Fluency

By ‘procedural fluency’, Kilpatrick et al. (2001: 121) referred to the “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them accurately, flexibly and efficiently”. Wong and Evans (2007: 825), citing Rittle-Johnson, Siegler and Alibali (2001), state that “developing students’ procedural knowledge had positive effects on their conceptual understanding, and conceptual understanding was a prerequisite for the students’ ability to generate and select appropriate procedures”. This follows the distinction made between

procedures taught with and without connections (Stein et al., 2000). Thus, we rated ‘unpacking’ of procedures, which reflects the structure of the discipline, higher than less conceptually founded explanations or demonstration, whether this was done by the teacher – ‘task controlled’ (Stigler, Gallimore & Hiebert, 2000) – or learners were guided to do so – ‘solver controlled’ (Stigler, Gallimore & Hiebert, 2000). Fluency in choosing appropriate procedures relies on knowing more than one and in understanding which one to use when and why (Ma, 1999), and was used as another indicator.

Opportunity To develop Adaptive Reasoning

‘Adaptive reasoning’ refers to the “capacity for logical thought, reflection, explanation, and justification” (Kilpatrick et al., 2001: 129). “In mathematics, adaptive reasoning is the glue that holds everything together, the lodestar that guides learning” (Kilpatrick et al., 2001: 129). The centrality of proof, conjecturing, refuting, reasoning in mathematics learning is widely argued (Niss & Jensen, 2002; Powell, Francisco & Maher, 2003; Stigler & Hiebert, 1997; Stylianides & Stylianides, 2008). However, adaptive reasoning also includes creating and understanding appropriate analogies; intuitive and deductive reasoning based on patterns, analogy and metaphor, and preformal reasoning such as reasoning from representation (Blum & Kirsch, 1991).

Thus, we took all instances of justification as indicative of facilitating adaptive reasoning, but when learners were encouraged to actively engage in justification, it was rated highly, whereas using inappropriate analogies (cf. Christiansen, forthcoming) was given a low rating.

Opportunity To Develop Strategic Competence

Strategic competence is “the ability to formulate, represent and solve mathematical problems. [...] Students need to encounter situations in which they need to formulate the problem so that they can use mathematics to solve it” (Kilpatrick et al., 2001: 124). Engaging in problem-solving may further enhance conceptual understanding and adaptive reasoning (Samuelsson, 2010).

Problem-solving has been widely discussed, including developing models for, and engaging the heuristics involved in problem-solving (De Corte, Verschaffel & Masui, 2004; Higgins, 1997; Polya, 2008; Schoenfeld, 1992). Any problem-solving or problem-posing activity whether using pictures, flow charts, lists or other was considered indicative of the opportunity to develop strategic competency, but in formulating rating criteria, we distinguished between more and less appropriate heuristics, whether the teacher reduced the problem to something solvable by standard procedures (cf. Taylor & Taylor, 2013), and the extent to which choosing appropriate problem-solving strategies was engaged.

A rubric for OTDMP

Table 1 shows the rubric capturing the external language of description for OTDMP.

Table 1: Descriptors for rating the opportunity to develop each strand of MP

Components of OTDMP	Rating 1 (Low)	Rating 2 (Medium)	Rating 3 (High)
OTDCU	No link between concepts or representations. Representations do not 'capture' central aspects of concepts.	Some links between concepts and/or representations.	Clear explanations of concepts. Connections to other concepts indicated. At least two representations explicitly connected.
OTDPD	Inconsistent messages of ability, effort or performance. Real-world situations described, but not explicitly related to mathematics.	Occasional positive reinforcement of effort, performance or ability. Encouragement of interest. Some attempts to stress sense-making. Real-world situations mentioned and used to motivate mathematics, but connections are only partially explicit.	Demonstrated sensitivity, respect and interest in learners' responses and questions. Learners encouraged to persevere and learn. Opportunities fostered to develop links between real-world situations and mathematics.
OTDPF	Only one procedure shown, with no justification. Procedures may not be performed fluently by teacher.	Opportunities offered to perform procedures appropriately and fluently. Some reasons for the procedure given. Alternate procedures not explored.	Why, when and how a procedure is applied is clear. Coherent sequencing in development of a procedure. Different procedures may be compared.
OTDAR	Justifications given, but invalid, with reference to authority or through inappropriate analogies.	Reasoning explicit and valid. Justifications sometimes given by teacher.	Justifications occur frequently. Learners encouraged to justify.
OTDSC	Inappropriate heuristics; problem-solving reduced to algorithm.	A single heuristic appropriate to the topic.	Multiple heuristics to solve problems. Opportunities to choose flexibly among these engaged.

Method

The project was started in 2009 as part of the KwaZulu-Natal Provincial Treasury Project, and linked with a project overseen by Carnoy and Chisholm (Carnoy et al., 2012). Forty primary schools were sampled from one education district in KwaZulu-Natal, using stratified (according to DoE quintile) random sampling to comprise 75% less-resourced schools and 25% better resourced schools. The final number of schools was 39. All Grade six teachers in the sampled schools participated in the study. Research assistants were trained in data collection and were often paired with experienced researchers. The original project had a post-positivist orientation, assuming measurability of several variables such as the teachers' pedagogical content knowledge. However, the part of the project reported on in this article engaged the data from an interpretivist orientation.

One lesson with each teacher was video recorded, mostly in the beginning of Grade 6 though some later in the year. Other data were collected from teachers, learners and principals. In this article, we focus on the videos. Some teachers did not want to have lessons recorded and, as a result, only 30 lessons were recorded, which must be borne in mind when considering the findings.

Each lesson was viewed in intervals of 5 minutes, with each segment being assessed for the five strands of OTDMP. This gave rise to a total of 242 five-minute segments coded.

Besides assigning a rating for the strength of the strand, it was indicated whether the OTDCU was through the teacher stating the concept or developed in other ways; whether the OTDPF was through a procedure given or developed by learners, and whether the OTDPD concerned perseverance, confidence or applications. Other aspects were also coded for, but are not engaged in this article.

Findings

Table 2 shows the frequencies of the ratings for each strand. There were no instances in any of the recorded lessons that warranted a rating of 3 for any of the strands.

Table 2: Frequency of the ratings of the strands of OTDMP in the sample

OTDMP	OTDCU			OTDPF			OTDSC			OTDAR			OTDPD		
Rating	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
Frequency	24	17	0	142	81	0	3	1	0	10	9	0	27	22	0
Presence	41			223			4			19			49		

Figure 1 illustrates how many of the lessons analysed were coded for each of the opportunities. This is similar to the presence of the strands of MP found in the North-West Province and Botswana (Carnoy et al., 2012: 108), which gives us reason to believe that this is a fairly representative picture. It gives the impression that OTDMP is frequent, but this disguises the generally low rating of these occurrences. Such rating cannot be determined for the Carnoy et al. study.

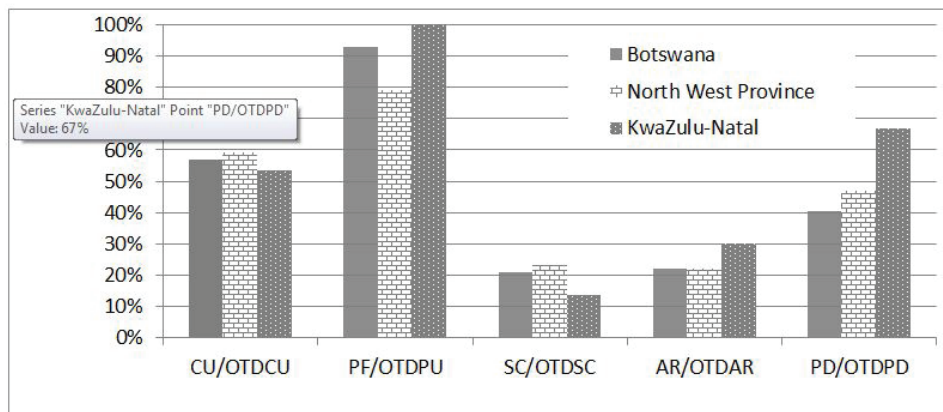


Figure 1: Percentage of lessons coded for each of the five strands of OTDMP, compared to the data from Botswana and North-West Province, as read from the graph in Carnoy et al. (2012: 108)

It appears that there are more opportunities to develop productive disposition in the KwaZulu-Natal classrooms. Perhaps this is due to the inclusion of both opportunities to develop confidence and opportunities to appreciate the applicability of mathematics, which could have been given a different weighting in the two studies.

Overall, opportunities to develop operations with numbers, calculate areas or perimeter of regular 2-dimensional figures, find averages in statistical data, draw bar charts from given data, and other procedural work was dominant in lessons. It was only in slightly more than over half of the observed lessons that the opportunity to develop conceptual understanding was present, and the other strands were scarcer.

The connectedness and intertwined nature of the strands of mathematical proficiency makes it relevant to consider to what extent the opportunity to develop strands were present in the same lesson. Just over half of the lessons observed showed three opportunity strands present, whereas five of the thirty lessons exhibited a single strand only. Two lessons contained the opportunity to develop all five strands of proficiency.

OTDCU was noted in 16 of the sampled lessons in a total of 41 five-minute segments. In 32 of the instances, the teacher simply stated or directed learners' attention to a concept. In the remaining 9 segments, a concept was formulated through discussion or demonstration. All of these instances occurred in three lessons.

In one lesson, learners were introduced to the concept of decimal fractions. Learners counted in tenths from 1.0 to 1.9. There was no link made to other representations, and the counting procedure could reinforce the common misconception that real and rational numbers have successors the same way integers do (Roche, 2005). This is in line with Mhlolo et al.'s (2012: abstract) finding that the “teachers’ representations of mathematical connections were either faulty or superficial in most cases” (abstract).

A rating of 2 was given to a lesson where learners constructed various 3-dimensional shapes. Use of these figures was made to connect the shapes with corresponding concepts (faces, edges and vertices). Four of the total 13 OTDCU with a rating of 2 appeared in this particular lesson. This reflects the peculiar situation in South Africa where the variation between schools is much greater than the variation within schools (ie., between learners), compared to other SACMEQ countries (SACMEQ II, 2010).

OTDPF dominated the mathematics lessons. Sixty-nine per cent of coded opportunities were opportunities to develop procedural fluency. In all but one of these lessons, a procedure was demonstrated by the teacher. Generally, the lessons unfolded with first presenting examples and then assigning similar problems for the learners to complete.

Only four instances of OTDSC were identified. One each of the following heuristic strategies was recorded: pictures, lists, trial and error, consulting similar problems.

There was some indication that there is a correlation between learner achievement gains and OTDAR, yet this is not a reliable correlation due to the infrequency of the strand: 19 incidences of reasoning, justification and proving occurred across 9 lessons.

Forty-one instances of OTDPD occurred. The majority, 31 of these, considered real-world examples. However, approximately half of these had a rating of 1, because the links to mathematics were not made explicit. This is likely to lessen the impact of the opportunity to develop productive disposition, as well as provide entry into mathematics (Hoadley, 2007; Maton, 2013).

Encouraging learners to persevere, praising learners’ efforts, or instilling confidence in the mathematics with which they are working was seldom recorded. Many lessons showed episodes in which chorus answers were expected, with teachers acknowledging correct or incorrect answers to the class as a whole. Instances in which teachers coaxed learners to correct their own mistakes or praised a novel solution were not observed.

Conclusion

We have found the external language of description presented in this article useful in our analysis of classroom teaching. Our findings using this instrument were compatible with those from the study of teaching in North-West Province and Botswana, but the rubric allows us to interrogate the quality of the OTDMP in more depth and detail. One concern is how to code for the ways in which the various strands of OTDMP relate; this requires further attention.

Applying the external language of description to the data from 30 Grade 6 mathematics lessons in KwaZulu-Natal showed that, while opportunities to develop MP exist, they are limited both in range and in quality. The quality was apparent in the ratings, and in the subsequent coding within the strands of OTDMP, where concepts and procedures were stated, not explored, and where the 'glue' of mathematical proficiency – adaptive reasoning – was given little room to develop.

During mathematics lessons, the strand of procedural fluency was clearly prevalent in the Grade 6 KwaZulu-Natal teachers' current practice. However, without OTDCU, it likely remains procedures without connections (Stein et al., 2000), and thus its impact on the development of mathematical proficiency is likely to be limited. When given narrow procedural questions all the time, students do not get the opportunity to work on a level that is right for them, where they can contribute and be challenged. Incidentally, this not only reduces opportunities to learn, but also likely leads to lack of germane cognitive load, thus limiting schemata construction (Artino, 2008). Finally, it means that the teaching methods are not appropriate for un-streaming or un-tracking learners, and as such is likely to continue to reproduce inequalities both within schools and after (cf. Boaler, 2011).

Acknowledgements

Data collection for this research was funded by the KwaZulu-Natal Provincial Treasury. The views expressed in this article do not necessarily reflect the views of the funders or the authors of the full report accessible at <https://www.dropbox.com/sh/lfbew0p2sqrhpd5/5KFkJ7dEsW>. Dr N. Mthiyane, University of KwaZulu-Natal, oversaw data collection. Thank you to the other participants, and to the teachers who allowed researchers to video record lessons.

References

- Adler J & Patahuddin SM 2012. Quantitative tools with qualitative methods in investigating mathematical knowledge for teaching. *Journal of Education*, (56): 17-43.
- Artino AR Jr 2008. Cognitive load theory and the role of learner experience: An abbreviated review for educational practitioners. *Association for the Advancement of Computing in Education Journal*, 16(4): 425-439.
- Ball DL, Hill HC, & Bass H 2005. Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(3): 14-46.
- Ball DL, Thames MH & Phelps G 2008. Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5): 389-407.
- Bandura A 1977. Self-efficacy: Toward a unifying theory of behavioral change. *Psychological Review*, 84: 191-215.
- Barmby P, Harries T, Higgins S & Suggate J 2009. The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, 70(3): 217-241.
- Bernstein B 1996. *Pedagogy, symbolic control and identity: Theory, research, critique*. London: Taylor & Francis.
- Blum W & Kirsch A 1991. Preformal proving: Examples and reflections. *Educational Studies in Mathematics*, 22(2): 183-203.
- Boaler J 2002. The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms. *For the Learning of Mathematics*, 22(1): 42-47.
- Boaler J 2011. Changing students' lives through the de-tracking of urban mathematics classrooms. *Journal of Urban Mathematics Education*, 4(1): 7-14.
- Bobis J, Anderson J, Martin A & Way J 2011. A model for mathematics instruction to enhance student motivation and engagement. *Motivation and disposition: Pathways to learning mathematics, National Council of Teachers of Mathematics Seventy-Third Yearbook*, 2011.
- Bossé MJ & Bahr DL 2008. The state of balance between procedural knowledge and conceptual understanding in mathematics teacher education. Retrieved on 4 November 2011 from <http://www.cimt.plymouth.ac.uk/journal/bossebahr.pdf>.
- Carnoy M, Chisholm L & Chilisa B (eds) 2012. *The low achievement trap*. Cape Town: HSRC Press.
- Christiansen IM (forthcoming). When conceptual blending isn't: Examples from mathematics education. In W Hugo (ed), *Conceptual blending* (pp. 115-123). Cape Town: Pearson.
- Christiansen IM & Aungamuthu Y 2012. Language issues, misconceptions and confusion in abundance: A qualitative analysis of grade 6 learners' responses on a mathematics test. *Education as Change*, 16(1): 51-67.
- Christiansen IM & Bertram C 2012. Editorial: Special issue on teacher knowledge and teacher learning. *Journal of Education*, (56): 1-16.

- De Corte E, Verschaffel L & Masui C 2004. The CLIA-model: A framework for designing powerful learning environments for thinking and problem solving. *European Journal of Psychology of Education*, 19(4): 365-384.
- Dowling P 1998. *The sociology of mathematics education: Mathematical myths/pedagogic texts*. Vol 7. London: Falmer Press.
- Dweck CS 2006. *Mindset: The new psychology of success*. New York: Ballantine Books.
- Gilbert MC 2008. *Applying contemporary views of mathematical proficiency to the examination of the relationship of motivation and mathematics achievement*. Unpublished PhD, University of Michigan.
- Hattie J 2003. *Teachers make a difference: What is the research evidence?* Paper presented at the Australian Council for Educational Research Annual Conference on Building Teacher Quality. Retrieved on 5th May 2012 from <http://www.educationalleaders.govt.nz/Pedagogy-and-assessment/Evidence-based-leadership/Measuring-learning/Teachers-Make-a-Difference-What-is-the-Research-Evidence>
- Higgins KM 1997. The effect of year-long instruction in mathematical problem solving on middle-school students' attitudes, beliefs, and abilities. *The Journal of Experimental Education*, 66(1): 5-28.
- Hoadley U 2007. The reproduction of social class inequalities through mathematics pedagogies in South African primary schools. *Journal of Curriculum Studies*, 39(6): 679-706.
- Hugo W 2013. *Cracking the code to educational analysis*. Cape Town: Pearson.
- Kilpatrick J, Swafford J & Findell B (eds) 2001. *Adding it up: Helping children learn mathematics*. Washington: National Academy Press.
- Krauss S & Blum W 2012. The conceptualisation and measurement of pedagogical content knowledge and content knowledge in the COACTIV study and their impact on student learning. *Journal of Education*, (56): 45-65.
- Langa M & Setati M 2007. *Investigating the use of learners' home languages to support mathematics learning*. Paper presented at the the 13th Annual National Congress of the Association for Mathematics Education of South Africa, White River, Mpumalanga, 2nd-6th July.
- Lesh R, Post T & Behr M 1987. Representations and translations among representations in mathematics learning and problem solving. In C Janvier (ed), *Problems of representation in the teaching and learning of mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.
- Ma L 1999. *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
- Maton K 2013. Making semantic waves: A key to cumulative knowledge-building. *Linguistics and Education*, 24(1): 8-22.
- Mhlolo MK, Venkat H & Schäfer M 2012. The nature and quality of the mathematical connections teachers make. *Pythagoras*, 33(1). Retrieved on 23rd July 2013 from <http://dx.doi.org/10.4102/pythagoras.v33i1.22>
- Middleton JA, Leavy A, Leader L & Valdosta G 2013. A path analysis of the relationship among critical motivational variables and achievement in reform-oriented mathematics curriculum. *RMLE Online*, 36(8). Retrieved on 23rd July 2013

from [http://www.amle.org/Publications/RMLEOnline/Articles/Vol36No8/
tabid/2823/Default.aspx](http://www.amle.org/Publications/RMLEOnline/Articles/Vol36No8/tabid/2823/Default.aspx)

- Moloi M & Strauss J 2005. *The SACMEQ II Project in South Africa: A study of the conditions of schooling and the quality of education*. Harare: SACMEQ.
- Moodley VG 2008. *A description of mathematical proficiency, in number skills, of grade ten learners in both the Mathematics and Mathematics Literacy cohorts at a North Durban school*. Durban: University of KwaZulu-Natal.
- Niss M & Jensen TH (eds) 2002. *Kompetencer og matematikl ring: Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Vol 18. Copenhagen: Undervisningsministeriet.
- Pape SJ, Bell CV & Yetkin IE 2003. Developing mathematical thinking and self-regulated learning: A teaching experiment in a seventh-grade mathematics classroom. *Educational Studies in Mathematics*, 53: 179-202.
- Pillay V 2006. *Mathematical knowledge for teaching functions*. Paper presented at the the 14th Annual SAARMSTE Conference, University of Pretoria, South Africa, 9th-12th January.
- Polya G 2008. *How to solve it: A new aspect of mathematical method*. Princeton: Princeton University Press.
- Powell A, Francisco J & Maher C 2003. An evolving analytical model for understanding the development of mathematical thinking using videotape data. *The Journal of Mathematical Behavior*, 22(4): 405-435.
- Rayner V, Pitsolantis N & Osana H 2009. Mathematics anxiety in preservice teachers: Its relationship to their conceptual and procedural knowledge of fractions. *Mathematics Education Research Journal*, 21(3): 60-85.
- Richland LE, Zur O & Holyoak KJ 2007. Cognitive supports for analogies in the Mathematics classroom. *Science*, (316): 1128-1129.
- Rittle-Johnson B, Siegler RS & Alibali MW 2001. Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93: 346-362.
- Roche A 2005. Longer is larger - or is it? *Australian Primary Mathematics classroom*, (10): 11-16.
- SACMEQ II 2010. Visualization of research results on the quality of education. Retrieved 1st June 2011 from [http://www.sacmeq.org/visualization-research.
htm#visualization2](http://www.sacmeq.org/visualization-research.htm#visualization2).
- Samuelsson J 2010. The impact of teaching approaches on students' mathematical proficiency in Sweden. *International Electronic Journal of Mathematics Education*, 5(2): 61-78.
- Sanni R 2009. Mathematical proficiency and practices in Grade 7. *Learning and Teaching Mathematics*, 7: 25-28.
- Schoenfeld AH 1992. Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

- Schoenfeld AH & Kilpatrick J 2008. Toward a theory of proficiency in teaching mathematics. *The International Handbook of Mathematics Teacher Education*, 2: 321-354.
- Shulman LS 1986. Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2): 4-14.
- Siegfried JM 2012. *The hidden strand of mathematical proficiency: Defining and assessing for productive disposition in elementary school teachers' mathematical content knowledge*. San diego, CA: University of California/San Diego State University.
- Smith K & Klein M 2010. *Preparing preservice teachers to step up to the intention of the new Australian mathematics curriculum: Can a productive disposition be reignited?* Paper presented at the Australian Teacher Education Association Conference, Townsville, Queensland, Australia, 4-7 July 2010.
- Spaull N 2011. A preliminary analysis of SACMEQ III. *Stellenbosch Economic Working Papers*, (11/11). Retrieved on 23rd July from www.ekon.sun.ac.za/wpapers/2011/wp112011/wp-11-2011.pdf.
- Stein MK, Smith MS, Henningsen MA & Silver EA 2000. *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.
- Stigler JW, Gallimore R & Hiebert J 2000. Using video surveys to compare classrooms and teaching across cultures. Examples of lessons from the TIMMS video studies. *Educational Psychologist*, 35(2): 87-100.
- Stigler JW & Hiebert J 1997. Understanding and improving classroom mathematics instruction. *Phi delta kappan*, 79(1): 14-21.
- Stylianides GJ & Stylianides AJ 2008. Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning*, 10: 103-133.
- Suh JM 2007. Tying it all together: Classroom practices that promote mathematical proficiency for all students. *Teaching Children Mathematics*, 14(3): 163-169.
- Takahashi A, Watanabe T & Yoshida M 2006. Developing good mathematics teaching practice through lesson study: A U.S. perspective. In T Miyakawa, K Aoyama & K Chino (eds), *Progress report of the APEC project: "A Collaborative Study on Innovations for Teaching and Learning Mathematics in Different Cultures among the APEC Member Economies" - Innovative Teaching Mathematics through Lesson Study* (pp. 129-136). Tsukuba, Japan: University of Tsukuba.
- Taylor N & Taylor S 2013. Teacher knowledge and professional habitus. In N Taylor, S van der Berg & T Mabogoane (eds), *Creating effective schools* (pp. 202-230). Cape Town: Pearson.
- Thompson PW 1994. Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26(2-3): 229-274.
- Van der Berg S, Burger C, Burger R, De Vos M, Du Rand G, Gustafsson M, Moses E, Shepherd D, Taylor S, Van Broekhuizen, H. & Von Fintel D 2011. *Low quality education as a poverty trap*. Stellenbosch: Stellenbosch University.

- Wong M & Evans D 2007. Students' conceptual understanding of equivalent fractions. Paper presented at the 30th Annual Conference of the Mathematics Education Research Group of Australasia, 2nd-6th July.
- Yackel E & Cobb P 1996. Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4): 458-477.
- Zandieh MJ 1997. The role of a formal definition in nine students' concept image of derivative. In S Berenson, K Dawkins, M Blanton, W Coulombe, J Klob, K Norwood & L Stiff (eds), *Proceedings of the 20th annual meeting of the North American chapter of the international group for the Psychology of Mathematics Education* (pp. 136-141). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.