The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners in a selected school in the Pinetown District.

By

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A thesis for the degree of

Doctor of Education (Mathematics Education)

In the School of Education

Faculty of Arts and Design

Durban University of Technology (Indumiso Campus)

February 2022

APPROVED FOR FINAL SUBMISSION

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ABSTRACT

Mathematics results in South Africa continue to show an area of weakness. Trigonometry is currently an examinable section in the subject mathematics and is assessed in grade 12 examinations.

This study made use of concrete mathematical manipulatives in exploring conceptual understanding of grade 12 Mathematics learners in learning to solve 3-dimensional trigonometric problems. The research questions addressed in this study are: How will Mathematical models help learners to adopt more active approaches towards the learning of three-dimensional trigonometric problems amongst Grade 12 learners? How will the use of models improve the learning of Mathematics?

The use of sound teaching and learning principles fosters an environment where learners are motivated to reach their full potential. The study was informed by constructivist ideas. Grade 12 learners of mathematics participated in the research, engaging in qualitative assessments in the form of activity worksheets and semi-structured interviews.Results and analyses reveal that using manipulatives is beneficial to learnerswhen solving 3D trigonometric problems. The findings in this study are important for the South African education system. The changing school climate and the improving of learning strategies with the use of mathematics manipulatives in Mathematics is very important in improving learner performance and the Mathematics results in our country. Underperformance in Mathematics has persistently been a challenge in SouthAfrican schools. This study was conducted to add to current research in the debates about the use of manipulatives in mathematics in classrooms in secondary schools. In addition, this study was carried out to: determine how Mathematical models/manipulatives helped grade 12 learners to adopt more active approaches towards the learning of three-dimensional trigonometric problems and whether the use of models does improve the learning of mathematics. The research elicited sufficient data to support Piaget andBruner's theory which suggests that the use of manipulatives aids in constructing new mathematical knowledge, providing a link from concrete to abstract.

Key words: manipulatives in teaching, constructivism,concrete, abstract representations in mathematics, mathematics activity worksheets

DEDICATION

Many thanks go:

- To my husband Dr Ivan Niranjan, my mentor, my pillar of strength, mysoul mate and my daughters Kiara and Alka Niranjan;

- To my late dad, Krassan Rugnandan and my late mum Gyanmathie Rugnandan, who always provided for their children. My parents always encouraged and supported me in my studies;

- To my late brothers Dhamjee Rugnandan and Naran Rugnandan who always showed a keen interest in my education;

- To my sister Indira Singh, who also played an important role in my education.

  It is through the support of my family that I have arrived at this point in my education.

ACKNOWLEDGMENTS

I would like to acknowledge Professor Deonarain Brijlall who identified the potential for a doctoral thesis in this area. He continuously guided me through the process and always provided constructive and timeous feedback. Thank you for the support, care and encouragement offered to me during this journey.

Appreciation is extended to Doctor Veronica Masuku from the University of Kwa Zulu Natal for her assistance, guidance and advice during the course of my study. On several occasions Dr Masuku assisted with the administrative aspects of the study and constantly encouraged me to persevere with the study.

A special thank you to the grade 12 Mathematics learners who participated in this research study. In addition, I would like to thank the Principal of the school for granting me permission to conduct research at the school.

A special thank you to my husband and daughters who made have many sacrifices. Thank you for being supportive, understanding, caring and patient as I spent several long hours with my studies. For that I am eternally grateful.

DECLARATION

I, Caresse Niranjan, declare that:

i. The research reported in this thesis, except where otherwise indicated, is my own work.

ii. This thesis has not been submitted previously for any degree or examination at any university or other higher education institution.

iii. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

Signed: 29 March 2022 Date:

31-03-2022

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Supervisor: Prof D. Brijlall Date

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GLOSSARY

AC	Abstract Conceptualization
ACE	Advanced Certificate in Teaching
AE-	Active Experimentation
CAPS	Curriculum and Assessment Policy Statement
CE	Concrete Experience
CRA	Concrete, Representational, Abstract
DoBE	Department of Basic Education
DHET	Department of Higher Education and Training
3D	Three Dimensional
ELT	Experimental Learning Theory
FET	Further Education and Training
GDP	Gross Domestic Product
HOTS	Higher Order Thinking Skills
HEIs	Higher Education Institutes
4IR	Fourth Industrial Revolution
LOTS	Lower Order Thinking Skills
NBPTS	National Board for Professional Teaching Standards
NCTM	National Council of Teachers of Mathematics
NSC	National Senior Certificate
OECD	Organisation for Economic Co-Operation and Development
PCK	Pedagogical Content Knowledge
RO	Reflective Observation
SAGM	Subject Assessment Guidelines for Mathematics
SAQMEQ	Southern and Eastern African Consortium for Monitoring Educational Quality
STEM	Science, Technology, Engineering and Mathematics
TIMMS	Trends in International Mathematics and Science Study
VUCA	Volatile, Uncertain, Complex and Ambiguous
AC	Abstract Conceptualization

LIST OF APPENDICES

CHAPTER ONE: INTRODUCTION

1.1 Overview

This chapter provides an overview of the study. The background and purpose of this study is explained first. The motivation for conducting this study and the nature of Mathematics with respect to the study are discussed. Research questions and key terminology in the study are presented and explained. Thereafter National Performance in the Eleven most popular subjects (2015-2019) and performance trends are analysed. This is followed by summaries of successive chapters.

1.2 Background

In the researcher's experience and discussions held with other educators, the use of the sine rule, cosine rule and area rule continues to be complicated and confusing to learners when applying them to solve three-dimensional problems. Solving three-dimensional trigonometric problems is one of the topics taught to grade 12 learners at high school. Many learners find it difficult to decompose the problem and select correct and logical methods andformulae to solve problems.

The teaching and learning setting in South African classrooms has been transformed by policy-makers revising curricula in accordance with the Curriculum and Assessment Policy Statement (DoBE, 2012). According to Van Laren (2012: 203) the revision plans are expected to improve teaching and learning in mathematics. This study addresses an alternate approach to teaching trigonometry by developing and using mathematical three-dimensional modelsto extend teachers pedagogical content knowledge for teaching Grade 12 three- dimensional (3D) problem solving in trigonometry. The study will address the challenge cited by the Minister of Education, the challenge being that teachers' knowledge and the supply of quality learning support material is inadequate in the area of trigonometry amongst other teaching subjects (Motshekga, 2012: 2). The resultant effect is poor performance by learners in Mathematics.

1.3 Purpose of Study

The rationale of this study focused on how to increase learners' performance in Mathematics through the use of 3 dimensional models. The results in Mathematics in South Africa continue to show an area of weakness. Trigonometry is currently an examinable subject and is assessed in grade 12 examinations.

According to Brijlall & Niranjan (2015: 363) manipulatives are tools used in mathematics instruction that, when used effectively, can positively assist pupils to grasp mathematical concepts taught in secondary schools.

This study offered an alternative approach to teaching trigonometry by refining and developing mathematical models to refine and improve teachers' pedagogical content knowledge grade 12 trigonometry.

Motshekga in DoBE (2017: 1) acknowledges that Mathematics analysis conducted reveals the weaknesses in learners' responses. The analysis of the misconceptions and error patterns exposed in learners' responses can inform instructional practice. These identified weaknesses can allow teachers to refine their teaching strategies appropriately. The Department of Education will, through interventions, continue to capacitate teachers in designing responsive and suitable instructional programmes that will effectively focus on the areas of weakness identified in Mathematics.

Swan (2006: 162) argues that traditional 'transmission' methods in which explanations, examples and exercises dominate do not promote robust, transferrable learning that endures overtime or that may be used in non-routine situations.

Trigonometry is now examinable and will be assessed in the Grade 12 examination. Therefore, providing an active learning environment assists learners to participate in the learning activity, thereby enhancing learning.

1.4 National Performance in Eleven most Popular subjects 2015-2019

The National Senior Certificate (NSC) School Subject Report (2019: 4) presents performance in eleven gateway subjects. The report is to be used by school managers, subject advisors, district planners and curriculum specialists to analyse specific subjects overall performance within a school, cluster, circuit or district level. The report assists education stakeholders to identify thespecific subject that is displaying poor performance and to provide and ensure appropriate interventions are immediately introduced.

Mathematics clearly reveals a downward trend and all stakeholders need to identify factors that have contributed to the poor performance and to arrive at solutions to best address the challenges that exist. If poor performance is detected early then it would allow for stakeholders to put in place effective interventions early enough to see a positive impact, thus enabling high performance levels and also increasing performance. Figure 1.1 presents the performance of learners in the most critical subjects. In figure 1.2 the National performance in the eleven most popular subjects from 2015-2019 is presented. Figure 1.3 reveals the steady decline in the number of Mathematics candidates from 2015- 2019.

The table embedded in figure 1.1 below shows the performance of learners in the eleven most critical subjects. These subjects are Mathematics, Physical Science, Mathematical Literacy, Life Sciences, History, Geography, English first additional language, Economics, Business Studies, Agricultural Sciences and Accounting.

	Year														
	2015			2016			2017			2018			2019		
Subject Description	Total Wrote	Achieved30-100%	% Achieved at 30% and Above	Total Wrote	Achieved30-100%	% Achieved at 30% and Above	Total Wrote	Achieved30-100%	% Achieved at 30% and Above	Total Wrote	Achieved30-100%	% Achieved at 30% and Above	Total Wrote	Achieved30-100%	% Achieved at 30% and Above
Accounting	140 474	83 747	59.6	128 853	89 507	69.5	103 427	68 318	66.1	90 278	65 481	72.5	80 110	62 796	78.4
Agricultural Sciences	104 251	80 125	76.9	106 386	80 184	75.4	98 522	69 360	70.4	95 291	66 608	69.9	92 680	69 132	74.6
Business Studies	247 822	187 485	75.7	234 894	173 195	73.7	204 849	139 386	68	192 139	124 618	64.9	186 840	132 571	71
Economics	165 642	112 922	68.2	155 908	101 787	65.3	128 796	91 488	71	115 169	84 395	73.3	107 940	74 796	69.3
English First Additional Language	543 941	528 157	97.1	547 292	533 235	97.4	503 151	488 572	97.1	498 959	485 112	97.2	489 072	477 560	97.6
Geography	303 985	234 209	77	302 600	231 588	76.5	276 771	212 954	76.9	269 621	200 116	74.2	271 807	218 821	80.5
History	154 398	129 643	84	157 594	132 457	84	147 668	127 031	86	154 536	138 570	89.7	164 729	148 271	90
Life Sciences	348 076	245 164	70.4	347 662	245 070	70.5	318 474	236 809	74.4	310 041	236 584	76.3	301 037	217 729	72.3
Mathematical Literacy	388 845	277 594	71.4	361 865	257 881	71.3	313 030	231 230	73.9	294 204	213 225	72.5	298 607	240 816	80.6
Mathematics	263 903	129 481	49.1	265 810	135 958	51.1	245 103	127 197	51.9	233 858	135 638	58	222 034	121 179	54.6
Physical Sciences	193 189	113 121	58.6	192 618	119 427	62	179 561	116 862	65.1	172 319	127 919	74.2	164 478	124 237	75.5

Source: Figure 1.1 Table showing the National Performance in 11 most popular subjects 2015-2019

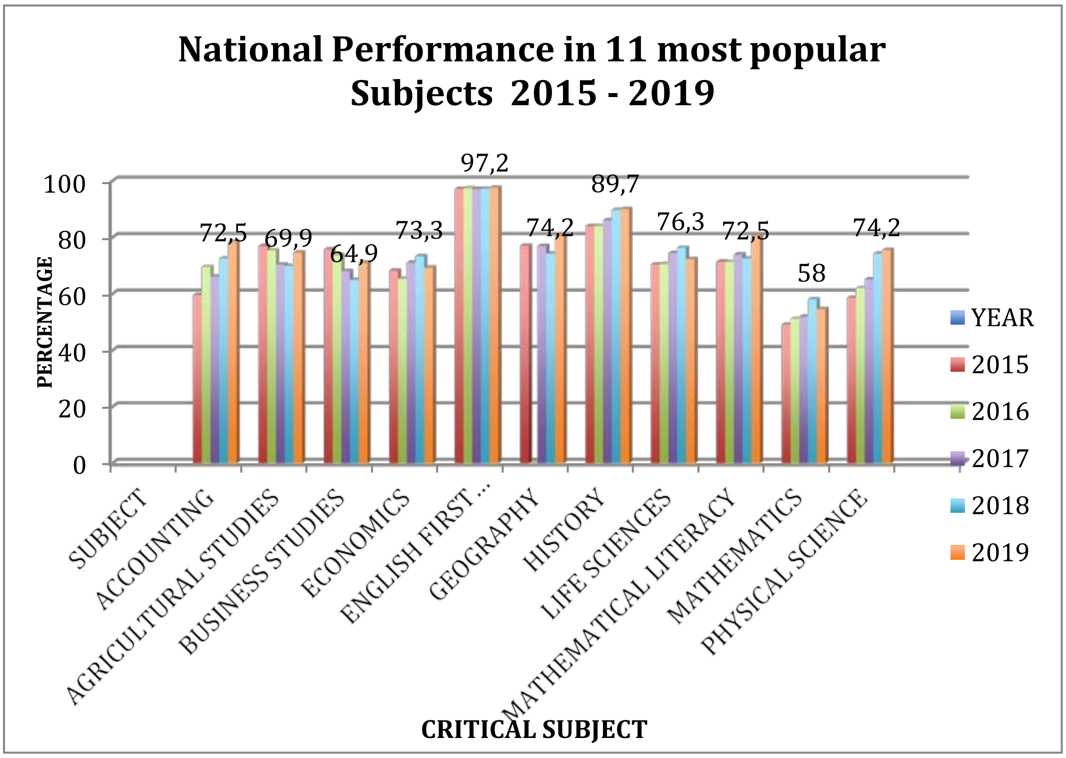


Figure 1.2 showing the National performance in the 11 most popular subjects from 2015-2019

The data from Table 1.1 was put onto a bar graph, Figure 1.2, to present the performance of learners in mathematics as compared to the other 10 subjects in the past five years. The subject Mathematics depicts the lowest bars indicating that it is the subject in which there is the poorest performance.

Urgent interventions in teaching are needed if results in Mathematics are to improve.

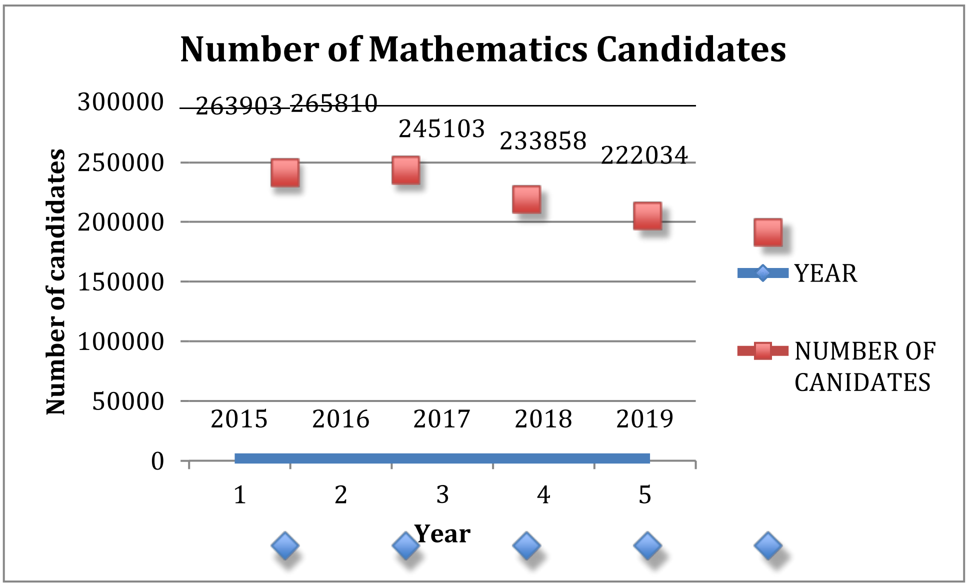


Figure 1.3 Number of Mathematics Candidates from 2015- 2019

From the graph figure 1.3 above, it is clear that there is a steady decline in the number of candidates studying Mathematics.

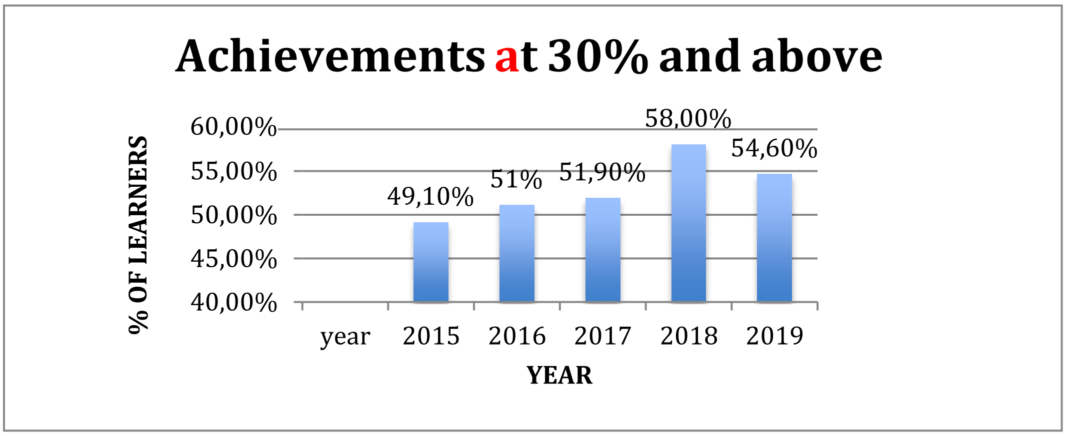


Figure 1.4 Achievement at 30% and above

The graph Figure 1.4 shows the percentage of learners who obtained a 30% and above pass. This average over the past five years is 52,92%. This would implythat 47,08%, nearly half the grade 12 learners, had not met the minimum requirements (failed). This clearly indicates that performance in Mathematics as a critical subject, is poor and requires immediate intervention to improve performance.

1.5 Performance Trends (2015-2019)

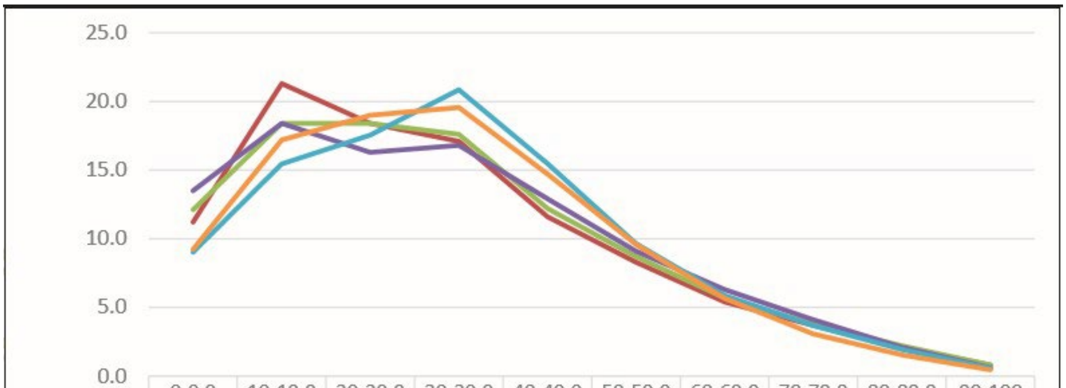
A comparative analysis of the performance of grade 12 learners from the year 2015 to 2019 is discussed below. The number of learners who wrote, the number and percentage of learners who obtained 30% and above, the number and percentage of learners who obtained 40% and above is discussed and analysed. Tables and graphs are used to display the data in order to allow for clear and easier understanding, interpretation and observation of trends that may exist, particularly in annual changes or medium term changes.

Year	No. wrote	No. achieved at 30% and above	% achieved at 30% and above	No. achieved at 40% and above	% achieved at 40% and above
2015	263 903	129 481	49,1	84 297	31,9
2016	265 912	136 011	51,1	89 119	33,5
2017	245 103	127 197	51,9	86 096	35,1
2018	233 858	135 638	58,0	86 874	37,1
2019	222 034	121 179	54,6	77 751	35,0

Table 1.5 Overall Achievement Rates in Mathematics

The number of learners who wrote the Mathematics examination in the year 2019 compared to 2018 decreased by 11 824 when compared to the number who wrote in 2018 (NSC diagnostic report, 2019).The performance of learners in 2019 showed a decline at the 30% level from 58% to 56,4%. Performance at 40% level also revealed a decline from 37,1% to 35%.

According to the NSC diagnostic report (2019: 177) performance in the 2019 examination revealed a lack of understanding of basic concepts across some topics in trigonometry, euclidian geometry and analytical geometry in the curriculum thus a decline in the performance at both the 30% and40% level of achievement. Teaching and learning of basic concepts must not be overlooked by past examination papers, which are beneficial resources for revision. Performance will improve if attention is given to strengthening the content knowledge in Trigonometry and leaners' exposure to complex and problem solving questions across all topics in the curriculum, beginning in the earlier grades.



	0-9.9	10-19.9	20-29.9	30-39.9	40-49.9	50-59.9	60-69.9	70-79.9	80-89.9	90-100
2015	11.2	21.3	18.4	17.1	11.6	8.3	5.4	3.7	2.1	0.8
2016	12.1	18.4	18.4	17.6	12.2	8.7	5.7	3.8	2.2	0.8
2017	13.5	18.4	16.3	16.8	12.9	9.1	6.3	4.1	2.1	0.6
2018	9.0	15.4	17.5	20.9	15.5	9.6	5.9	3.7	1.9	0.6
2019	9.2	17.2	19.0	19.6	14.7	9.6	5.6	3.1	1.5	0.4

Figure 1.6 Performance Distribution Curves in Mathematics (Percentage)

Source (National Senior certificate Report Part 1, 2019)

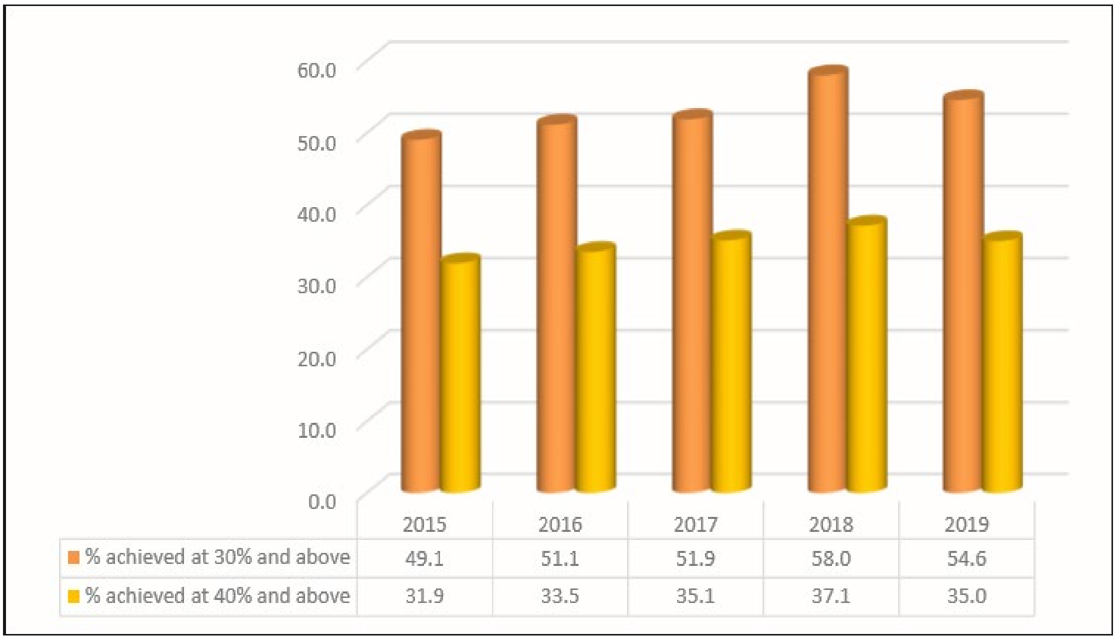


Figure 1.7 Overall Achievement Rates in Mathematics (percentage) Source (National Senior certificate Report Part 1, 2019)

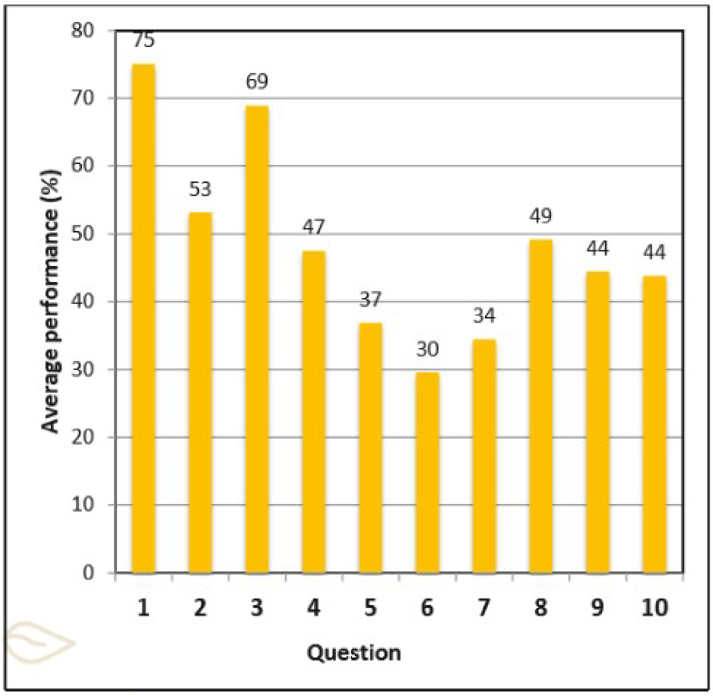
As per Figure 1.7 the number of learners who obtained a 40% pass was lower than the 30%pass. It is clear that results and the quality of passes require improvement.There is clearly a need to capacitate teachers to develop reactive and appropriate instructional programmes to successfully address the areas of weakness that exist in Mathematics.

1.6 Diagnostic Question Analysis for Mathematics (Paper 2 – 2019)

Learners struggled with concepts that needed deeper conceptual understanding. Questions which required interpretation of information or justification, posed the greatest challenges.

The graph Figure 1.8 is based on data from a candidate's sample that was randomly selected. According to the National Certificate Diagnostic Report (2019) the graph might not accurately reflect national averages. It is useful in assessing the challenges that learners experienced in each question.

Questions 6 and 7 contained trigonometry and 3D problems. From the graph below it is evident that learners performed most poorly in trigonometry



Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry

Figure 1.8 Average Percentage Performance per Question for Paper 2 -2019

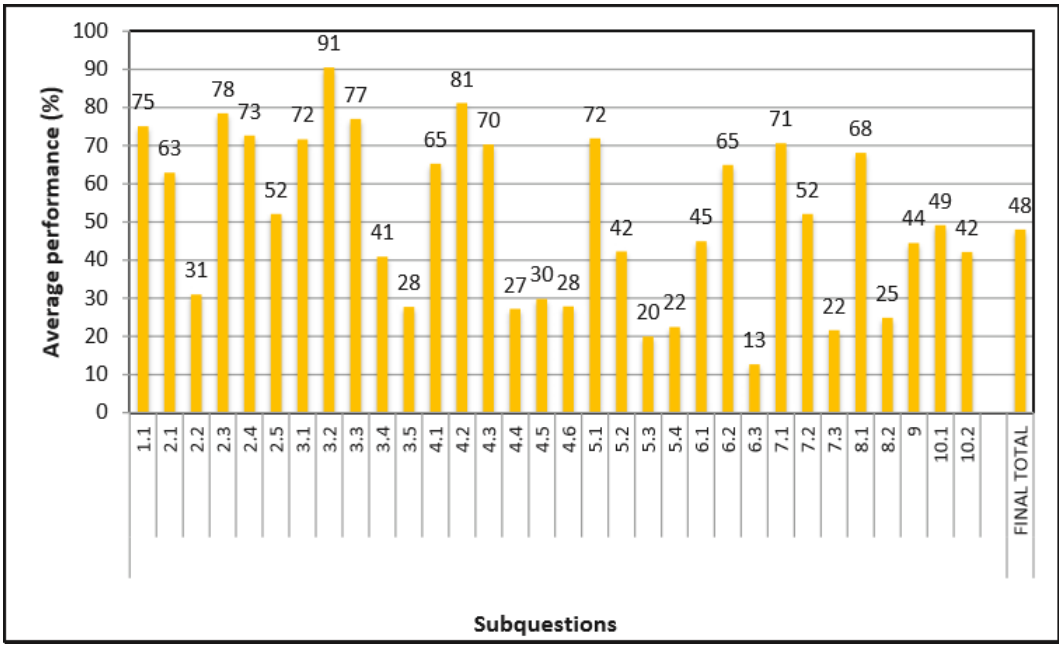


Figure 1.9 Average Percentage Performance per Sub question for Paper 2-2019

The report suggests that for improvement to occur, learners need to be exposed to numeric questions on 3-D problems. This allows for a transition for learners to develop strategies for solving such problems and gain confidence to move from numeric type questions to non-numeric and higher order questions. The implication was that teachers need to develop more effective strategies to be used when solving right - angled triangles and triangles that are not right angled. Hence learners need to be taught the conditions that determine which rule should be selected in order to solve the problem.

1.7 NSC Diagnostic Reports For Paper 2

1.7.1 Diagnostic Report NSC Paper 2 -2018

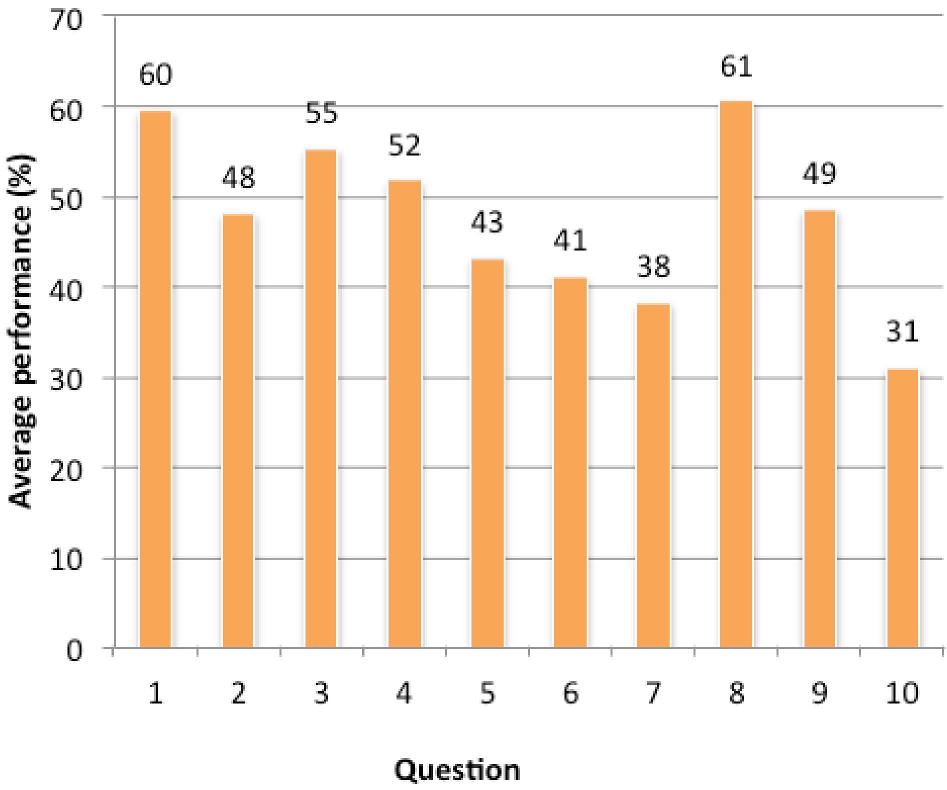


Figure 1.10 Average percentage performance per Question for Paper 2
Source: 2018 National Senior Certificate – Diagonistic Report (Motshekga in DoBE 2018, 143)

Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry

Question 7 in figure 1.10 above was based on 3-D problem solving. Learners were unsuccessful in making the link between the angle of elevation and the angle of depression.

Several learners correctly used the cosine formula but were unable to make the double angle substitution thereafter. Some learners failed to write down the cosine formula correctly, i.e. they left out the squares. Candidates demonstrated poor algebraic manipulation skills when squaring terms and removing brackets (Motshekga inDoBE 2018, 143).

It is evident from the above NSC reports that intervention strategies have to be put in place to ensure learners become more confident in solving 3D trigonometric problems. This study made use of manipulatives to aid students to visualise the 3-D figure and to extract the triangles in the horizontal, vertical and slant planes. The use of manipulatives aids learners in deeper conceptual uderstanding . The manipulatives make learning more meaningful and provide a link from concrete to abstract.

Diagnostic report NSC paper 2 (2017: 1) acknowledges that Mathematics analysis conducted reveals the weaknesses in learners' responses. The analysis of the misconceptions and error patterns exposed in learners' responses can inform instructional practice. These identified weaknesses can allow teachers to refine their teaching strategies appropriately. The Department of Education will, through interventions, continue to capacitate teachers in order to design responsive and suitable instructional programmes that will effectively focus on the areas of weakness identified in Mathematics.

This study provides teachers with an alternate approach to teach using manipulatives to assist learners understand abstract concepts such as the sine rule, cosine rule and area rule which are all merged to solve high level3D-trigonometric problems.

1.7.2 Suggestions For Improvement

Motshekga in DoBE (2018:150) provide the following suggestions for improvement:

Teachers need to develop strategies to be used when solving right-angled triangles and triangles that are not right- angled. Teach learners the conditions that decide which rule should be used to solve the question. Learners must refer to the formula sheet to ensure that formulae are copied correctly.

In Grades 10 and 11, learners should be exposed to problems that involve a combination of shapes in 2-D. This should develop the skill of identifying common sides and angles in composite shapes. It is also useful for learners to draw 3-D shapes. This will give them the opportunity to understand the different perspectives involved.

Learners should be encouraged to highlight the different triangles using different colours. This would allow them to identify the common sides and angles. Teachers should show learners how to deconstruct composite shapes into several triangles. Educators are to, initially, expose learners to numeric questions on solving 3-D problems. This makes it easier for learners to develop strategies on how to solve such questions. Once learners have gained confidence with numeric type questions, they should then be exposed to non-numeric and higher order questions.

1.7.3 Diagnostic report NSC Paper 2 - 2017

The figure 1.11 below shows the diagonstic Question analysis for Mathematics Paper 2 -2017

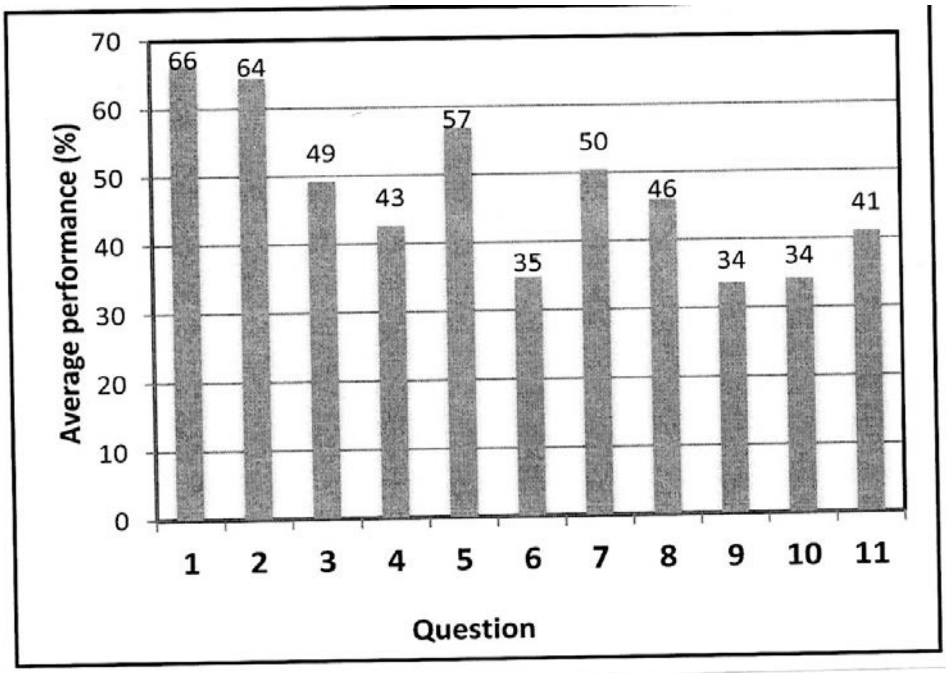


Figure 1.11 Average percentage performance per Question for Paper 2
Source: 2017 National Senior Certificate – Diagnostic Report (Motshekga in DoBE 2017: 164)

Q1 Data Handling

Q2 Data Handling
Q3 Analytical Geometry
Q4 Analytical Geometry
Q5 Trigonometry
Q6 Trigonometry
Q7 Trigonometry
Q8 Euclidean Geometry
Q9 Euclidean Geometry
Q10 Euclidean Geometry
Q11 Euclidean Geometry

The National Senior Certificate Diagnostic Report (2017) states that question seven in figure 1 above, were based on trigonometry 3D problem solving.
The report stated that learners experienced difficulty in visualising the different planes in the sketch. Learners were unsuccessful in linking the two right – angled triangles (NSC Diagnostic Report, 2017). In addition, the report revealed that learners were unable to state correcttrigonometric ratios in the triangles. Learners continuously mixed up angles and sides. Several learners were confused with the orientation of 3-D shape and were unable to use the cosine rule correctly.

1.7.4 Improvement suggestions

Motshekga in DoBE (2018: 170) provides the following suggestions for improvement :

Teachers need to develop strategies to be used when solving right-angled triangles and triangles that are not right-angled. Learners must refer to the formula sheet to ensure that formulae are copied correctly. In Grades 10 and 11, learners should be exposed to problems that involve a combination of shapes in 2-D. This should develop the skill of identifying common sides and angles in composite shapes. Learners should be encouraged to highlight the different triangles using different colours. This would allow them to identify the common sides and angles. Teachers should show learners how to deconstruct composite shapes into several triangles.

Initially, teachers must expose learners to numeric questions on solving 3-D problems. This makes it easier for learners to develop strategies on how to solve such questions. Once learners have gained confidence with numeric type questions, they should then be exposed to non-numeric and higher order questions.

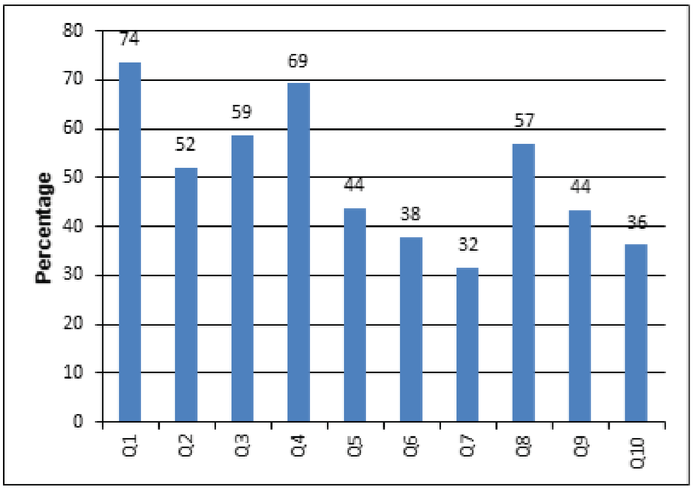
The manipulatives designed by the researcher offered an intervention to the approach to teach 3-D trigonometry problems. The manipulatives accommodated more than one learning style. Learners engaged with the manipulatives and were better able to see the different planes. Learners socialized, discussed and exchanged ideas. The social interaction allowed for new knowledge to be constructed.

1.7.5 Overview Of Learner Performance In Paper 2

Motshekga in DoBE (2018:163) states that individual performance in the paper varied from very poor to excellent. However, there seemed to be a slight improvement in the overall performance in paper 2. Learners performed well in data handling and analytical geometry. It appears as though learners are struggling with trigonometry and Euclidean geometry. The number of learners who did not attempt these questions is a cause for concern. Integration of topics is still a challenge to many candidates. Mathematics cannot be studied in compartments and it is expected that candidates be able to apply knowledge from one section to another section of work. It is evident that many of the errors made by candidates in answering this paper have their origins in a poor understanding of the basics and foundational competencies taught in the earlier grades. Candidates continue to struggle with concepts in the curriculum that require deeper conceptual understanding.

1.7.6 Diagnostic report NSC Paper 2 - 2016

The figure below shows the diagnostic Question analysis for Mathematics Paper 2 -2016



Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q8	Euclidean Geometry
Q10	Euclidean Geometry

Figure 1.12 : Average percentage performance per Question for Paper 2

Source: 2016 National Senior Certificate – Diagnostic Report (Motshekga in DoBE 2017: 165)

The National Senior Certificate Diagnostic report (2017) states that question seven in figure 1.12 above, was based on 3D problem solving. This question was poorly answered by most learners with some not even attempting to answer the question. This indicates that learners have adopted a negative attitude towards solving three dimensional word problems and particularly if the angles and sides are given in variables instead of numerical values. Learners experienced difficulty in visualising the different planes in the sketch. Many learners displayed confusion with the 3-D orientation of the shapes. Learners experienced difficulty in selecting correct sides and angles. A small number of learners were able to select the correct rule to be applied.

Motshekga in DoBE (2017: 1) states that the Department of Basic Education, will through targeted interventions, continue to capacitate teachers to develop responsive and appropriate instructional programmes that will successfully address the areas of weakness.

Motshekga in DoBE (2017: 174) provided recommendations for improvement which included the need for teachers to develop strategies to be used when solving right-angled triangles and triangles that are not right-angled. Teachers should use models to show learners the different planes in a 3-D shape. This would assist learners to identify the different triangles and place angles and side in perspective.

It is clear from the above NSC reports that intervention strategies have to be put in place to ensure learners become more confident in solving 3D trigonometric problems. This study makes use of manipulatives to aid students to visualise the 3-D figure and extract the triangles in the horizontal, vertical and slant planes. The use of manipulatives aids learners in deeper conceptual understanding. The manipulatives make learning more meaningful and provide a link from concrete to abstract.

Merrill, Devine, and Brown (2010: 16) affirmed that improving and enhancing content knowledge requires teachers of mathematics to use three-dimensional (3D) solid modeling in mathematics classrooms to improve learners' understanding of mathematical concepts and principles. When students visualize, they see the relevance and this promotes rigour. Weng (2011: 52) reiterates that visible and visual 3D dynamic design has the potential to increase learning interests in mathematics, by improving learner's interdisciplinary and multimedia design abilities.

When students visualize, they see the relevance and this stimulates rigour. Herbst and Chazan (2011: 1) collaborate this view by stating that artefacts can be used in activity systems for learners to interact with the object and other learners and the teacher thus enriches the lesson.

The use of manipulatives in teaching mathematics permitted learners to construct their own cognitive models from abstract mathematical ideas and processes. Manipulatives offered a common language to communicate these models to other learners and to the educator. It further engaged learners to increase interest and enjoyment in mathematics (Marasigan, Balba, Dela Cruz & Manalo, 2019).

1.8 Motivation

This study was motivated by:

- The researcher's personal experiences and interest in teaching strategies;
- The understanding of the sine rule, cosine rule and area rule;
- Difficulties and challenges experienced in understanding the sine rule, cosine rule and area rule and their application to successfully solve three dimensional problems.

1.8.1 The Researcher's Personal Interest And Experience

The researcher has been a mathematics teacher at several public schools in South Africa, from the year 1998 to date.

Trigonometry is the science of the numerical relations between the sides and angles of triangles. Trigonometry is considered to be an important component of mathematics. Learners continue to experience difficulties in solving 3D trigonometric problems. The researcher has observed several weaknesses in her learners' responses. These weaknesses include incorrect selection and application of the sine rule, cosine rule and area rule, poor algebraic skills to make a variable the subject of the formula, poor calculator skills and lack of visualization to see triangles in the vertical plane, horizontal plane and slant plane. Learners are more inclined towards procedural methods rather than conceptual understanding in trigonometry.

1.8.2 The Understanding of the Sine Rule, Cosine Rule and Area Rule

The researcher has observed that learners exhibit a lack of knowledge when applying the sine rule, cosine rule and area rule. Learners are able to solve a problem if it requires the use of solely one rule. When challenged with a complex multi procedure three - dimensional problem, they are unable to solve problem successfully.

1.8.3 Difficulties and challenges experienced in understanding the sine rule, cosine rule and area rule and its application to successfully solve three-dimensional problems

When learners are presented with a 3-D problem, some learners do not make an attempt to solve the problem and arrive at a solution. It is unfortunate that presently learners have adopted a negative attitude towards problem solving of 3-D trigonometric problems. Learners experience great difficulty visualising the various planes in the sketches. They display confusion with the three dimensional orientation of the shape which leads to difficulty in choosing the correct angles and sides. Further more, learners experience difficulty in selecting the correct rule to be applied.

1.8.4 Learners' preference of procedural methods rather than conceptual understanding in solving three dimensional trigonometric problems

Trigonometry can be as an inseparable part of mathematics in the secondary school and can be described as a product of geometrical realities, algebraic techniques and trigonometric relationships.

Orhun (2004) states that mathematics education is based on problem solving, application of knowledge and manipulations problems and when learners encounter word problems it appears that their non-systematical and incomplete knowledge results in faults and conceptual mistakes. When developing and analyzing problems, explaining results and confirming processes are not fully comprehended, this results in learners surrendering creativeness and this leads to learning by heart.

It is, therefore, imperative that the development of teacher education programmes includes aspects of the pedagogical content knowledge (PCK) required for mathematics teachers. Brijlall & Isaacs (2011) scrutinized the link between mathematics content knowledge and classroom practice of two university lecturers. They concluded that for effective teaching to take place a strong link between these two aspects was necessary. Cochran, Derutter & King (1993) claim that field knowledge and pedagogical knowledge of teachers plays an important role in teaching and learning. Teachers are confident when they have a strong pedagogical knowledge and this enables teachers to assist their learners in constructing new knowledge.

Khashan (2014:193) believes that teachers of mathematics possess a low level of conceptual knowledge and appear more confident to apply procedural knowledge rather than conceptual knowledge.This implies that teachers lack sufficient understanding of the components of mathematical knowledge and its structure. He further adds that this finding may be attributed to the fact thatwhat learners had learned in mathematics are merely rules and procedures, which are memorized by heart and applied as algorithms in order to solve problems without really understanding these rules and procedures or the relationships that exist between them. In schools, learners spend most of their time dealing with procedures with minimal focus on conceptual knowledge upon which these procedures have been constructed. Khashan (2014) further adds that this can be attributed to conventional methods that exist in our educational institutions by which, currently, teachers were taught along their earlier learning cycles, distinctly focusing on procedural knowledge at the expense and sacrifice of conceptual knowledge. This resulted in not allowing teachers to address conceptual problems. Acquiring conceptual knowledge of mathematics requires acquiring teaching strategies that would focus on assisting learners to identify relationships that exist between mathematical ideas, understanding bonding between these relationships and how they are formulated, in order to produce an integrated and coherent set and on how to apply these ideas in the world of mathematics.

Darey, Terzinha, Peter and Christina (2012) concur with Khashan and state that learners are dealing with mathematical content mostly as procedural knowledge, without concentrating on conceptual knowledge. This results in building their mathematical experiences more saturated with procedural knowledge rather than conceptual knowledge.

Similarly Miqdadi, Ruba, Malkawi, Amal, Zoughbi and Ali (2013), in their study reveal that the conventional techniques applied in teaching mathematics focused mainly on procedural aspects, rather than focusing on conceptual structure. Teachers often encouraged learners to learn by means of applying computation processes without understanding and concentration.

1.9 Research Questions and Focus of Enquiry

This study made use of concrete mathematical manipulatives in exploring conceptual understanding of grade 12 Mathematics learners in learning to solve 3-dimensional trigonometric problems. The research questions addressed in this study are:

- How do Mathematical models help learners to adopt more active approaches towards the learning of three-dimensional trigonometric problems amongst Grade 12 learners?
- How will the use of models improve the learning of Mathematics?

Aims and objectives

The aim of this study was to find out whether the use of models in teaching 3 dimensional trigonometry problems improved the cognitive understanding in learners.

The research questions for this were formulated with the following objectives in mind:

- the effect of models on learners' academic achievement,
- the impact of concept images in assisting students to make the necessary mental construction that will lead to conceptual understanding of trigonometric concepts,
- the identification of difficulties that learners encounter in learning the concepts, which may be the barrier in making the required mental constructions,
- the evaluation of the level of mental constructions that learners make while interacting with the models
- the assistance needed by learners to identify planes in 3D shapes?

1.10 Terminology and Concepts

The concepts and terms used in this study are outlined below.

1.10.1 Concrete Model

Sowell (1974) states that three kinds of materials exist. These include concrete, pictorial, and abstract. Concrete materials are movable and can be manipulated by the learner. Pictorial are materials that are visual and include charts, pictures and diagrams. Numerals and words are considered to be abstract materials.

1.10.2 Artefacts

Fernandes, Carron and Ducasse (2018:1) explain that when educators prepare a learning activity, the educator introduces special tools, which serve as auxiliaries between the learner, concept and the activity. These auxiliaries aid learners to construct a mental representation of the concept. These tools are known as artefacts. These artefacts can consist of physical objects such as tokens of varying colours, dice or three- dimensional models, or they can be aids to a pen-and –paper format by drawing a grid or a series of objects. In most cases the artefact is an established object used in such a way so as to provide support in the learning process. Learners internalise the concept in a step-by step-process as the artefact serves as an intermediate link between the learnt concept and the internal representation.

1.10.3 Manipulatives

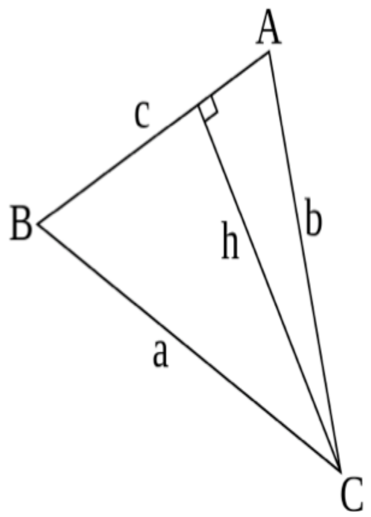
Manipulatives are physical objects that can be used to actively engagelearners in the hands-on learning of mathematics. Manipulatives can be utilized to introduce, practice or remediate a mathematical concept.Manipulatives can also be used in all grades and areas of mathematics instruction such as number and operations, algebra, geometry and measurement.

Marasigan,Balba, Dela Cruz & Manalo (2019: 73) describe manipulatives as being physical objectives that learners and teachers use to illustrate and discover mathematical concepts, whether made specifically for Mathematics or for other purposes. More recently virtual manipulative tools are available for use in the classroom. Manipulatives are treated as a tool for teacher modelling and demonstration. A Mathematical manipulative is defined as any material or object from the real world that children move around to show a mathematical concept. They are concrete hands on models that appeal to the senses and can be touched by students. These materials should relate to a learner's real world.

McNiel and Jarvin (2007, 309) define manipulatives as concrete objects used to help students understand abstract concepts in the domain of mathematics. This is supported by Furner, Yahya and Duffy (2005:17) who believe that the use of manipulatives provides teachers with great potential to use their creativity to do further work on mathematical concepts as an alternative to merely relying on worksheets. Students are learning Mathematics in an enjoyable way, making connections between the concrete and the abstract.
In this study the word manipulative is used to refer to the mathematical model.

1.10.4 Sine rule

The Sine Rule can be used in any triangle (not just right-angled triangles) where a side and its opposite angle are known.
Given the following triangle with corresponding side lengths a, b and c:



The **sine rule** or **law of sines** is the following identity:

$\frac{a}{sssnA} = \frac{b}{\sin B} = \frac{c}{\sin c}$

We will prove the first identity

The second equality can be proved similarly.

By drawing the height of the triangle from vertex to the opposite side, we can express the height in two different ways:

- First, we have sin A = $\frac{opposite}{hypotenuse} = \frac{h}{b}$ which implies that

  h= b sin (A)

- Also sin B = $\frac{opposite}{hypotenuse} = \frac{h}{a}$ which implies that

  h = a .sin(B)

By equating these values of h, we have

a .sin(B) = b sin(A)

$$\frac{a}{sssnA} = \frac{b}{sssnB}$$

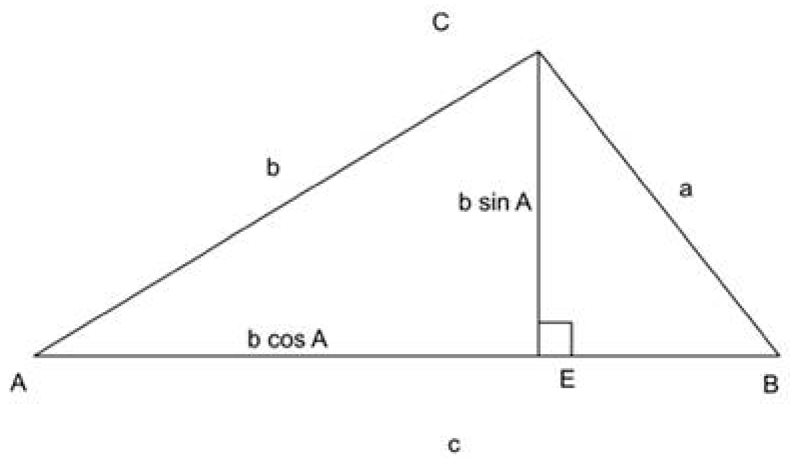
By drawing the height from the other two vertices, we can similarly show the second equality.

Source: Adapted from Abbort, Bouman, Bruce, Du Toit, Pillay, Schalekamp & Smith (2013)

1.10.5 Cosine Rule

The Cosine Rule can be used in any triangle where you are trying to relate all three sides to one angle.

Proof



Using the Pythagoras Theorem on triangle BCE. The sides of this triangle are a, b sin A, c – b cos A, respectively. So, by Pythagoras

$a^2 = (b \sin A)^2 + (c - b \cos A)^2$.

Hence

$a^2 = b^2 \sin^2 A + (c^2 - 2bc \cos A + b^2 \cos^2 A)$.

But now we can apply the fact that $\sin^2 A + \cos^2 A = 1$ to give

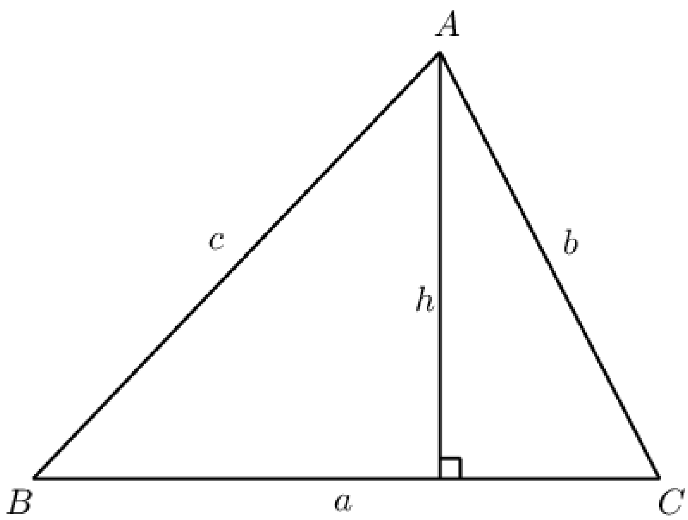
$a^2 = b^2 + c^2 - 2bc \cos A.$

Source: Adapted from Abbort *et al* (2013)

Type equation here.

1.10.6 Area Rule

For any Δ*ABC* with *AB*=*c*, *BC*=*a* and *AC*=*b*, we can construct a perpendicular height (*h*) from vertex *A* to the line *BC*



In Δ*ABC*

$$\sin B = \frac{h}{c}$$

therefore $h = c \sin B$

we know that area $\Delta ABC = \frac{1}{2}$ x a x h

$= \frac{1}{2}$ x a x c sinB

$= \frac{1}{2}$ ac sin B

Alternatively we could write that sinC = $\frac{h}{b}$

h= b sinC

and, then we would have area $\Delta ABC = \frac{1}{2}$ x a x h

$= \frac{1}{2}$ x a x b sinC

Similarly, by constructing a perpendicular height from the vertex B to the line AC, we can also show

Area of$\Delta ABC = \frac{1}{2}\, bc \sin A$

The area rule in any ΔABC

Area $= \frac{1}{2}$ bc sinA

$= \frac{1}{2}$ ac sinB

$= \frac{1}{2}$ ab sinC

Source: Adapted from Abbort *et al* (2013)

1.10.7 Trigonometry- Solution of Triangles in Three-Dimensions

Yiadom (2014:1) **s**tates that two-dimensional space occupies a single plane, three-dimensional space occupies three planes. The three planes are horizontal, vertical and inclined. The sine, cosine and area rules can also be used to solve problems in three-dimensional space. The diagram below illustrates the three different planes for an object in three dimensions

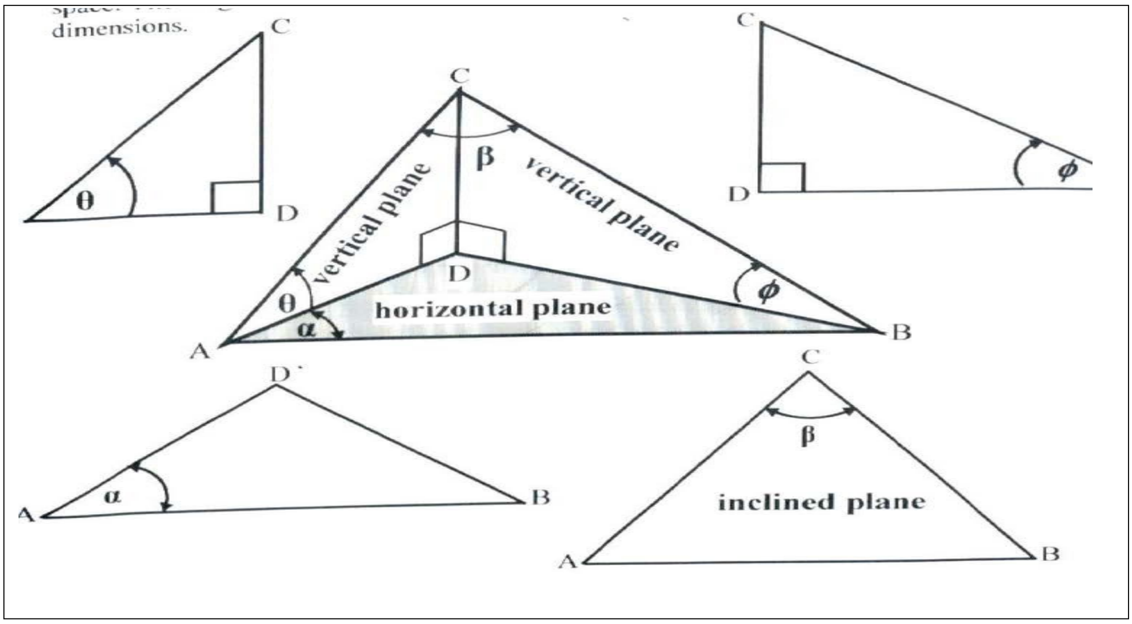


Figure 1.13 Vertical plane, Horizontal plane and inclined plane. (Source: Yiadom, 2014)

Yiadom (2014:1) further explains that three-dimensional problems are often solved by taking a series of triangles in different planes and applying to each separately the area, sine and cosine rules. In cases of right angle triangles the definitions of sine; cosine; tangent are used. It is always important and sufficient to start with the triangle with the most information such as two sides and an angle.

1.11 Combining Trigonometric Skills – Use of sine rule, cosine rule and area rule to solve three-dimensional problems

Choosing The Appropriate Technique-

Sometimes more than one technique can be utilized to solve trigonometric problems but one would want to select the most efficient technique and easiest method to save time. The flow chart shows how to decide which method to use.

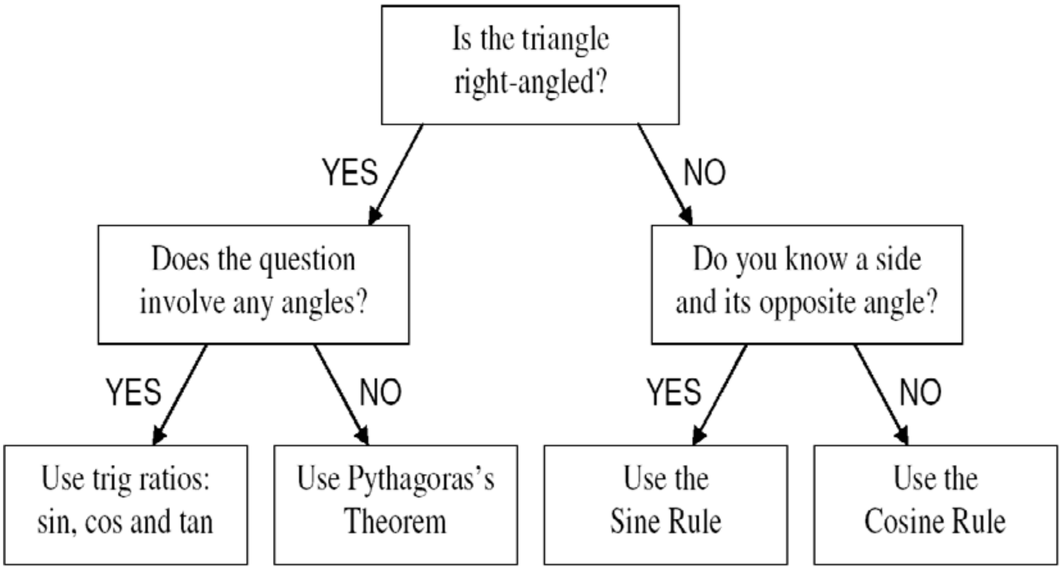


Figure 1.14 Flow Chart

Resource: Reynolds (2010)

1.12 Pedagogical Factors Affecting The Learning of Trigonometry

Kagenyi (2016: 31) states that the pedagogical factors that affect the learning of trigonometry include: inappropriate teaching and learning methods applied, shortage or inadequate trigonometry teaching and learning resources, teacher and student attitudes, classroom atmosphere and teacher teaching behaviour to learning process of trigonometry, inappropriate interaction patterns during teaching and learning process of trigonometry.

The introduction of manipulatives will provide a different learning method, add to the trigonometry resources and have an impact on learners' attitude towards solving three-dimensional trigonometric problems.

1.13 Structure of the Thesis

In acquiring an appropriate approach to this thesis, the following structure was chosen and implemented. This thesis comprises of eight chapters, the bibliography and appendices. The chapters are as follows:

Chapter One introduces the background and purpose of this study. In addition the National performance in the eleven most popular subjects from 2015 to 2019, performance trends, diagnostics question analysis for Mathematics paper 2, suggestions for improvement, motivation to conduct this research, terminology and key concepts, sine rule, cosine rule and area rule, trigonometric skills, pedagogical factors affecting the learning of trigonometry are discussed.

Chapter Two

This chapter begins with the history of Mathematics in South Africa and 4IR, highlighting international and national studies conducted. Thereafter, the state of Mathematics currently is described. Major systemic problem areas impacting achievement in Mathematics are discussed. The challenges experienced amongst Mathematics learners, content knowledge gaps that exist in mathematical knowledge, challenges teachers experience and the impact of the Gross Domestic Product (GDP) of the country follow. Pedagogical content knowledge is addressed. Research on the factors associated with Mathematics performance is explained.

Thereafter, the context of the study is defined. Performance trends of Grade12 mathematics Paper 2, from 2016 to 2020, are interrogated. An overview of learner performance in Paper 2 is presented and suggestions for improvement recommended. The descriptions of the different learning styles that exist and which apply to this study are discussed. In this chapter the literature and relevant studies on the use of manipulatives in Mathematics are cited and discussed. The 2018,2017 and 2016 National Senior Certificate (NSC) Diagnostic report for Paper 2 are analysed, offering an overview of learner performance in Paper 2. Suggestions for improvement are discussed and the importance of introducing manipulatives into the classroom. Thereafter, a comprehensive literature review and findings are summarized. Recent South African and international studies are deliberated. In the study the advantages, drawbacks, benefits and mistakes of using manipulatives are discussed. The chapter concludes with a summary of Lesh's Translation Model and its application to this study.

Chapter Three presents the theoretical framework for this research study. The theories of Piaget and Vygotsky and Kolb's Experiential Learning Theory (ELT) have underpinned this research. In this study, information processing constructivism was used. Constructivism as a paradigm suggests that learning is an active process. Knowledge is constructed and not directly perceived through the senses.

Chapter Four

This chapter begins with a discussion of the research methodology employed. Methodological issues relevant to this study are considered. The critical research questions and research instruments are explained to be in keeping with the dictates of some experts in the field of education research. The interpretive paradigm is explained in detail and proven to co-inside with the theoretical framework adopted for this study. The data capture strategies are aligned to a qualitative approach used in this study. In addition, reliability and validity are discussed.

Chapter Five

This chapter provides details of a pilot study undertaken by the researcher in a school in the Pinetown District, in search of validation of the research instruments used in this research and thus placing it in context. The pilot study made use of mathematical models/manipulatives to teach grade 12 learners to solve three-dimensional trigonometric problems. In this chapter, discussions and results on the pilot study are reported. In the pilot study qualitative methods were employed and data collected through the use of mathematical models, activity worksheets, semi structured interview schedules, observation and video/audio recordings that were administered tograde 12 mathematics learners (n=9).

Chapter Six

This chapter discusses the analysis of the data. Qualitative methods were applied in this study and the data was retrieved through activity sheets, semi-structured interviews and observations. The learner responses in the activity sheets are analysed to ascertain how the purposely designed manipulative, enhanced the learning of trigonometry among grade 12 learners and how the use of manipulatives impacted the learners' understanding and learning solving three-dimensional Trigonometry problems. The semi -structured interview was designed to acquire an in-depth insight into the learners' knowledge of and skills in solving three-dimensional trigonometric problems. The researcher observed the learners while they interacted and engaged with the manipulatives.

Chapter Seven

This chapter discusses learners' responses and reveals that the use of manipulatives aids the understanding of mathematical concepts when taught using activities containing 3D trigonometric problems, which includes the transition from concrete to abstract. The chapter offers graphs, tables, transcripts and written responses of grade 12 Mathematics learners solutions and responses to solving 3D trigonometric problems using area rule, sine rule, cosine rule and basic trigonometric ratios along with application of theorem of Pythagoras. This information serves to explore the conceptual understanding of trigonometric concepts and the influence the mathematical manipulatives have on the grade 12 learners understanding.

In this chapter the results are analysed via thematic content analysis. Themes that emerged from the data obtained are discussed in detail and related to similar research carried out. An over view of the study is given on the advantages of manipulatives and how they improve performance and increase understanding of abstract concepts. The research questions are addressed. Thematic content analysis and steps in the analysis process are discussed. From the study nine themes emerged and they are discussed in detail. Learning modes, utility of manipulatives in relation to mathematics itself and the analysis flow chart are explained.

Chapter Eight

This chapter presents an overview of the study, conclusions to the thesis, contribution to research and recommendations, limitations of the study and a summary and concluding remarks of the thesis. The final chapter captured the researchers views and thoughts of the study.

1.14 Synopsis

In this chapter the researcher discusses the rationale that prompted interest in conducting this research.

The next chapter two discusses the literature reviewed for this study.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

The previous chapter provided the background, purpose and motivation for the study. The analysis of the ational performance of the eleven critical subjects and trends were discussed. Terminology and concepts used in the study were clearly explained.

This chapter begins with the history of Mathematics in South Africa and 4IR, highlighting international and national studies conducted. Thereafter the state of current mathematics is described. Major systemic problem areas impacting the achievement in Mathematics are discussed as well as the state of current mathematics teaching and learning practices in South African schools. The challenges experienced amongst Mathematics learners, content knowledge gaps that exist in mathematical knowledge, challenges teachers experience and the impact of the Gross Domestic Product (GDP) of the country are also viewed. Pedagogical content knowledge is addressed. Research on the factors associated with mathematics performance is explained and the use of manipulatives is defined.

Thereafter the context of the study is defined. Performance trends of Grade12 Mathematics (Paper 2) from 2016 to 2020 are interrogated, learner performance in Paper 2 is presented and suggestions for improvement recommended. The descriptions of the different learning styles that exist and which apply to this study are discussed. In this chapter the literature and relevant studies on the use of manipulatives in Mathematics are cited and discussed. The 2018, 2017 and 2016 National Senior Certificate (NSC) Diagnostic reports for paper 2 are analysed, offering an overview of learner performance in paper 2. Suggestions for improvement are discussed and the importance of introducing manipulatives into the classroom. A comprehensive literature review and findings are summarized. Recent South African and international studies are deliberated. In the study the advantages, drawbacks, benefits and mistakes of using manipulatives are discussed. The chapter concludes with a summary of Lesh's Translation Model and its application to this study.

2.2 History of Mathematics in South Africa and 4IR

Education and mathematics competencies and skills have become the most central elements that impact on the development of any nation, particularly in science and technology. The fundamental importance of mathematics to humans could be explained in terms of the interrelationship between mathematics and development of humans to advance the cause of humans (Madonsela, Ndlovu, Brijlall, 2020: 1303).

International and national studies have shown that South African learners have poor mathematical skills (Bansilal, James & Naidoo, 2010). The performance in the Grade Eight learners in the Trends in International Mathematics and Science Study (TIMMS) that was carried out in 1995, 1999 and 2003 showed that South African learners scored the lowest amongst forty countries (Howie, 2001; Howie, 2004; Reddy, 2006). Dempster & Reddy (2007) and Vithal (2008) state that even though there wascriticism against some procedures used in these studies, the results are still concerning. Howie (2004) claims that the reason of this situation appears to be around one factor, which is education quality.

Jojo (2019) claims that in South Africa the teaching of Mathematics has been shown to be among the worst in the world. The unacknowledged poor quality of mathematics teaching methods that exist in public schools had disadvantaged learners and has affected learner entry to both higher education and modern knowledge intensive work skills.

Bloch (2009), Christie, Butler & Potterton (2007), Fleisch (2008), Kraak (2004) and Motala& Pampallis (2005) highlight the following major systemic problem areas impacting achievement in Mathematics:

a) The history of apartheid,
b) The negative impact on education as a result of poor decisions made at political level;
c) The increasing and endemic poverty levels in society particularly those affecting the black population;
d) The large differences in both quality and quantity of teaching and learning that has occurred in advantaged private and former model C schools and in disadvantaged public schools in rural and township areas;
e) The poor management of several schools by principals and school governing bodies;
f) The existence of poor infrastructure in many schools;
g) Insufficient training of several teachers who are under qualified in both content knowledge (what is to be taught) and didactic knowledge (how to teach), not enough contact time spent by teachers and learners in schools and lack of adequate support to schools by education departments at national, provincial and district levels.

In summary, a common thread can be seen from the literature as to the factors impacting the poor Mathematics performance.

In the fast growing developments in digital technology, Artificial Intelligence (AI) and Biotechnology is denoted as the 'Fourth Industrial Revolution' (or 4IR) (Schwab, 2016), signifying one aspect of a world that has been described as 'VUCA': Volatile, Uncertain, Complex, and Ambiguous (Organisation for Economic Co-operation and Development [OECD], 2018). Moodley (2021) states that this VUCA world, technological developments converge with interconnected social, economic, and political trends at a local and global level creates complex opportunities and challenges.

In South Africa, global trends and drivers stand to exacerbate existing challenges, such as youth unemployment, poverty, and inequality. These stubborn historical challenges could be intensified, rather than eased, by trends such as automation, urbanisation and digitisation. The effects of these trends can be seen in the global decline in demand for skills, which can be automated or digitized, such as routine and manual tasks. An increase for tasks that only humans can do well follow such as non-routine, interpersonal tasks (Fadel *et al.* 2015). Research suggests that there is a presence of skills mismatch between those skills employers require and those of school-leavers in South Africa (Department of Higher Education and Training [DHET], 2019). This, given the rate of change in the workplace globally, does not bode well for the future unless significant shifts are made towards more relevant education that speaks to these demands. As a result of these developments, there is an increased need for education systems to equip learners with a broad range of 'competencies for a continuously changing world' in order to provide success in this uncertain future that prevails (Moodley, 2021). Perumal (2014) states that those involved in Education in South Africa need to address not only the current demands of the 4IR but, also more importantly and holistically, to acknowledge the transformative role that education can play in a society that has for many years been damaged.

2.3 State of current mathematics teaching and learning practices in South African schools

The relatively poor Mathematics performance robs South Africa of the required number of learners who will continue with their studies and move on to study actuarial science, engineering and other Mathematics requiring qualifications at various tertiary institutions. The consequences are fewer than required graduates to fill the positions available. A shortage of skilled personnel to drive the relevant economic sectors which impacts the Gross Domestic Product (GDP) of the country positively (Mosiane, 2019).

The low Mathematics results expose the challenges in Mathematics education throughout the basic education sector in South Africa (Collett & Steyn, 2017). Olivier (2018) adds that challenges amongst Mathematics learners include the learners' lack of interest and self-directed learning (SDL), inability to learn at the levels they are being taught, low confidence and a presence of significant content knowledge gaps in their mathematical knowledge. The challenges amongst teachers are the lack of skills and confidence to teach the mathematics curriculum effectively and the widespread reliance of several on traditional teacher-centered pedagogies (Collect & Steyn, 2017; Olivier, 2016).

Venkat and Spaull (2015) state that there exists a large body of assessment data that points to poor performance in Mathematics across all the levels of the schooling system in South Africa. This data includes classroom observations and local small-scale studies (Ensor, Hoadley, Jacklin, Kuhne, Schmitt, Lombard & Van den Heuvel- Panhuizen, 2009; Schollar, 2008) and nationally representative assessments of mathematics achievements which include Southern and Eastern African Consortium for Monitoring Educational Quality (SACMEQ), Trends in International Mathematics and Science Study (TIMMS) and the National School Effectiveness Study (NSES) (cited in Venkat and Spaull 2015). Venkat and Spaull (2015) confirm that the analysis of these studies and assessments reveal a very low and highly unequal performance of learners in South Africa.

The continuous low student performance over the past decade has sparked an increasing interest into understanding how teacher characteristics, pedagogical practices and content knowledge fit within these patterns of poor academic performance and development (Taylor & Taylor, 2013; Carnoy, Chisholm & Chillisa, 2012). Venkat and Spaull (2015) claim that a common finding in these studies reveals the existence of several South African mathematics teachers who lack fundamental understandings of mathematics. Engelbrecht and Harding (2015) claim that the reasons for the unpreparedness of mathematical students are several and these include the socio-economic status of the learner, low quality teaching, under qualified mathematics teachers, poor decision-making authorities' influence in designing and planning curriculum and finally the continuous pressure to meet the social transformation and skills of the new South Africa.

Jojo(2019) advocates a shift in the mindset of mathematics teachers which must be encouraged. The Department of Education should reconsider the reopening of training colleges to ensure quality mathematics teachers can be trained. Performance of the subject must be excluded from politics and be controlled by conceptual knowledge of the subject. In this way learners are not only taught for meeting promotion requirements but rather understanding that would enable learners to connect and apply learnt and known mathematical concepts to enhance their livelihoods. A system utilized in Singapore that empowers and forces teachers of Mathematics to serve for two years before they become permanent can turn the tables around in South Africa.

We live in an age of rapid social change and improvements have gained momentum and information and communication are significant in every moment of life (Koparan, 2017). Several communities, researchers and institutions emphasize the importance of educating teachers to use proper technologies and materials in teaching mathematics (Niess, 2006; Association of Mathematics Teacher Educators, 2006; International Society for Technology in Education, 2008; National Council of Teachers of Mathematics, 2000).

2.4 Pedagogical Content Knowledge

In the years following Shulman's seminal 1986 address introducing the notion of pedagogical content knowledge (PCK), most scholars and policy makers have assumed that knowledge not only exists but also contributes to effective teaching and student learning. Standard documents including those of National Council of Teachers of Mathematics (NCTM) and the National Board for Professional Teaching Standards (NBPTS) note the importance of teachers holding knowledge of " students as learners" (NTCM, 2000: 17).

A thorough understanding of mathematical content knowledge, including knowledge of the teaching of the subject, is needed in order to produce effective mathematics teachers (Hill, Ball and Schilling, 2008). Venkat and Spaull (2015) claim that a common finding in studies carried out by Taylor & Taylor (2013) and Carnoy, Chisholm & Chillisa (2012) reveal the existence of several South African Mathematics teachers lacking fundamental understandings of mathematics. Teaching and learning is compromised in South African mathematics classrooms when teachers lack mathematical content knowledge and pedagogical content knowledge. Hill, Ball and Schilling (2008) offer a measure of the domain between the subject matter knowledge and pedagogical content knowledge. Figure 2.1 summarizes their framework of teaching and knowledge.

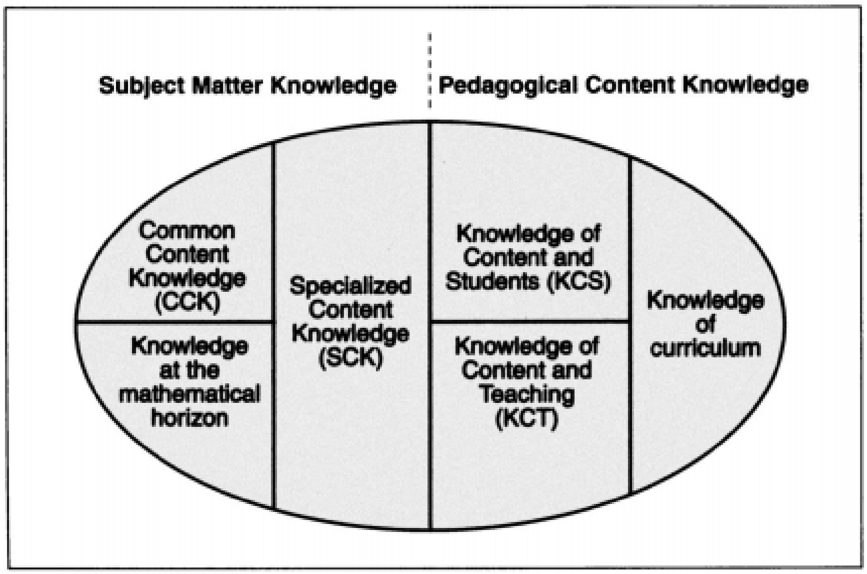


Figure 2.1 Domain Map for Mathematical Knowledge for Teaching (Source Hill, Ball & Schilling, 2008)

Bansilal, Mkhwananzi & Brijlall (2014) claim that several studies point to the problem that South African mathematics teachers have a poor content knowledge of their subject. Hugo, Wedekind and Wilson (2010) reported on a study of teaching and learning mathematics in Kwa- Zulu Natal primaryschools and discovered that none of the teachers were able to achieve 100%for tests on the curriculum which they were currently teaching. An alarming 24% of the respondents obtained less that 50%. The analysis of the Southern and Eastern African Consortium for Monitoring Educational Quality (SAQMEQ) 2010 results showed that the top 5% of grade 6 learners (559 learners) had scored higher marks on the exact same mathematics test that the bottom 12,5% of Grade 6 educators (62 teachers) in the sample achieved(Spaull, 2011).

Bansilal *et al* (2014) conducted a study with 253 FET Grade 12 mathematics teachers from Kwa-Zulu Natal to investigate their content knowledge. The results of this study raised concerns about the teaching of mathematics by FET mathematics teachers whose knowledge of school mathematics was weak and in addition teachers explanations were often incoherent and illogical. The results of this study show that the teachers performed well at level I type questions (average of 84%) but as the cognitive levels increased, the teachers' responses became insufficient (average of 47% for level 3 and 26% for level). Bansilal *et al* questions how teachers can set fair assessments that cover all four taxonomy level for their learners when they themselves are experiencing difficulties in solving questions based on levels 3 and 4 of the taxonomy. The results show that teachers do not have adequate knowledge of school mathematics, this needs urgent addressing. They advocate that it should be compulsory for under qualified teachers to complete a foundation programme that is based on school level Mathematics content before being allowed into the formal teacher qualification programme of the ACE (Advanced Certificate in Teaching).
Research on teachers' pedagogical content knowledge shows that that there is much more to be learnt than knowing the common content knowledge (Ball *et al*, 2008). Several researchers agree that professional development

programmess should include a focus on content knowledge and pedagogical knowledge (Basilal & Rosenberg, 2011; Ono & Ferreira, 2010; Kriek & Grayson, 2009; Peressini, Romagnano, Knuth & Willis, 2004).

In South Africa the great demand for effective professional development of practicing teachers was highlighted when many colleges were shut down during the 1990s, while some were incorporated into higher education institutes (HEIs) in 2001. The closure of theses colleges resulted in several graduates pursuing studies at HEIs to improve and upgrade their qualifications. This legacy of deep inequality in education provisioning in SouthAfrica has led to a demand for effective professional development programs. Along with the several school curriculum changes, professional development programmes frequently served three purposes that included upgrading, retraining and opening up pathways to higher education (Bansilal *et al*, 2014).

According to Ball *et al.* (2008) the term Mathematical Knowledge for Teaching" refers to " the mathematical knowledge needed to carry out the work of teaching mathematics'.

Ball *et al* (2008) has provided three domains of mathematical knowledge for teaching. These include specialized content knowledge, which is mathematical knowledge that is used by teachers in teaching. The second is knowledge of students and content which is knowledge that combines knowledge of content and students. In this domain teachers are expected to anticipate student common misconceptions and errors, have the ability to interpret students' incomplete thinking and to predict what students will do to specific tasks and what would they find challenging and interesting. The third knowledge of teaching and content domain is knowledge about instructional sequencing of specific content and significant examples for highlighting important mathematical issues.

2.5 Research on Factors Associated with Mathematics

performance

The creation of an environment conducive to learning is vitally important in the academic achievement of learners. Such an environment extends beyond the classroom and school to include the home. It is from these environments that learners draw both tangible and intangible information that impacts their educational experience (Visser, Jaun & Feza, 2015, 1).
Visser,Juan, Feza (2015) state that the consensus amongst South African studies is that the scarcity or availability of key school resources affects educational outcomes, with higher level of resources being related to better educational outcomes. Spaull (2013: 2011) has identified two types of schools in our system. He classifies them as the wealthy functional schools and the poor dysfunctional schools and the roles they have in offering quality education to its learners. He goes on to say that poorly resourced schools also have teachers who have poor qualifications whilst the better resourced schools attract teachers of a good quality and who have higher qualifications.

Attwood (2001) claims that the achievement in secondary schools is influenced by several factors. These factors include learners' abilities, perceptions and attitudes, peer and parent influences and socio economic status, poor school learning environment, past racial discrimination, learning culture and the low expectations of teachers and principals. Van der Berg & Burger (2003) claim that some studies have shown a strong connection between disadvantaged school environments and learners' weak achievement in Mathematics. They further claim that learners from the Western and Northern Cape province in South Africa, which comprises of mostly white populations and affluent communities, tend to perform better and lead in grade 12 results whilst learners in the Limpopo Province, that has a large African population, usually perform poorly and rank last.

Many researchers have showed another contributory factor towards the poor performance, that being teachers' background and content knowledge. A study conducted by Ogbonnaya and Osiki (2007) that was carried out in Lesothowith grade 10 mathematics educators, investigated the relationship between mathematics achievement and teachers background, teaching practices and professional development. The results of their study revealed that teachers' background such as subject majors, studied, experience and qualifications has an influence and were a strong predictor of learners' achievements in Mathematics. In a similar study by Simkins, Rule and Berstein (2007) they highlight the importance of enhancing and improving the quality and supply ofmathematics and science teachers in South African schools in order to improve the results in these gateway subjects.

Howie (2003) conducted a research project with more than 8000 pupils in 200 schools in South Africa and discovered that pupils' proficiency in English strongly impacted their success in Mathematics. In South Africa learners taught in a language different from their home language experience a negative impact on their performance in Mathematics. These studies show that learners will perform better if taught in their home language. Wabiri & Taffa (2013) argue that South Africa is a multilingual country and this variable needs to be acknowledged by educationalists in order to benefit learners. Regrettably the socio-economic differences that exist in the South African education system are strongly connected to ethnicity and gender.

Makgato (2007) carried out a study that focused on both teachers' andlearners' insights as to the reason why grade 11 and 12 Mathematics and Physical Science learners perform poorly. He discovered that the main contributing factors included teachers' content knowledge and time management, students' motivation and interest, teaching strategies and parents' commitment to children's education. Jaiyeoba and Atanda (2011) conducted a similar study in Nigeria and investigated nine school quality factors that could possibly impact student's performance in Mathematics. From the nine variables it was established that instructional materials and adequate toilet facilities were considered to have added significantly to students' achievements. A study in Kenya was conducted by Mbugua, Kibet, Muthaa and Nkonke (2012) which investigated the factors contributing to the performance in Mathematics by form 3 students in secondary schools. The contributing factors to the poor performance comprised of inadequate teaching and learning materials, under-staffing, and lack of motivation and poor attitudes by both teachers and students.

2.6 Research Perspectives of the use of Manipulatives

Manipulatives were first used, in the nineteen century when Friedrich Froebel became the first person to make use of concrete objects in the instruction of Mathematics (Fröbel & Lilley, 1967). He had a strong belief that it was possible to teach mathematical concepts by allowing his students to interact with concrete, movable, physical objects that would equip them to establishpatterns and identify geometric forms found in nature. Years later it was Brunerwho also recommended the use of concrete materials when he had hypothesized that learning occurs by transitioning through the three stages ofrepresentation namely enactive (concrete), iconic (representation) and symbolic (abstract) (Bruner, 1966). The use of physical objects to discover and explore a mathematical concept aligns with the first stage of Bruner' three- stage model, this being the enactive stage which involves learners interactingwith a hands –on, tactile exploration of the topic before proceeding towards amore abstract understanding of a concept. Piaget (1970) likewise recommended the use of physical objects in his theory of cognitive development. This is the third stage of his theory and is the concrete operational stage that involves learners applying logic to solve problems that apply to concrete objects.

Educators have used a variety of teaching materials to help their learners understand concepts and to learn. Since ancient times, people of various different civilizations have made use of physical objects to assist them in solving everyday mathematical problems and it has been a continued aim for educators and education, per se, to make provision to equal access tolearning, irrespective of learners' condition, abilities or disabilities (Ipapo, Hipona, Salvedia & Martin, 2021).

For a long time the use of manipulatives has either been encouraged or discouraged by researchers. Some researchers state that manipulatives do assist learners to better understand the abstract concepts contained in mathematics (Sowell,1989), whilst others (McNeil, Uttal, Jarvin & Sternberg, 2009; Jarvin, McNiel & Sternberg, 2006; Ambrose,2002) disagree that manipulatives are effective.

In contrast to these criticisms McNeil & Jarvin (2009) claim that the use of manipulatives posses several benefits which include:

- Providing an additional resource in learning mathematics
- Assisting learners to make the link with real-world knowledge
- Aiding in understanding and increase memory

Even though manipulatives prove to have several benefits, there is no guarantee of success if teachers use them incorrectly, that is primarily for fun ineffectively. Studies against the use of manipulatives show that teachers appear to see activities that use manipulatives as playtime (Green, Piel & Flowers, 2008). The study carried out by Moyer (2001) which comprised of ten school teachers revealed that teachers discovered the use of manipulatives to be rewarding and enjoyable with learners but they could not identify the value of the use of manipulatives as aids for mathematics learning.

In addition, the use of manipulatives is the requirement for dual representation or understanding manipulatives as both concrete objects and as symbols of mathematical concepts (Uttal, Scudder & De Loache, 1997). McNiel & Jarvin (2007) state that acquiring dual representation skills requires additional cognitive resources that are absent in developing children. Boultan-Lewis (1998) explains that whilst children may posses the ability to manipulate the object and give suitable names to them, they still continue to lack the ability to find the link between the mathematical concepts represented by the object to its tangible symbol.

Unlu (2017) conducted a study with teachers and concluded that they wanted to use manipulatives in their lessons as manipulatives made teaching easier and aided to concretize abstract mathematical concepts in the lesson. Teachers who declined the use of manipulative in their lessons, displayed the lack of skill in the use of manipulatives, difficulty in the preparation and utilization of manipulatives. A study carried out by Piskin-Tunc, Durmus & Akkaya (2012) with secondary school mathematics teachers revealed that teachers believed that using concrete manipulatives and virtual learning objects in mathematics teaching would result in an increase in learners' success.

The study of Marshall and Swan (2008) revealed that teachers believed that the use of manipulatives in mathematics lessons increases children's mathematics learning. In addition, the results showed that if mathematics manipulatives are to be effective, then it is important that mathematics manipulatives be part of a carefully planned mathematics programme. In particular, teachers should have adequate knowledge of Mathematics, students and manipulatives.

Golafshani (2013) investigated the beliefs of mathematics teachers on the use of manipulatives in mathematics teaching and the effects it had on student's learning. The findings revealed that teachers were eager and willing to incorporate manipulatives in the mathematics lessons. The use of manipulatives had a direct impact on student learning. Whilst these studies show that teachers have a positive approach to the use of manipulatives, there exist studies with teachers who think that manipulatives are not needed for lessons. Studies by Ciftci, Yildiz & Bozkurt (2015) and Kutluca & Akins (2013) reveal that teachers did not believe that concrete manipulatives could be used in all topics Mathematics. In addition, Mathematics teachers in secondary also did not think that manipulatives could be incorporated in every subject of a mathematics course. Kucukgencay, Peker & Acar (2020) examined teachers access statuses to manipulatives and discovered that teachers generally had a positive view regarding the use of manipulatives in lessons but the challenges that existed prevented them from using manipulatives. The greatest challenge experienced is that teachers have limited access to manipulatives. In addition, the opportunity to access manipulatives for teachers and to design appropriate manipulatives may have an impact on the frequency with which manipulatives are used in lessons. They recommend that teachers should have infinite access to manipulatives and teachers should also possess adequate knowledge to be able to design their own manipulatives for their mathematics lessons. Additionally, Johnson, O' Meara & Leavy (2021) claim that manipulatives provide learners with an additional lens through which to view the mathematical concept and offers an additional resource to assist in the development of their understanding.

The above differing research perspectives hold some truth and is key to find a common ground.

2.7 Context of The Study

A continuous dilemma for educators of Mathematics is the concerns of how to assist learners in understanding abstract concepts such as the sine rule, cosine rule and area rule, which are used to solve complex 3-D trigonometric problems. Learners appear to understand the sine rule, cosine rule and area rule in isolation, however, when exposed to complex problems, which require multiple procedures, they experience immense difficulty and anxiety. Trigonometry has presented several challenges to learners (Demir, Sutton-Brown & Czerniak, 2012 and Maharaj, 2008). Learners do not appreciate and understand the historical use, use in daily life and benefits offered. Dundar (2015) maintains that many learners who come across or experience the application of the sine, cosine, tangent and cotangent in high school education for the first time, are unable to see the link and relate these concepts to their real-life situations and are unable to identify the origins. Learners will grasp trigonometric concepts once real-life applications of trigonometry and the importance are imparted to the learners. Brijlall & Niranjan (2015) state that one of the most vital factors in trigonometry is the transformation of the cosine and sine rule into a sum that becomes a beneficial idea in problem solving activities. Maharaj (2008) claims that the teaching of trigonometry theorems and concepts are crucial for the development of logical, creative and analytical thinking skills of learners. Trigonometry is the prerequisite for the understanding of more complex and advanced concepts and forms the basis of advanced learning in mathematics.

The aim of teaching mathematics is to help learners acquire cognitive abilities to be able to evaluate daily life scenarios using problem- solving techniques. Mathematical educationists recommend that multiple representations should be used in teaching mathematics as these representations assist learners to acquire flexible and multivariate skills in problem solving. This acquisition of skills improves their understanding of mathematical concepts (Brijlall and Maharaj, 2015). Brijlall (2014) claims that the existence of several factors on school performance, is largely as a result of unacknowledged poor teaching and learning of Mathematics in many schools. He argues that this poor teaching and learning leads to learners performing poorly in Mathematics. Demir, Sutton-Brown & Czerniak (2012) add that understanding trigonometric links is difficult for learners and that the traditional methods of teaching trigonometry do not overcome learners' difficulties. Not surprisingly educators, educationalists, researchers and the minister of Basic Education, Angie Motshekga have called for better techniques to assist learners learn mathematical concepts. The work carried out by psychologists such as Jerome Brunner, Jean Piaget, Zolton Dienes and Richard Skemp is now beginning to influence mathematical pedagogy.

Orhun (2010) describes trigonometry as being a branch of mathematics that deals with the relationships of the angles and sides of triangles, which makes up a vital background to the solution of problems that exist in several disciplines. Siyepu (2015) concurs with Orhun and states that trigonometry is used often in mathematical explanations, ideas and concepts one of which includes trigonometric ratios applied to describe the existence of relationships of angles and sides in right –angled triangles.

Research has shown that learners have not fully developed clear concepts in solving 3-D problems with the knowledge of trigonometry. Trigonometry has been frequently seen as a difficult subjects which learners struggle with (Siyepu, 2015).

In teaching Mathematics, the researchers primary concern is with concept construction as compared to memorization of facts. Policy-makers have transformed the South African teaching and learning setting by revising curricula in accordance with the Curriculum and Assessment Policy Statement (Department of Basic Education 2012:1). According to Van Laren (2012: 203) this revision is expected to improve teaching and learning in mathematics. Maor (1998: 37) reiterates that achievement in trigonometry is powerfully dependent on geometric concepts, as learners need to connect and identify measurements in drawings of shapes to numerical ratios.

This study attended to and addressed the Minister of Education's challenge that teachers' knowledge and the provision of quality learning support material is insufficient in the area of trigonometry which results in poor performance by learners in Mathematics (Motshekga: 2012).

Trigonometry in the past and present forms a considerable component of the Grade 12 examination. Supplying an interactive learning environment that allows for learners to participate in the learning activity will thus surely enhance learning. The research undertaken was to illustrate real life scenarios and trigonometric models and the application of these concepts. These were investigated to establish whether they improved concept development from concrete to abstract and if deeper conceptual understanding occurred. These real life situations included hot air balloons above the ground, vertical towers, triangular right prisms, goal posts in Moses Mabhida soccer stadium, the Great Pyramid at Giza in Egypt and a rectangular birthday card placed on a table.

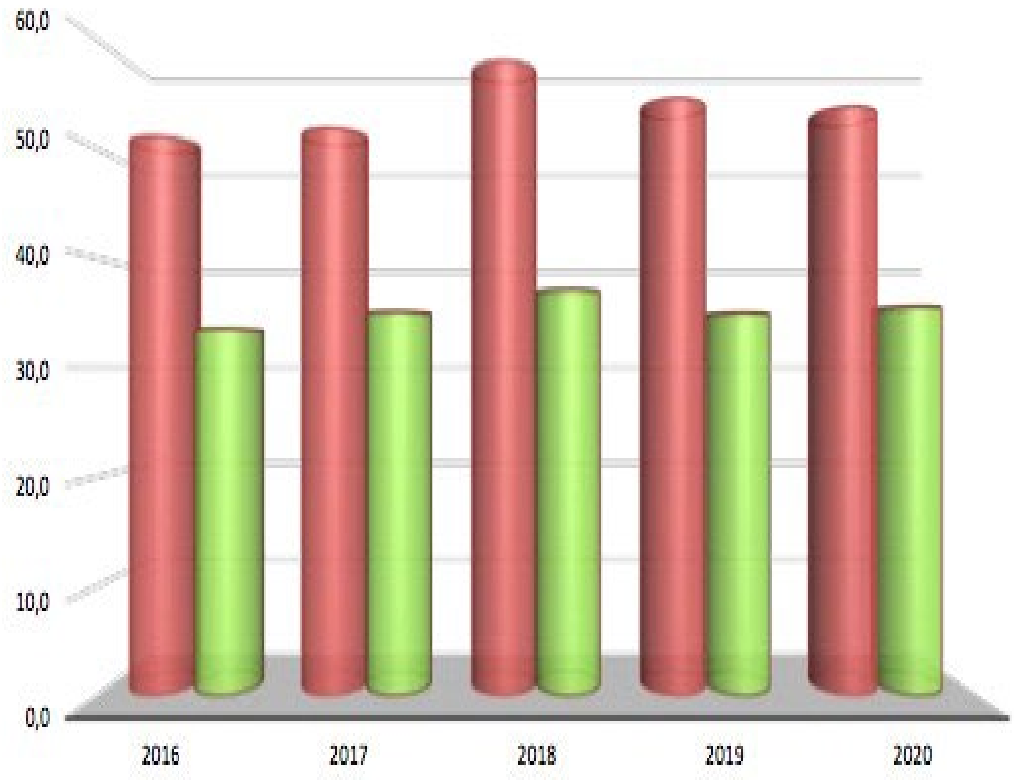
Problem solving plays an important role in education. Poon (2012: 449) states that one of the aims of teaching through problem solving is to motivate students to improve and build on their thought processes as their experience permits them to reject some ideas and become more conscious of others. In addition to developing their knowledge, students also obtain an understandingof when it is suitable to use particular strategies. Hidayah and Prayoga (2021)claim that learners' attitude towards Mathematics is a major contributory factorthat affects learners mathematics achievement and learning activities. Their findings inferred that the causative model of learners' attitude towards Mathematics with the forming indicators in discovery learning with virtual and concrete manipulatives proved to be good.

Collaborative learning refers to the instructional activity learners engage in when working together and interacting intensively so as to exchange ideas to achieve a common goal (Laal & Ghodsi, 2012). Carbonneau, Wong & Borysenko (2020) state that learning with concrete manipulatives is highly dependent on learners' interpretation of the manipulatives and that collaborative learning is a strategy that can be easily included in classrooms.

2.8 Performance Trends of Paper 2 (2016–2020)

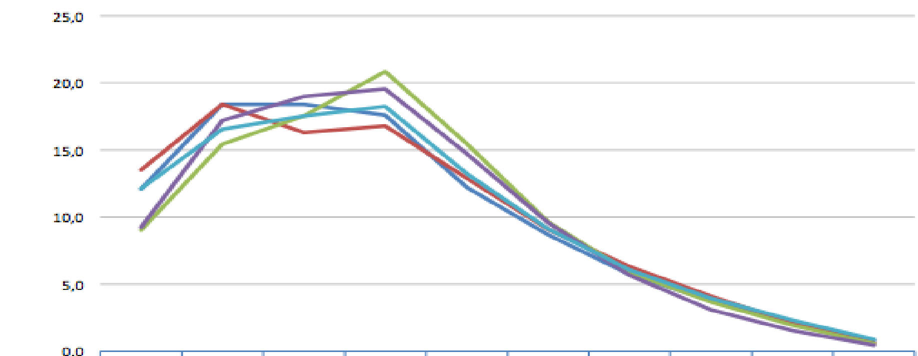
The number of candidates who wrote the Mathematics examination in 2020 increased by 11 281 in comparison to the numbers in 2019. The performance of the candidates in 2020 showed a slight decline at the 30% level from 54,6% in 2019 to 53,8% and a slight increase at the 40% level from 35,0% in 2019 to 35,6%.

Table 2.1 Overall achievement rates in Mathematics



	2016	2017	2018	2019	2020
% achieved at 30% and above	51,1	51,9	58,0	54,6	53,8
% achieved at 40% and above	33,5	35,1	37,1	35,0	35,6

Figure 2.1.1. Performance distribution curves in Mathematics (percentage)



	0-9.9	10-19.9	20-29.9	30-39.9	40-49.9	50-59.9	60-69.9	70-79.9	80-89.9	90-100
2016	12,1	18,4	18,4	17,6	12,2	8,7	5,7	3,8	2,2	0,8
2017	13,5	18,4	16,3	16,8	12,9	9,1	6,3	4,1	2,1	0,6
2018	9,0	15,4	17,5	20,9	15,5	9,6	5,9	3,7	1,9	0,6
2019	9,2	17,2	19,0	19,6	14,7	9,6	5,6	3,1	1,5	0,4
2020	12,1	16,5	17,5	18,2	13,2	9,1	6,1	3,9	2,3	0,9

According to DBE NSC Diagnostic report (2020: 182) performance in the 2020 examination revealed a deficiency in the understanding of basic concepts across some topics in the curriculum. It appears that candidates are becoming over-reliant on past examination papers. While past examination papers may serve as a valuable resource for revision, the teaching and learning of basic concepts cannot and should not be overlooked. It was pleasing to note that the candidates' answering of routine questions in Euclidean Geometry shows continuous improvement. Performance will be further enhanced if attention is given to the following areas: strengthening the content knowledge in Trigonometry and learners' exposure to complex and problem solving questions across all topics in the curriculum, starting in the earlier grades.

2.9 Overview of Learner Performance in Paper 2

Individual performance in the paper varied from very poor to excellent. Integration of topics is still a challenge to many candidates. Mathematics cannot be studied in compartments and it is expected that candidates must be able to apply knowledge from one section to another section of work. It is evident that many of the errors made by candidates in answering the Trigonometry questions in this paper have their origins in a poor understanding of the basics and the foundational competencies taught in the earlier grades.

In general, candidates need to exercise caution with algebraic manipulation skills since overlooking certain basic principles or practices results in the unnecessary loss of marks. Although the calculator is an effective and necessary tool in Mathematics, learners appear to believe that the calculator provides the answer to all their problems. Some candidates need to realise that conceptual development and algebraic manipulation are often impeded because of the dependence on a calculator (DBE NSC Diagnostic report, 2020)

Luneta & Giannakopoulos (2019) state that teachers also experience challenges and difficulties related to a concept or skill in Mathematics, where this lack of knowledge plays a major and important role in the creation of misconceptions (Ozkan & Ozkan, 2012). Several studies in mathematics claim that misconceptions are like “snowballs” (Makonye & Luneta, 2014). Misconception will continue to transpire among students in more complex learning topics. If this is not addressed then existing knowledge related to mathematical concepts that are not mastered by learners will have a negative effect on new learning topics. The reason being that each new topic will consist of a variety of new mathematical concepts. These misconceptions need to be identified immediately by teachers in order for misconceptions to be at a minimum before the commencement of the following topic (Rohani, Riyan & Effandi, 2014). It can be concluded that the teaching and learning process plays a vital role in assisting learners to eradicate misconceptions. Therefore, it is important that teachers employ a variety of teaching strategies by focusing on possible misconceptions that their learners could potentially encounter (Chu, Shahrill & Tan, 2016).

The study conducted by Hamzah, Maat & Ikhsan, (2021) which involved articles that determine misconceptions in trigonometry and ways to eliminate them, produced findings that learning trigonometry, using manipulative materials and digital form software, could eliminate misconceptions. Ulyani and Qohar (2021) found in their study that using manipulative materials in the teaching of trigonometry could enhance learners understanding of trigonometric concepts. In their study the manipulative materials made by learners included rotation of a circle, three types of trigonometric triangles and a comparison of sine, cos and tan. The findings revealed a positive outcome. Learners provided a positive feedback in each aspect that was assessed. In addition, the quality of learning outcomes can improve through the use of manipulative material. Ulyani & Qohar (2021) claim that learning through manipulative materials allows for the learning of trigonometry to become easier to understand. Further-more learning that includes elements of play, allows for improvement of students' understanding in trigonometry (Bernard, Sumarna, Rolina & Akbar, 2019; Prabowo, Usodo & Pambudi, 2019; Ibrahim & Ilyas, 2016; Jorda & De los Santos, 2015; Yusha'u, 2013). The finding in the study of Hamzah, Maat and Ikhsan (2021) reveal that learning trigonometry using manipulative materials and digital form software can eliminate misconceptions.

2.10 Suggestions for Improvement of Mathematics Paper 2

A careful analysis of the information that is given will give learners some idea of the concepts required in solving a triangle. Teachers need to develop strategies to be used when solving right-angled triangles and triangles that are not right-angled. Learners should be taught the conditions that decide which rule should be used to solve the question. Learners should be encouraged to highlight the different triangles using different colours. Initially, expose learners to numeric questions on solving 3-D problems. This makes it easier for them to develop strategies on how to solve such questions. Once learners have gained confidence with numeric type questions, they should then be exposed to non-numeric and higher-order questions (Mathematics National Senior Certificate 203 Diagnostic Report 2020 – Part 1).

2.11 Learning Styles

Learning styles can be described as individual characteristics that explain an individual's preference in a learning environment and an individual's learning style has a major impact on their academic achievement (Kablan & Kaya, 2013; Cheng, Cheng & Chen, 2012; Kurbal, 2011). Studies that have been conducted on learning styles have shown that various teaching methods have varying impact on learner academic performance in learners who have varying learning styles (Tulbure, 2011; Kvan & Jia, 2005). Dana- Picard, Kidron, Komar & Steiner (2006) state that the traditional teaching strategy known as lecturing is often used in mathematics education, which does not always accommodate all learners learning style (White, 2012). It is, therefore, imperative that educators tailor their teaching instruction to accommodate varying learning styles in the classroom (Bhatti & Bart, 2013). Thus, in order to meet the needs of learners with different learning styles, it is important that different learning methods be incorporated in mathematics education (Kablan, 2016).

One of the well-known theories that explain the relationship between learning style and academic achievement is David Kolb's Experiential Theory (ELT) (Kolb, 1984). Kolb's ELT consists of four modes of effective learning namely concrete experience (CE), reflective observation (RO), abstract conceptualization (AC) and active experimentation (AE).

Figure 2.2 displays the learning styles and learning modes that stem from different combinations of modes.

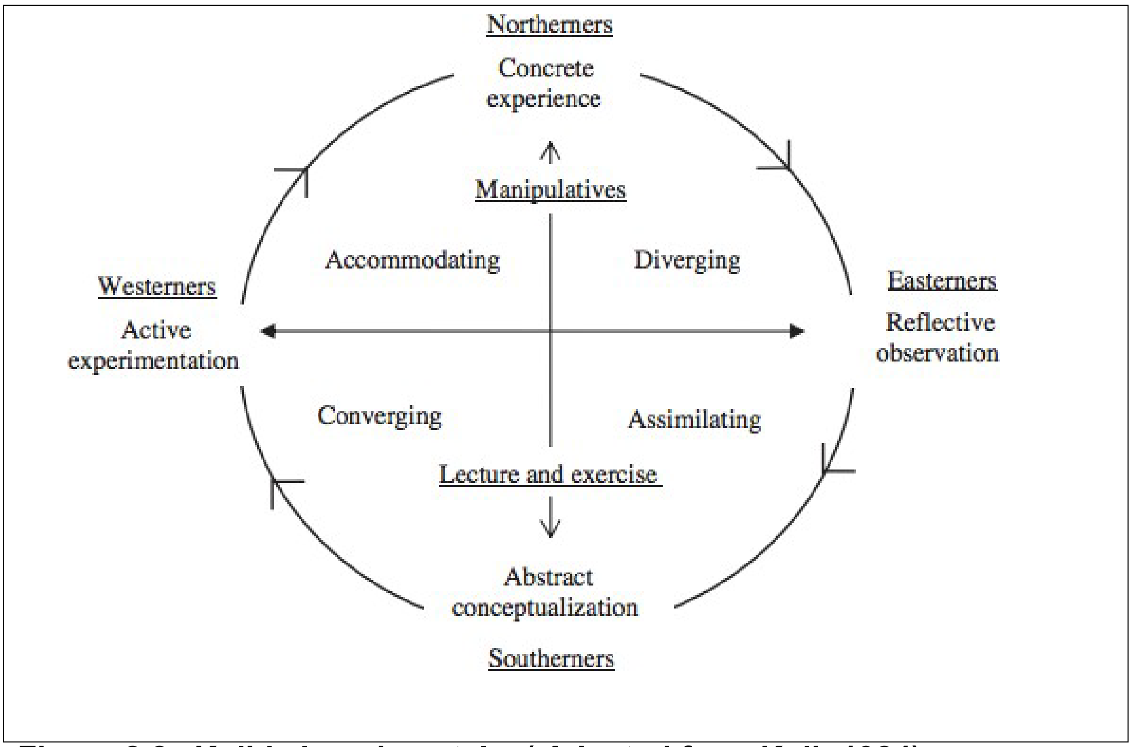


Figure 2.2 Kolb's learning styles(Adapted from Kolb,1984)

Kolb explained that an individual's learning style is established by the blend of all four learning modes that an individual prefers and each mode is further divided into one of two dimensions. The first dimension (AC-CE) is known as the abstract-concrete dimension. This suggests how one perceives and comprehends new information. In new situations some individuals choose to answer using concrete methods that involve their senses and feelings. On the other hand, others prefer abstract methods that require thinking and analyzing. The second dimension (AE-RO), the active reflective continuum, addresses how an individual processes new information. Sutliff & Baldwin (2001) state that some individuals opt to transform new information actively by doing whilst others prefer reflecting via observation. David Hunt and his colleagues (1987) (as cited in Kolb & Kolb, 2005) classified four more learning styles, namely, the northerner, easterner, southerner and westerner. The northerner learning style incorporates the RO and AE dialectics and specializes in concrete experience. It is a combination of characteristics and abilities of the accommodating and diverging styles. The southerner learning style is a combination of elements from the assimilating and converging styles. It is not rigid with respect to the RO and AE dimensions and specialises in AC (Kolb,1984). Kablan (2016) adds that that the most common instructional methods applied in mathematics instruction are lectures and exercises.

Academics and researchers over the years have formulated various models for learning styles among individuals. The majority of these models are based on five or seven learning styles. Terminology may be different, however, the core of theses models remains unchanged. Mantle (2001,1) describes the seven learning styles as:

(1) ***Solitary*** (intrapersonal) is where the learner prefers to work alone and use self-study. They pursue their own interest and have a clear deep understanding of themselves. They appear to stand out from the crowd and enjoy being independent and original. They perform well in self-placed instruction, individualized projects and working on their own. Learners enjoy their own space and must be encouraged to socialize.

(2) ***Social*** **(**interpersonal) is where the learner prefers to learn in groups or with other people. They adapt easily to social situations, have several friends and make excellent leaders. These learners are described as being empathetic, and patient which makes them popular among classmates. This type of learner excels in a group situation as they compare, share, relate, discuss and interview each other.

(3) ***Logical*** *(*mathematical): The learner is very mathematically inclined and enjoys solving mathematical related problems. They are very logical and direct. These learners prefer using logic and reasoning and learn best by categorizing, classifying and working with abstract patterns and relationships.

(4) ***Physical*** (kinesthetic) is where the learner prefers using body, hands and sense of touch. This type of learner constantly walks around, touches everything and uses body language to convey their feelings. They require active education. They would prefer to play sport or do craft instead of reading a book.

(5) ***Verbal*** (linguistic) is where the learner prefers using words, both in speech and writing. This type of learner enjoys reading, writing and story telling. They are able to easily memorise places, dates, names and trivia. They possess the ability to repeat back everything that you have told them. These learners learn best by saying, hearing and seeing words.

(6) ***Aural*** (auditory-musical): This type of learner prefers using sound and music and is best at noticing details, pitches and rhythms that escape a normal listener. They learn best via music, melody and rhythm. To aid in memorization techniques, they write songs and rap narratives.

(7) ***Visual*** (spatial) is where the learner prefers using pictures, images and spatial understanding. They are good at working with colours and pictures. They are artistic however they often experience problems expressing it.

Individuals have a specific learning style preference depending on the situation. Most people will therefore have a combination of learning styles. A mathematics educator is more likely to use the teaching style similar to their own preferred learning style and experiences. The task of now teaching large classes with learners preferring their different learning styles becomes much more demanding. Rahman and Ahmar (2017: 74) state that the development of one's personality usually depends on learning styles and is often influenced by environmental, emotional, social influence and individual feelings. How to learn is different for each person. Some learners need to see more, some need to hear and some have to do something to the body using a series of activities. Koparan (2017: 26) claims that better and permanent learning is more likely to be achieved with the inclusion of more sense organs in the learning process.

2.12 Learning Styles Specific To This Study

This study focuses on three specific learning styles namely visual, auditory and kinesthetic.

Learning style is an important modality in the learning process. Gilakjani and Ahmadi (2011) claim that modalities are often linked with learning styles. It is an association with how learners make use of their senses in the process of learning. Three types of modalities exist. These include visual, auditory and kinesthetic (Gholami & Bagheri, 2013; Gilakjani & Ahmadi, 2011). In addition, Larbi & Mavis (2016) claim that one of the major tasks of a teacher of mathematics is how to ensure the subject is meaningful to learners. Very little or no learning occurs when the environment is threatening and, therefore, requests educators to create a productive and conducive environment. Mathematics therefore needs to be presented in a manner that satisfies learners learning styles and thought processes.

Rahman & Ahmar (2017) state that the development of a person's personality is dependent on learning styles and is shaped by the emotional, environmental, social influence and individual feeling. Everyone has a differentway of learning. Learners need to hear, see more or do something to the bodyusing a set of activities. Rahman & Ahmar (2017) claim that the learning styleof each person consists of three types of visual style, auditory style and kinesthetic style. The various types are listed below in detail:

2.12.1 Visual Learning Style

The visual learning style is a style where learners learn most effectively by seeing images of what they are learning. These learners learn through reading and are orientated to printed texts. Visual learners tend to think and learn best in pictures and visual images. Learners who possess a visual learning style are neat, systematic, good planners, meticulous to detail, excellent spellers and pay attention to their physical appearance and attire.

Shabiralyani, Hasan, Hamad & Iqbal (2015) claim that visual aids arouse the interest of learners and assists teachers to teach concepts in an easy manner. Visual aids are made up of instructional aids that are utilized in classrooms in order to support and foster students learning process. Visual aids assist teachers to establish, clarify, and correlate and co-ordinate specific conceptions, understandings and supports teachers to make learning active, more real, motivating, encouraging and significant in their classrooms. The research conducted by them revealed that individuals tend to easily forget but correct application of visual aids assists in retaining concepts permanently, learning better when exposed to different visual aids. Visual aids propagate the real image when learners hear and see clearly; visual aids build interesting environments; visual aids promote vocabulary of learners; visual aids encourage conceptual thinking and provide direct experience to learners and visual aids assist teachers to obtain some time and make learning more permanent.

2.12.2 Auditory Learning Style

In the auditory learning style learners learn better when they hear what they are learning. Learners who have an auditory learning style tend to talk to themselves, are easily distracted by commotion, move their lips and read out aloud, are happy to read aloud and listen, can repeat and mimic the tone and find difficulty in writing but enjoy storytelling;

2.12.3 Kinesthetic Learning Style

Kinesthetic learning style is a style of learning that involves moving, experiencing and experimenting. These learners tend to speak slowly, responds to physical attention, touch people to obtain their attention, stand close when talking to people, are always physically orientated, have the early development of large muscles and learn through manipulation.

2.13 How These Learning Styles Apply to the Study?

Teaching in a way that combines different learning styles is probably the most appropriate and beneficial option in South African schools. The combination of the kinesthetic, visual and auditory learning styles would work well in teaching 3D trigonometry mathematics.

One of the most suitable methods of efficient teaching in mathematics education is via manipulative supported teaching. Since manipulative-supported education targets more than one sense organ, it is persistent, quick and meaningful and requires little time to ascertain what is required by the learners to plan and prepare the lesson (Tertemiz, Celik and Dogan, 2014).

The use of manipulatives allows for information to be received visually and kinesthetically (Hunt, Nipper & Nash 2011: 1). According to Oladejo, Olosunde, Ojebisi & Isola (2011) manipulative teaching material is considered to be any object from the real world that children can rotate, paly or build a model of to demonstrating a scientific concept. They are concrete and hands on models that gravitate to all senses and learners can touch. These manipulative teaching materials must be linked to a learner’s world.

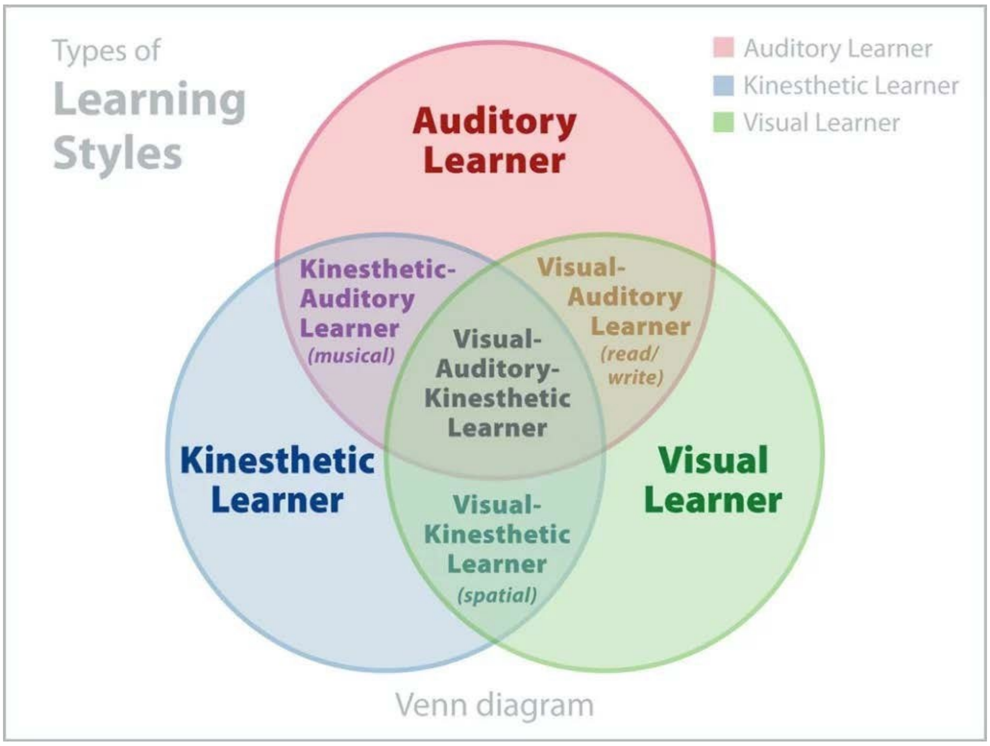


Figure 2.3 shows the 3 learning styles that are most relevant to this study and the overlapping of the learning styles

In this study the use of manipulatives catered for learners with different learning styles. Some learners gravitated to one specific style while others had a combination of learning styles. When learners engaged with the manipulatives, they were able to interact with the teaching aid. Learners created meaning through negotiating, diagnosing and challenging misconceptions and beliefs. Marasigan *et al* (2019) adds that cognitive processes are assisted by manipulatives and another advantage that exists is that manipulatives engage students and increase both interest and enjoyment in Mathematics. In his study learners who engaged with manipulatives stated that they were more interested in Mathematics. Thus manipulative usage encouraged a learning environment that promoted engagement and understanding.

Marasigan *et al* (2019) argues that every individual has his or her ownstrengths and weaknesses and it is, therefore, imperative for Mathematics teachers to provide learning materials for learners to utilize to obtain a better learning of the subject content. Similarly Larbi & Mavis (2016) add that manipulatives are beneficial to learners of Mathematics and provide a tool for teachers to use in their instruction when introducing mathematical concepts and also to assess learners' understanding. Learners each have a unique way of learning. Senses are introduced in learning when manipulatives are utilized as they act as visual representations of mathematical concepts. Apart from satisfying the needs of learners who enjoy learning this way, it gives teachers innovative ways of introducing mathematical topics.

Hillman, Hillman, Duprat & Cargile (2018) maintain that the use of physical manipulatives has proved to be effective in classrooms as they are multi-sensory and can represent ideas and concepts in many ways. In addition Cope (2015) and Kocaman (2015) claim that manipulatives not only contributeto the learners cognitive aspect but also develops the psychomotor skills by attending to the learners' sense of sight, touch and hearing.

2.14 Higher – Order Thinking Skills

Anderson & Krathwohl, (2001) and Krathwohl(2002) state that Bloom's taxonomy has been revised and has changed to (1)remember, (2) understand, (3) apply, (4) analyse, (5) evaluate and (6) create as shown in figures 2.2 and 2.3. Armstrong (2016) provides a detailed explanation of the indicators at the various stages and theses include (1) Remembering: be able to recall and recognize; (2) Understanding: be able to interpret, exemplify, classify, summarise, compare, infer and explain; (3) Applying: be able to implement and execute; (4) Analysing: be able to organize, differentiate and attribute; (5) Evaluating: be able to critique and check and (6) Creating: be able to plan, generate and produce. Figure 2.2 shows the various levels of Blooms Taxonomy.

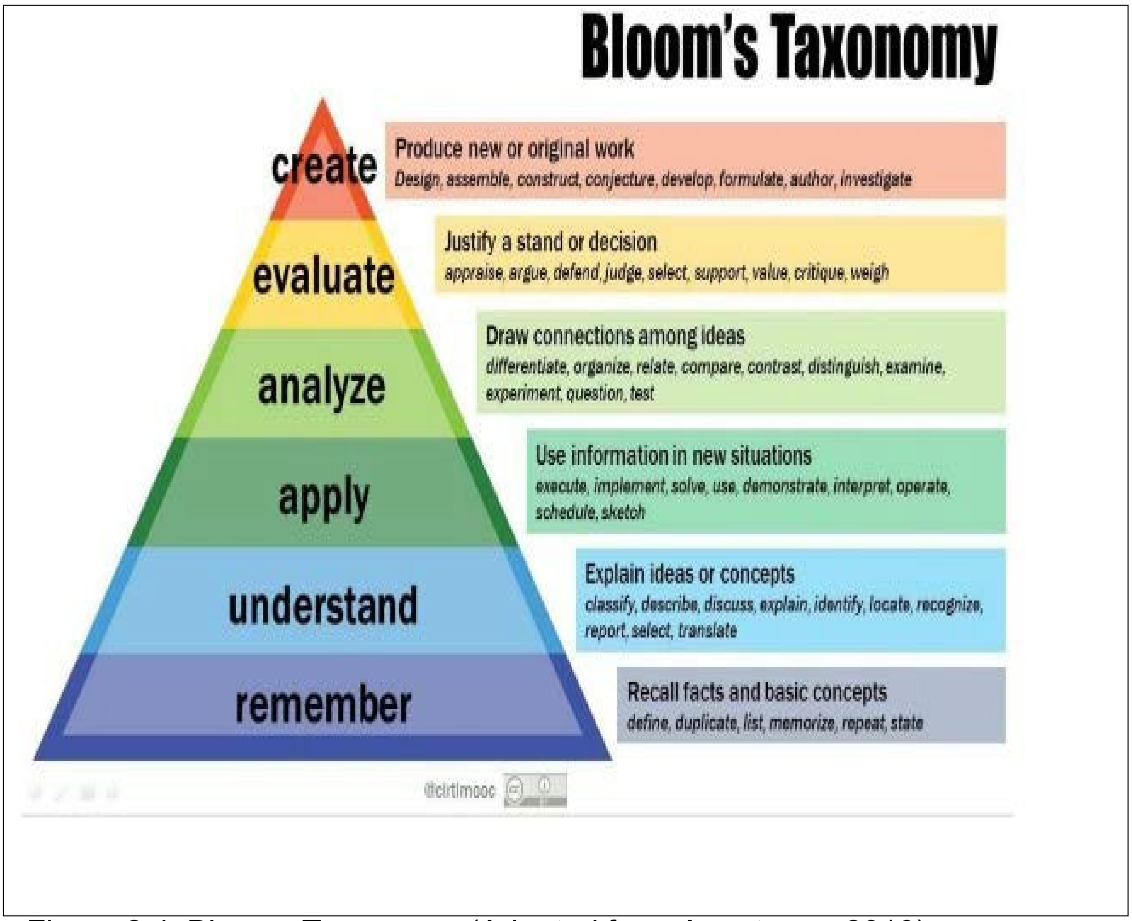


Figure 2.4 Blooms Taxonomy (Adapted from Armstrong, 2010)

In addition Hidayah, Isnarto, Masrukan, Mohammad Asikin & Margunani (2021) maintain that a learner in the final stage (creating) implies that they posses the skills of preceding stages. The curriculum mandates that according to the revised Bloom's Taxonomy, learning must develop higher- order thinking skills. The taxonomy has in addition made provisions for factual,procedural, conceptual and metacognitive knowledge.

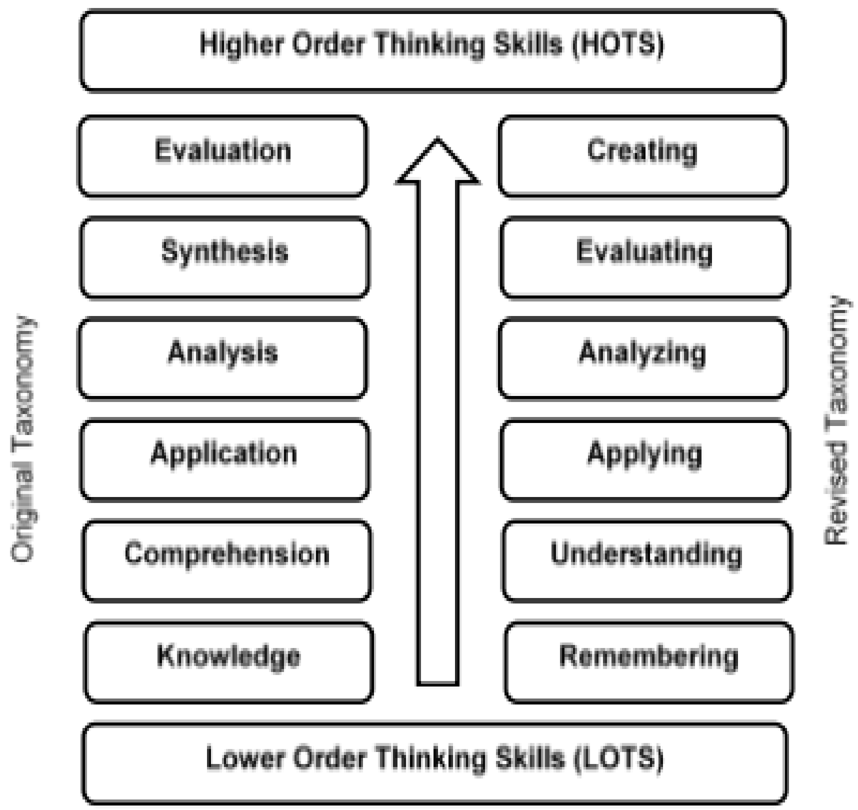


Figure 2.5 Revised Bloom taxonomy (Source Hidayah *et al*, 2021, 540)

Learning may commence from the lower order thinking skills and proceed to the higher order thinking skills. Bloom's levels regarded as HOTS comprises of analysis, evaluation and creation (Liu, 2010). Some researchers (Furner & Worrell, 2017) argue that manipulatives could assist learners understanding mathematical objects better but also that is not guaranteed. Ojose (2008) asserts that learners in the cognitive level of operational concrete require the use of manipulatives in mathematics learning.

The use of manipulatives will have a greater effect when combined with a set of questions to direct learners' discovery through activities (Hidaya *et al*, 2018). It has been proven that questioning is an effective means of scaffolding the higher-level thinking processes of mathematics learners (Lee & Chen, 2015). Hidayah *et al* (2021) advocates that good questioning coupled with manipulatives aids in fostering higher order thinking in learners.

2.15 Manipulatives are Important in the Mathematics Classroom

Manipulatives can be described as concrete objects with which learners interact, which they can touch, see and hear and that act as a visual representation of mathematical concepts that are abstract. Smith (2009) describes manipulatives as" physical objects that are used to teach as teaching tools to engage learners in a hands-on learning experience of mathematics." Smith further adds that manipulatives enable the teacher and learner to detach from the transitional classroom environment and usual instructional style by giving learners a practical experience. Strom (2009), Reimer and Moyer (2005), Day and Hurell (2019) and Stiegelmeier and Moore (2019) claim that research has shown that learners of varying ages obtain enjoyment when taught Mathematics through participatory and interactive methods that allow for the utilization of manipulatives.

According to learning theory based on theorists such as Jean Piaget, children are considered to be active learners who master concepts by progressing through the levels of knowledge, namely concrete, pictorial and abstract. Interaction with manipulatives allows for learners to firstly explore concepts at the concrete level of understanding. When learners engage with manipulatives, they begin to construct understanding and start internalizing mathematics processes and procedures. Jean Piaget and Jerome Bruner have revealed that children learn concepts better when their thinking moves through the three hierarchical levels namely enactive(concrete), iconic(pictorial) and the symbolic(abstract) (Kurniawan, Budiyono, Sajidan & Siswandari, 2020; Ubuz & Erdogan, 2019; Furner & Worrell, 2017).

Figure 2.4 below shows the three experiential stages that Bruner (1966) proposed:

a) The **enactive (direct sensory) experience** where learners adopt an active part in their learning through the manipulation of their surroundings and environment, b) **Iconic representation of experience** where enacted experiences are embodied via film clips, charts, diagrams and c) **symbolic representation** comprising of written language symbols that include words and mathematical symbols.

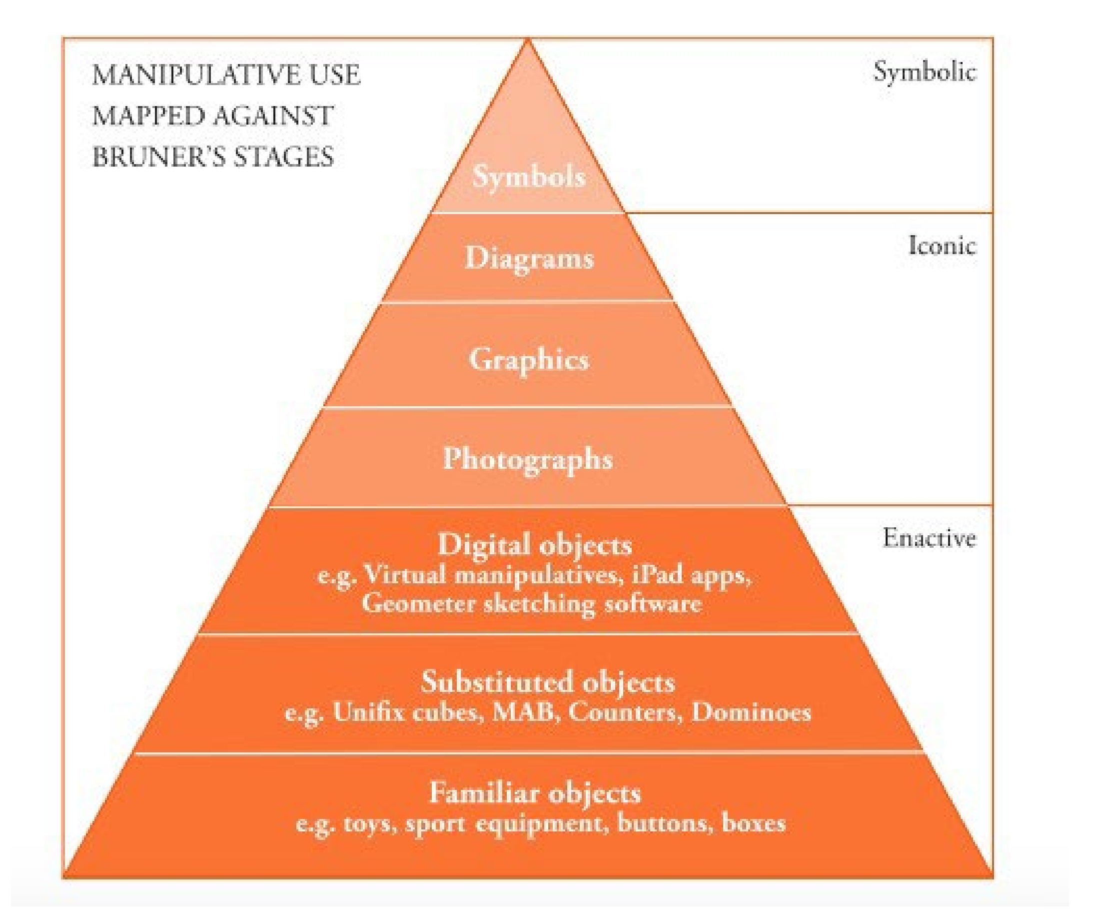


Figure 2.6 Mapping mathematics materials to Brunner's experiential learning. (Source Larkin, 2016: 13)

Literature has long advocated the use of manipulatives in classrooms to enhance learners' understanding of mathematics concepts. Ndlovu and Chiromo (2019) states that teaching Mathematics while using manipulatives assists learners conceptualize mathematical concepts. Therefore, a great need for educators' competencies in using manipulatives to assist learners is required. In order to construct a solid foundation of a concept, the use of manipulatives to develop understanding of abstract mathematics is considered essential. A study carried out by Buasen, Lubrica, Oryan, Alimondo, Dolipas & Ocampo (2020) suggests that students be exposed to interactive devices like manipulatives to improve their academic performance in Mathematics and to stimulate their interest in following degrees that are Mathematics related. Students perceived that the inclusion of concrete objects to illustrate abstract concepts in Mathematics can enhance their performance in Mathematics. In addition Lee and Chen (2015) claim that manipulatives effectively engage the intellect of students, irrespective of their liking to Mathematics.

Ndlovu and Chiromo (2019) argues that because the use of manipulatives has been identified as one of the resources that assists learners learn abstract Mathematics, it is vitally important that pre-service teachers possess knowledge and know how to use manipulatives themselves to solve problems so that they will effectively apply the use of manipulatives in their teaching.

Contrary to popular belief, manipulatives are not instructional magic in Mathematics learning. Reimer & Moyer (2005) and Day & Hurrell (2017) claim that it is crucial for teachers and researchers to acknowledge that presenting a manipulative to a learner, does not guarantee that learning will take place. The use of manipulatives can result in confusion if they are not correctly used and guided by the educator.

Maboya, Jita & Chimbi (2020) assert that some mathematics teachers believe that the use of manipulatives lessens their workload. These researchers argue that if manipulatives are not correctly used and if proper guidance by the teacher is absent or lacking, then manipulatives can cause confusion and create misconceptions in learners' understanding. This results in more work for the teacher to clear the confusion and eradicate misconceptions. Larbi & Mavis (2016) advises that care must be taken when using manipulatives otherwise learners think that two mathematical worlds exist, namely manipulative and symbolic. They concur with Day and Hurell (2017) and reiterate that merely presenting manipulatives in the mathematics classroom does not ensure meaningful learning. Carbonneau, Zhang & Ardasheva (2018) assert that manipulatives are not a panacea to all students learning. The instructional efficacy of a mathematics manipulative does not lie in the physical, concrete object but rather in the teacher's knowledge and ability to assist learners in making a link and generating meaning between the objects being manipulated and the abstract concept being taught. Moyer and Jones (2004) state that teachers who have poor content knowledge, frequently use manipulatives as toys and proceed to teach using rote learning. Van der Walle, Karp,Lovin and Bay-Williams (2014) explain that the misuse of manipulatives by learners could be avoided if learners are given free time to interact with the manipulative before commencing to solve of the problem. They further add that teachers misuse manipulatives by telling learners "to do as I do" with a manipulative. This amounts to learners being taught a rote mathematical procedure. Carbonneau *et al* (2012) discovered that the most engaging manipulatives were the perceptually rich ones. Larkin (2016) argues that merely giving a learner a manipulative does not imply the learner will be successful with the mathematical concept. Larkin discovered that learners find links between objects ad mathematical concepts when they use a perceptually rich manipulative that is scaffolded.

It is evident that ultimately the successful use of manipulatives is dependent on the teacher's content knowledge of mathematics, instructional beliefs, lesson planning, creativity, else teachers will continue with didactic instruction and the incorrect use of manipulatives.

2.16 Overview Of Literature On Manipulatives

The history of manipulatives for teaching mathematics extends for least two hundred years. More recent important influences have included Montessori, Piaget, Zoltan Dienes and Jerome Bruner. These innovators and researchers has highlighted the importance of real learning experiences and the use and need for concrete tools as an important stage in development of understanding. There is a long and solid research history in the use of manipulatives in teaching Mathematics. Manipulatives allow for learners to construct their own cognitive models for mathematical ideas and processes that are abstract. In addition manipulatives offer a common language to communicate these models to other learners and educators (Marasigan *et al,* 2019).

Sulistyawati, Puspitasari, Saidah & Rofiqoh (2021) claim that Mathematics can be considered as one component that has a significant role in the world of education. Mathematics necessitates learning as Mathematics involves solving everyday life problems, thinking clearly, logically, critically and forms a basics science to study others (Mahmudah, Ahyan & Rasidi, 2018). Sulistyawati, *et al* (2020) argue that there exists several problems in learning Mathematics. This includes the lack of maximum use of media in the process of learning and the difficulty of the learning material delivered by the educator, which results in poor understanding of learners. They suggest that educators apply innovative solutions, so that initially complicated learning of Mathematics becomes easy and reduces fear and laziness in learning among the learners. They provide a solution that educators can apply and that is the use of varied learning media, which includes the use of manipulatives. Making use of manipulative media can aid learners understand and prove abstract mathematical ideas, solve mathematical problems and allow for learning to become more challenging, motivating and interesting (Yensi, 2020; Larbi and Mavis, 2016). In addition, applying the use of manipulative media could increase (1) the activities of both the educator and the learner in the classroom environment, (2) learning outcomes, (3) motivation, (4) learner'sunderstanding a mathematical concepts, (5) attitude towards mathematics, (6)help learners to concretize abstract idea, (7) increase learner confidence, (8) and support higher order thinking skills (Hidayah, Isnarto, Masrukan & Asikin, 2020; Isnaniah & Imamuddin, 2020; Putra, 2020; Syam, Akib & Syamsuddin,2020; Ulyani & Qohar, 2020; Anggoro, 2019; Palupi, 2019).

Problem solving plays an important role in education. Poon (2012: 449) states that one of the aims of teaching through problem solving is to motivate

students to improve and build on their thought processes as their experience permits them to reject some ideas and become more conscious of others.
In addition to developing their knowledge, students also obtain an understanding of when it is suitable to use particular strategies.

In teaching Mathematics the researcher's primary concern is with concept construction as compared to memorization of facts. Policy-makers have transformed the South African teaching and learning setting by revising curricula in accordance with the Curriculum and Assessment Policy Statement (Department of Basic Education 2012:1). According to Van Laren (2012: 203) this revision is expected to improve teaching and learning in mathematics. Maor (1998: 37) reiterates that the achievement in trigonometry is powerfully dependent on geometric concepts, as learners need to connect and identify measurements in drawings of shapes to numerical ratios.

Merrill, Devine, and Brown (2010,16) affirm that improving and enhancing content knowledge requires mathematics teachers to use three-dimensional (3D) solid modeling in mathematics classrooms to improve learners' understanding of mathematical concepts and principles. When students visualize, they see the relevance and this promotes rigor. Weng (2011: 52) reiterates that visible and visual 3D dynamic design has the potential to increase learning interests in mathematics, by improving learner's interdisciplinary and multimedia design abilities.

When students visualize, they see the relevance and this stimulates rigour. Herbst and Chazan (2011: 1) collaborate this view by stating that artifacts can be used in activity systems for learners to interact with the object and other learners and the teacher thus enriches the lesson.

According to Strom (2009: 22) the use of manipulatives has now encouraged several learners to answer questions in the class who would otherwise just shrug their shoulders. In addition she claims that the use of manipulatiives encourages conversation and understanding between the teacher and the learner. The use of manipulatives improved her success with learning amongst her disabled learners.

Buasen, Lubrica, Oryan, Alimondo, Dolipas and Ocampo (2020) claim that students declared that their interest and curiosity in the interactive manipulatives was increased after engaging with the various manipulatives. Further, their students claimed that their academic performance in Mathematics improved by interacting with the manipulatives they had used in class. Their study suggested that students need to be exposed to interactive devices, which would improve mathematical academic performance and also encourage and arouse interest in choosing a Mathematics related degree.

2.16.1 Related Studies

The use of manipulatives in teaching Mathematics has had a long history. Several research studies have been completed on the use of manipulatives in assisting learners to better understand mathematical concepts (Ndlovu, 2019; Hidayah, Dwijanto & Istiandaru, 2018; Furner & Worrell, 2017; Kontas, 2016; Larbi & Mavis, 2016; Sandir, 2016;). Researchers Iqbal, Shams & Nazir (2020) examined the effect of using mathematics manipulatives on learners' academic achievement. Their findings revealed that manipulatives had a positive effect and learners who had interacted with manipulatives obtained higher achievement scores.

There, however, exists contradictory research which shows that manipulatives had no positive effect and were not any more effective than using other traditional ways and methods of teaching (Eastman & Barnette, 1979; Fennema, 1972). In addition some studies in literature suggested that manipulatives had no effect on learners' mathematical achievement (Clements, 1999; Sowell. 1989).

One such study carried out by Eastman and Barnett (1979) made use of 78 university students. From this cohort, 39 students were put into an experimental group who were allowed to use manipulatives and the other 39 students were put in a control group and were not allowed to use manipulatives. Eastman & Barnett (1979,12) concluded that:

> "In no instance did the experimental (hands on) group perform better than the control group, either on the paper-scored tests or on the tasks where both groups were asked to demonstrate the ability to manipulate the materials."

A study conducted by Fennema (1972) resulted in the same conclusion. The study was made up of 95 learners, which ranged from 7 to 8 years of age. These subjects were divided into two groups, namely symbolic and concrete. The concrete group was permitted to use a manipulative, Cuisenair rods, to solve mathematical questions whilst the symbolic group was not allowed to use manipulatives. The results revealed that the subjects learned betterwithout the use of the manipulatives. Fennema (1972: 238) claims that:

> "It does not indicate that the concrete models are not always more effective than symbolic models"

Thus, these early studies provided evidence that the benefits and advantages of using manipulative to teach Mathematics was inconclusive. These studies demonstrated that manipulatives did not aid in improving learners in learning mathematical concepts.

However, most of the research is directed to showing the use of manipulatives as a positive tool to improve students' learning in Mathematics (Parham, 1983; Suydam & Higgens, 1977; Kieren, 1971). This literature impacts this current study. In this review I have provided a chronological perspective about the importance of the use of manipulatives in mathematics. Some of the research conducted was done as early as the 1960s and 1970s and it produced mixed results. Kieren (1971) simply summarized the findings. This posed a problem as no experimental process was used as part of the research and in addition no classroom observations were done. This resulted in no declarations on the belief that manipulatives were beneficial to students. This then left it open to readers to derive their own deductions about the effect manipulatives has on teaching. Suydam & Higgins (1997) published a detailed review of more that twenty studies about activity based learning in mathematical instruction. Their review revealed that using manipulative materials resulted in an increase in achievement at all levels of elementary school.

In the 1980s more research was conducted. Parham (1983) concurred with the research of Suydam and Higgins (1977) pertaining to the advantages and benefits of utilizing manipulatives in Mathematics. Additional research found that those learners who used manipulatives in their mathematics class, from kindergarten to college age, normally performed better than those who did not use manipulatives (Raphael & Wahltrom, 1989; Sowell, 1989; Driscoll, 1983). Furthermore Sowell (1989) discovered that teachers who are well informed about the use of manipulatives and who teach using concrete materials, have a positive impact on the student's attitudes towards Mathematics. In additionthe study done by DeLoache, Scudder and Uttal in 1997 where young childrenwere the subjects of using manipulatives in Mathematics, showed that concrete objects aid children in understanding challenging concepts andmathematical processes.

Over time the use of manipulatives in teaching mathematics has developed remarkably. Through several studies the use of manipulatives has proven to be beneficial in Mathematics. Ralphael & Wahlstrom (1989) discovered that those teachers who often used a variety of instructional aids, managed to teach more subject content and their students achievement in topics such as geometry, ratio, percent and proportion improved. Research carried out by Moyer (2001) revealed that teachers play a vital role in creating mathematical environments that offer learners representations that improved their thinking.

Moyer & Jones (2004) conducted a study that comprised of 10 female middle grade teachers. Their study revealed the valve and importance of empowering teachers and showing them the benefits of instruction when using manipulatives. This research further provided important findings and showed that learners used manipulatives to help in their learning. A further study by Swan and Marshal (2010) looked at different ways in which teaching and learning through the use of manipulatives happened. Swan and Marshal discovered that there were many merits and gains to be obtained by making use of mathematical manipulative materials.

More current studies (Peltier, Morin, Bouck, Lingo, Pulos, Scheffler,Suk, Mathews, Sinclair & Deardorff, 2020; Marasigan *et al*, 2019; Hidayah, Dwijanto & Istiandaru, 2018; Liggett, 2107; Larbi & Mavis, 2016) provide evidence that using manipulatives allows students to better understand abstract concepts contained in Mathematics and to obtain better results. Numerous researchers had recently conducted research to investigate the impact manipulatives had in mathematics instruction. The findings in those studies showed that manipulatives increased mathematics achievement (Clements, 1999; Sowell, 1989)

The study by Larbi & Mavis (2016) was done to investigate the effectiveness of introducing algebra tile manipulatives in junior high school learners' achievements. The study was made up of 56 learners from two schools who were divided into two groups, one being the control group and the other the experimental group. The findings of their research revealed the value and that those that were taught with manipulatives performed slightly better. The use of manipulatives improved learner's thinking processes and proved to be an effective approach in teaching and learning.

A research study by Liggett (2107) comprised of 43 Grade 2 learners, who were divided into a treatment group and a control group. The use of manipulatives in this study confirmed the hypothesis that the scores in the post test of the subjects in the treatment group was considerably higher than the scores of the subjects in the control group. The study confirmed that manipulatives do provide benefits to learners. Learners who used manipulatives performed better than those who did not.

In addition, research carried out by Hidayah, Dwijanto & Istiandaru (2018) aimed to find out the effectiveness and the practicality of the use of manipulatives, which were integrated along with written, and oral questions in solid geometry learning. The findings of their research verified that the use of manipulatives offered opportunities to learners to be attentive to and observant of the teacher's questions and statements. By using manipulatives learners were aided to think and to easily recall the concepts. They discovered that learners were excited and glad to join in the learning process, which made use of solid figure manipulatives, worksheets and statements and questions.

Furthermore Marasigan *et al* (2019) claims that his research revealed that the use of manipulative materials in mathematics classrooms provides learners with a thorough understanding of Mathematics by providing opportunities to learners to discover and apply concepts presented in the classroom. He further adds that the use of manipulatives in the mathematics classroom is far more effective when compared to the plain lecture method, which makes use of instructional materials that can be described as being traditional. Complex concepts are easily comprehended and learners learn comfortably while enjoying and experiencing fun while doing activities via mathematics manipulatives. Teachers need to incorporate mathematics manipulatives as a new strategy in teaching mathematics. Colgan (2021) posits that research on manipulatives and learning is unequivocal as concrete objects assist learners to visualize mathematics. Abstract concepts can be difficult and challenging for young learners however manipulatives make them relatable, as children are better able to unpack and make sense of the concept with their sense of touch and sight.

Although various researchers in the studies discussed above, emphasize the importance of the use of manipulatives. Holmes (2013) argues that teachers still continue to use traditional methods in their mathematics classes. This is due to their biases concerning manipulatives. Some of those prejudicesinclude they do not know how to use the manipulative, manipulatives are not economical in terms of money and time, duration of a class is insufficient and manipulatives cause cognitive confusion. All these prejudices lead to a decrease in the use of manipulatives by teachers in the higher levels of education where learners acquired thinking skills that are abstract as opposed to younger learners in earlier grades where manipulatives are used more frequently by teachers. Marshall and Swan (2005) add that because of the existence of teacher's lack of information about the usage and management of manipulatives, a decline in usage in post-primary education has resulted.

Hollard (2020) states that research has revealed that manipulatives are useful tools in the mathematics classroom, enabling learners to better understand abstract mathematical concepts. Yet the use of manipulatives varies between teachers, schools and stages of learning, with most use taking place in primary schools. Research indicates that several teachers are not equipped to effectively use manipulatives. They are unsure how to facilitate the creation of connections, either using manipulatives to a minimum effect or totally dismissing their use. He conducted research to determine perceptions of the role of manipulatives in secondary mathematics education and to what extent secondary teachers are able to use manipulatives. His findings confirm that teachers generally have a positive belief on the usefulness of manipulatives in secondary schools, however their teaching actions are not in keeping with their beliefs. There exists a contradiction. The data obtained in this study further exposes teachers lack in confidence and knowledge and how to best use manipulatives. Some feel forced to use manipulatives but lack sufficient skills. The studies of O'Meara, Johnson and Leavy (2020) and Tunc (2020) concur with the findings of Hollard.

Within the literature there certainly exists a dichotomy of viewpoints about the benefits of making use of manipulatives to improve mathematics learning. Some researchers are of the opinion that manipulatives have little to no effect on improving mathematics learning among learners while other researchers strongly believe that this is untrue. On the other hand there are researchers claiming that there is no positive correlation between mathematics achievement and the use of manipulatives because of the existence of prejudices of teachers against the use of manipulatives.
The studies have thus brought us to inconsistent conclusions on the effects of using manipulatives in mathematics classes, particularly for learners at higher levels of education who have started to develop abstract thinking skills. This situation brings us to the conclusion that the effects of manipulatives on the achievement and attitude of learners still need to be clearly clarified. The study by Maningo, Almerino & Garciano (2021) is one such study where the results of their study traverses from other findings which claim that the inclusion of manipulatives during the delivery of instruction has not completely impacted concept building. From the inferences of the pretests and posttests, it implies that neither one nor the other outperforms the other. Another study by Hurst and Linsell (2020) reveals results that indicated the participants where uncomfortable and unfamiliar with the manipulative (bundling sticks). Their study provided little indication of learners having learned multiplicative concepts through the use of the manipulatives in their teaching instruction. They add that that there are no guarantees when using manipulatives as they do not magically lead learners to mathematical learning. The use of manipulatives calls for precise and clear direction from the teacher.

It is clear from the studies discussed above that more findings reveal that the use of manipulatives and cooperative learning strategies helped in bringing about improvements in increasing understanding of abstract and difficult mathematical concepts.

2.17 Its Effects on Learners Making Sense of Abstract Concepts and Attaining a Better Understanding of Mathematical Concepts

Manipulatives aid movement from concrete to abstract. Several studieshave been conducted and agree with my findings. Cockett (2015: 48) states that manipulatives can be an important tool to assist learners to think and reason in a more meaningful manner. Manipulatives help students develop conceptual understanding of mathematical ideas by representing ideas in several ways Shaw (2002: 1). Similarly, Marasigan *et al* (2019) adds that the use of manipulatives in teaching Mathematics permits learners to build their own cognitive models for abstract mathematical ideas and processes.

Shaw (2002: 2) further explains that when there is less confusion, deeper understanding can begin to hold, develop and grow, thus laying the groundwork for future mathematics learning. It is also important to identify that when there is less confusion or conflict of mathematical ideas in a learner’s mind, then there are automatically fewer meaningless rulesto commit to memory.

Uttal (2003:4) argues that when using manipulatives, children experience difficulty-linking representations based on manipulatives with written and symbolic representations. Uttal discovered that learners could either use manipulative form or written form but could not combine the both together to assimilate meaning. He discovered that learners failed to connect the two representations and only succeeded if manipulatives or written representations were used separately. McNeil and Jarvin (2009:1) suggest that this problem emerges when teachers fail to explicitly make the link to their Mathematics purpose in the activity. Boggan, Harper & Whitmire (2010: 4) argue that using manipulatives in Mathematics is beneficial to learning and they discovered that manipulatives assist learners to learn by allowing them to move from concrete experiences to abstract reasoning.

Cockette (2015: 49) explains that using mathematical manipulatives is an important tool for aiding learners to develop Mathematics ideas. The benefits include increased learner interaction and enjoyment, which results in increased efficiency and understanding. Manipulatives may include visual objects, which learners can touch and play with, or virtual manipulatives that learners can see with technology. Manipulatives cater for various learning styles and abilities in the mathematics classroom. Cockette believes that whilst some research suggest learners cannot make the link between manipulatives and written form, the dominant finding if her research reveals that manipulatives are beneficial for developing mathematical concepts.

In a study done by Enki (2014), the participants indicated that as opposed to previous traditional instruction methods, learning through activities that made use of manipulatives provided them with pleasure and increased their motivation and allowed them to have fun while learning. Research findings in the study by Lange (2021) revealed strong evidence that manipulatives are beneficial for the classroom when learners apply them in the standards they are being taught.

2.18 Advantages, Benefits, Drawbacks and Mistakes of Using Manipulatives

Many prominent mathematics education researchers (Ma, 1999: Ball, 1992; Baroody, 1989) have written about the use of manipulatives and have urged caution in the way in which they are regarded. Baroody (1989: 4) stated that manipulatives" must be used judiciously and cautiously for good results" and that there is no guarantee that students learning will follow from their use. He proposed two criteria for teachers to follow when planning to incorporate manipulatives in their lessons. Firstly, does the manipulative have meaning for learners as it connects with their existing knowledge and secondly, does it require learners to reflect and think? The use of manipulatives has several benefits and advantages in teaching abstract mathematical concepts. Several research studies have been carried out to support its advantages. In addition, some research findings indicate that there does exist drawbacks and disadvantages in utilizing manipulatives.

2.18.1 Advantages

There are several advantages and benefits of using manipulatives in the mathematics classrooms. Several studies and research have been carried out. Some of the findings indicated that manipulatives make learning meaningful, manipulatives aid engagement and understanding, manipulatives cater for individual needs and manipulatives aid movement from concrete to abstract The findings of various authors are discussed. This research supported the research that had been conducted.

Cockett (2015: 48) states that manipulatives facilitate the creation of a learning environment that encourage engagement and enable understanding. Shaw (2002: 3) claims that using mathematics manipulatives and models provides several benefits. Just as a picture can be worth a thousand words, manipulatives can provide visual representations of ideas, assisting learners to know and to understand mathematics. Manipulatives enhance the abilities of learners at all levels to reason and communicate. Working with manipulatives deepens understanding of concepts and relationships, makes skills practice meaningful and leads to retention and application of information in new problem-solving situations. In turn, the valuable time spent on manipulatives and model-based lessons has the sustained, long-term effect of building learner confidence and deepening mathematics understanding.

Hunt, Nipper & Nash (2011:1) add the following advantages in using manipulatives:

They teach underlying values and skills. Such as problem solving, patience and attention span, critical thinking, creativity and concentration. **Students feel more capable and competent.** As they do things on their own, and discover things on their own, they feel less dependent to their teachers. **Learners will see real life applications of concepts.** Rather than teaching them concepts, manipulatives allow them to literally grasp each situation and they will feel the relevance of the concepts. **Manipulatives keep the students occupied** and attract much attention. It is **easier for students to understand and reflect on the topic.** Since everything happened under their control, it is easier for them to analyze what they did and it allows them to "play around with the concepts".

Brijlall & Niranjan (2015) confirm that their findings are parallel to that of Hunt, Nipper and Nash. The use of manipulatives allows for learners to emerge as active participants, to relate to real world circumstances in mathematics, allows learners to engage in the discussion of mathematical ideas and concepts, aids in transforming abstract ideas to concrete, are useful tools to solve problems, provide opportunities for learners to apply their own methods in solving mathematical problems, creates opportunities to work co-operatively, intrigues and motivates learners to learn and allows for the learning of mathematics to be interesting, exciting and enjoyable,

Using manipulatives is simple and movable. The tactile experience adds a dimension of learning. Manipulatives allows students to be more creative selecting pieces. Students have more control. The process is traceable and allows for trial and error. Units are easier to distinguish, making it easier to see and easier to relate to real-world applications. The use of manipulatives is less expensive than technology. Manipullatives allow learners to be more cognitive of the operations being carriedout. Using manipulatives requires more thinking. The use of manipulatives does not make the learner feel rushed. Learners can be more creative. The use of maipulatives allows teacher to involve the whole class in an interactive lesson.

Manipulatives break the concepts down in a way that cannot be forgotten, makes the concept adhere better, allows information to be receivedvisually and kinesthetically, assists learners understand the concepts better and clarifies misconceptions, thus building connections between mathematicalconcepts and representations which results in more precise and richer understandings.

According to Heddens (1986: 14), utilizing manipulative material will assist learners in several areas of learning. Learners can learn to relate mathematics symbolism to real world and real-life situations, work together harmoniously and in a cooperative manner to solve problems, engage in discussion of mathematical concepts and ideas, verbalize their mathematical thinking, become confident and make presentations in front of a large group, see that mathematical problems can be symbolized in a variety of ways. There exists several ways to solve a problem and learners can also solve mathematical problems independently without merely following the educators' directions.

Burns,Joyce and Gollin (1996) provide many advantages which assist in making abstract ideas concrete. A picture may be worth a thousand words but while children learn to identify animals from picture books, they surely may still not possess a sense of the size of the animal, texture of skin or the sounds the animal would make. Videos also fall short here. It is clear that there is no substitute for first hand experience. Manipulatives offer learners ways to construct and formulate models of abstract mathematical ideas. Manipulatives develop learner's confidence by allowing them a way to test and confirm their reasoning. Learner's understanding becomes stronger if they have physical evidence of how their thinking operates. Manipulatives appear to lift Mathematics out of textbooks. Whilst learners need to be confident and proficient in the language of Mathematics, everything from a plus sign to algebraic notations, symbols and words only represent idea. Ideas exist in children's minds and manipulatives aid them to construct an understanding of ideas that can then link to mathematical vocabulary and symbols. Manipulatives can be described as being useful tools for solving problems. When searching for solutions, engineers build prototypes of equipment; architects build models of buildings and doctors use computers to predict the impact of medical procedures. Likewise, physical manipulatives act as concrete models for learners to utilize when solving problems. Learning mathematics with the use of manipulatives is interesting, exciting and pleasurable. If learners are given a choice of working on a page of problems or using colourful and interesting shaped blocks to solve problems, then there is no contest. Manipulatives motivate and intrigue while assisting students learn.

2.18.2 Link from Concrete To Abstract (CRA)

Witzel (2005) describes the concrete representational, abstract (CRA) instruction as a process for teaching and learning mathematical concepts. Beginning with the manipulation of concrete materials, the process then transfers learners to the representational level and then climaxes at the abstract level. CRA instruction makes provision for students to make associations from one stage of the process to the next. When learners are first permitted to develop a concrete understanding of the mathematical concept, they then become more likely to successfully carry out a mathematics skill and genuinely understand mathematical concepts at an abstract level.

Manipulatives are considered to be effective in fostering the development and enhancement of conceptual understanding in mathematics as they assist learners to link and relate concrete ideas to abstract ideas (Uribe-Florez & Wilkins, 2010; Witzel and Allsopp, 2007). Jones and Tiller (2017) claim that by making use of hands–on, concrete manipulatives during mathematics teaching time could result in learners having a higher retention rate and develop a more positive attitude towards their education. Smith (2009) further adds that a well designed manipulative bridges or closes the gap between formal mathematics and informal mathematics.

2.18.3 Manipulatives Make Learning Meaningful Through Interaction and Collaborative Learning

Cockette (2015: 48) states that manipulatives can be an important tool to assist learners think and reason in a more meaningful manner. Manipulatives are able to facilitate the creation of a learning environment that engagement and enables understanding. In a study conducted by Enki (2014) learners compared former traditional teaching methods and preferred learning through activities that involved manipulatives as they derived pleasure, their motivation increased and they had fun while learning. Stein and Bovalino (2001) concur with Cockette and Enki and conclude that by making manipulatives available to learners, teachers create a more meaningful experience for the learner by providing a concrete form for which learners are able to see the significance.

Carbonneau, Wong & Borysenko (2020) explain that the use of concrete manipulatives in classrooms provides opportunities for learners to interact physically with abstract content, which they normally would be unable to visualize or touch. Burns (2007) and Cocket & Kilgour (2015) add that through this process learners become more engaged with the learning material and are more successful in interpreting abstract mathematical concepts based on concrete manipulatives(Willingham, 2017; Sarama & Clements, 2009).

Collaborative learning can be described as any form of instructional activity that occurs when learners work together and interact closely in order to exchange ideas and thoughts to achieve a common goal (Laal & Ghodsi, 2012; Prince, 2004). Extensive research on collaborative learning has been researched and has been proven to be effective in increasing learner engagement and learning performance (Van Leeuwen & Janssen, 2019; Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt & Wenderoth, 2014). Loes, An, Sachaie & Pascarella (2017) adds that collaborative learning is particularly useful as it introduces learners to various perspectives, which potentially helps their understanding of the material they are being taught or exposed to. Carbonneau, Wong & Borysenko (2020: 2) explain learning with concrete manipulatives depends highly on students' interpretation of the manipulatives. Collaborative learning is one strategy that can be easily integrated within the classroom. Collaborative learning may be particularly beneficial in the context of learning with perceptually rich manipulatives. Learners who become overly fixated on surface properties of objects may benefit from engaging in discussions with peers who are more adept at recognizing the abstract representation of the manipulatives.

Silva, Costa & Martins (2020) in their findings state that students were highly motivated and displayed strong enthusiasm in the involvement of tasks, fondness for the discovery process and collaboration learning that resulted in the sharing of solutions obtained for challenges that were given to them. The above findings align with the findings of Choden & Chalermnirundorn (2021) who carried out studies on the use of manipulatives and cooperative learning. From their findings it was clear that students possessed a positive perception towards the use of manipulatives and cooperative learning. In addition, the students expressed their enjoyment of the lessons when they were taught using manipulatives and using cooperative learning styles. They claimed that the use of manipulatives kept them engaged and aided them in developing an interest in Mathematics. They also added that it was easier to solve mathematical problems with the aid of manipulatives and the help from fellow classmates through the use of cooperative learning. They also stated that they had a better understanding of concepts when they learned and taught each other in groups. Learners expressed that they felt comfortable and motivated learning from their peers than through the teacher. The study carried out by Charbonneau *et al.* (2020) showed that by applying the use of manipulatives and cooperative strategies, learners were aided to work well in groups, which improved problem solving mathematical skills and stimulated a positive attitude and outlook towards mathematics learning. The use of manipulatives also assisted in improving learner performances and developed analytical and critical skills.

2.18.4 Manipulatives Aid Engagement and Understanding

Cockett & Kilgour (2015) state that manipulatives facilitate the creation of a learning environment that fosters engagement and allows for understanding to occur. Shaw (2002:1) states that using mathematical models and manipulatives provides several benefits. Manipulatives can offer visual representations of an idea, helping students to know and to understand Mathematics. Manipulatives improve the learners' ability at all levels to communicate and reason. Interacting with manipulatives deepens understanding of concepts and relationships and allows for skills practice to become meaningful.

Similarly Pham (2015) adds that manipulatives promotes communication skills among learners which in turn increases self-confidence, which results in greater understanding and deeper learning. Xie, Antle and Motamedi (2008) in their study linked engagement and enjoyment of tangible objects in the learning process.

A study conducted by Swirling (2006) reveals that the use of concrete or virtual manipulatives could enhance learners' learning when working with complicated concepts. Flourence (2012: 5) maintains that mathematics manipulatives can assist learners to engage for a longer duration enabling them to stay focused on specific tasks. She further asserts that lecture-based teaching can frequently seem boring but the utilization of manipulatives permits learners to be actively involved in learning. Cockett (2015: 48) maintains that by using mathematical manipulatives, learners' confidence to complete difficult mathematics problems increases.

Marasigan *et al.* (2109) points out that one of the best ways in which mathematical ideas may be developed or applied is through activities that contains physical materials or manipulatives. A study done by Spear-Swerling (2006) revealed that the use of both concrete and virtual manipulatives had the possibility to improve a student's learning when complicated concepts were encountered. Shaw (2002: 3) claims that several children view Mathematics as being difficult and a struggle so they instantly surrender and give up. Shaw recommends that that the use of manipulatives has the ability to counter this. " When students physically move manipulatives to show various relationships, their sense of touch is actively engaged". This, aids understanding by engaging the kinesthetic side of the learner. As per the study conducted by Willingham (2017) & Vang (2017) the achievement of learners improved with the introduction of manipulatives and increased understanding and the development of a positive attitude towards mathematics.

2.18.5 Manipulatives Cater for Individual Needs

Learners learn through different learning styles. Sundstorm (2012: 4) claims that manipulatives can be a useful tool in catering for learners with different learning styles such as kinesthetic and visual learners. Kinesthetic learners learn best by interacting and physically touching objects and engaging with the concrete object. The mathematics manipulatives permit children to interact and touch the object so as to obtain a real representation of abstract mathematical concepts.

Mathematics manipulatives for visual learners can be posters or flash cards, which allows learners to assimilate a clearer understanding of mathematics problems (Sundstorm, 2012). Cockett & Kilgour (2015) state that the use of manipulatives caters for learners' individual needs particularly those learners who struggle with mathematical concepts. Research shows that the use of manipulatives is particularly beneficial to teach low-achievers, learners with disabilities and English language learners (Boggan, Harper & Whitmire, 2010).

2.18.6 Manipulatives Encourage Perseverance and Increased Confidence in Problem Solving

Perseverance is a key process through which Mathematics can be learned with understanding. However, withstanding such uncertainty can be difficult for learners to endure and therefore necessitates support. Teachers are therefore required to offer support and provide some form of scaffolding so as to ease the experience.

The idea of struggle has long been recognized as a key to learning mathematics with understanding (Festinger, 1957; Dewey, 1910). Researchers (Carbonneau, Marley & Selig, 2013; Saitta, Gittings & Geiger, 2011; McNeil, Uttal, Jarvin & Sternberg, 2009) state that learner's activities that integrate the use of manipulatives may improve learning as they provide more opportunities and exposure for learners to interact and engage with each other. There has been a profound impact on learners' learning achievement when there is an increase in learner engagement in STEM (Science, Technology, Engineering and Mathematics) learning (Freeman *et al,* 2014). Lee (2014) claims that behavioral engagement can be considered to be one component of student engagement that examines and explores the amount of effort and perseverance a student applies in learning. Farrington, Roderick, Allensworth, Nagaoka, Keyes, Johnson & Beechum, 2012) define academic perseverance as being the likelihood that students will finish a task to the best of their abilities in spite of obstacles and challenges they encounter on their path. Carbonneau *et al*. (2020:3) claims that since manipulatives are concrete objects designed to firstly represent abstract concepts that may otherwise be challenging to grasp and secondly, increase student engagement while learning, argues that learning using manipulatives can potentially influence students' perseverance. Belenky & Schalk (2014) agrees with Carbonneau *et al*. (2020) by stating that learning with manipulatives may aid in promoting academic perseverance by increasing learners' interest and engagement in their learning. Research has revealed that the use of well-designed external knowledge representations such as manipulatives may capture the student's interest. The use of manipulatives to trigger interest among learners as a topic is important as students' level of interest is predictive of their commitment, their engagement and their perseverance in learning (Hay, Callingham & Carmichael, 2015).

Sengupta- Irving & Agarwal (2017) claim that it is expected that students are to learn Mathematics in such a manner that when they are exposed to challenging problems, they will persist and not surrender. The creation of opportunities for students to persist in problem solving is therefore argued as being important to effective teaching and to learners acquiring positive dispositions in mathematics learning.

2.18.7 Improvement of Spatial skills and visualization

Lohman (1996) defines spatial skills as a person's skills to generate, retain, retrieve and transform well-structured visual images. Spatial skills encompass cognitive skills associated with spatial visualization, mental rotation and spatial orientation (Uttal & Cohen, 2012). Spatial visualization is the process of constructing, maintain and manipulating 2D and 3D objects in one's mind (Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013; Cracow & Sorby, 2008). Mental rotation is described as the rotation of mental representations of 2D or 3D objects to determine their images from various viewing angles (Ha & Fang, 2016). According to Lin, Chen, & Lou (2014) spatial orientation involves the change of location in space in relation to two-dimensional (2D) or three dimensional (3D) objects that a person can see.

2.19 Mistakes and Drawbacks

Manipulatives have to be carefully planned

Even though manipulatives have several benefits, challenges do exist when using them in classrooms. They are time consuming to set up, lack availability and many teachers lack pedagogical knowledge (Pham, 2015 & Moyer 2001). Hunt, Nipper & Nash (2011:1) provides the following disadvantage of using manipulatives: “There are a lot to consider when using manipulatives. Is the difficulty or complexity just right for the students? Will they get and understand the underlying concept? Is this too fun or too boring? The numbers on the manipulatives cant not actually be seen so one may miss the concept, requires internal affirmation rather than external, no feedback on whether one is right or wrong and is not very challenging.”

There exists a lack of use of manipulatives by educators. Research on the use of manipulatives have yielded several benefits, however, evidence has shown that teachers’ conception of using manipulatives is limited in their classroom instruction (Marzola, 2006; Furner & Worrell, 2017). Ndlovu & Chiromo (2019) claim that the reasons for the lack of effective use of manipulatives have not been thoroughly researched and that literature shows that the lack of teacher knowledge or expertise in a specific area is the main reason why topics are not taught effectively. Puchner *et al.* (2013) assert that teachers experience problems in using manipulatives when teaching. Ndlovu & Chiromo (2019) further suggest that since the use of manipulatives has been identified as potential recources that can assist learners understand abstract mathematics, it is therefore vital that teachers posses the required knowledge and expertise necessary to use by themselves to solvemathematical concepts which they would apply in their teaching.

Whilst the use of manipulatives can be considered to be an attractive teaching tool to mathematics teachers, they need to be cautioned that manipulatives are not a "fool proof" tool (Puchner *et al.,* 2013). During mathematics instructions many mistakes and misconceptions can happen. In order to prevent this from occurring, it is important to become aware of what mistakes can be produced.

Reviewed research reveals that a common mistake made is that manipulatives are not transparent. This means that the mathematical concepts that are being taught using manipulatives are not automatically grasped or understood by learners (Thompson, 1994; Ball, 1992; Cobb, Yackel & Wood, 1992).

It is claimed that since manipulatives are designed by those who are already familiar with the mathematical concepts which the manipulative is designed to teach and thatteachers are easily able to see the concept the manipulative was designed to teach, it is automatically assumed that learners will be able to see it as well. Teachers fail to understand that learners may see the existence of other concepts in the manipulatives (Puchner *et al.*, 2008; Moyer,2001; Ball, 1992;). Thompson (1994) concurs with Moyer, Ball and Puchner *etal.* by adding that because learners may 'view' other concepts in the manipulative, teachers need to be totally familiar with the manipulative that is being used in the classroom during the lesson. Teachers must be conscious of the different kinds of representations and acquire the ability to identify whenthe learners are using those instead of the intended one. Teachers frequentlyassume that learners are applying the intended representation and this results in a collapse in communication.

Kilpatrick, Swafford & Findel (2001) claim that some learners experience difficulty in making the connection between the physical manipulative and the mathematical concept being taught. This does not imply that the learners are unable to 'visualise' a different representation but that the learners are able to see the connection, resulting in just one additional thing to learn, instead of the manipulative being an aid in understanding mathematical concepts.

Suh and Moyer (2007) argue that the loss of a connection can be linked to a cognitive overload when interacting with symbols and manipulatives at the same time. This results in learners experiencing difficulty in keeping a record of everything at one time. Teachers who do not apply the use of manipulatives correctly can also be seen as another related factor. Heddens (1986) concurs with Suh and Moyer by stating that teachers may not be guiding learners to concepts and Ball (1992) adds that teachers themselves may not understand the use of the manipulatives.

Communication is another area of concern where teachers encounter difficulty when using manipulatives. This could result in learner misconceptions going unnoticed. Moyer (2001) and Heddens (1986) verify that communication is an area where teachers make mistakes when employing manipulatives. They concur that there is a need for learners to be given an opportunity to communicate and reflect on their interaction with the manipulatives. This process makes provision for the learners to formalize their understanding of the concepts learnt and to indicate to the teachers any misconceptions. Resnick (1983) claims that learners will attempt to make sense of what they are leaning even if they are not presented with all the information. This brings about misguided, incomplete and incorrect theories, which may not get addressed and corrected should there be a lack of communication.

2.20 A Critique of Use of Manipulatives

The use of manipulatives in combination with various other methods can improve and increase understanding of mathematical concepts. However, educators should not rely entirely on the use of manipulatives as they can become ineffective. Learners could lose the chance for deeper conceptual learning if manipulatives are used without formal discussion, abstraction and mathematical conceptualization.

Mustafa al-Absi & Nofal (2010: 51) claim that their study revealed that using manipulatives in mathematics teaching and learning affects students' achievements in Mathematics, and that there were significant differences between the achievement of learners who where taught by using manipulatives in comparison to those learners who where taught using traditional methods. They further add that when educators utilize manipulatives, they are able to assess learners' abilities by observing their performance, which is a good alternative assessment method, which enhances students learning and increases their achievement levels. Munday (2019) posits that manipulatives are useful in scaffolding students to build mastery of mathematics, which leads to a sense of achievement by learners. Manipulatives allows for the building of a strong foundation in Mathematics as they interact with the manipulative which, results in hands-on experience and are able to construct connections amongst the different aspects of mathematical skills, which in-turn builds confidence and perseverance to ensure challenging problems and tasks are executed.

While conducting this research, the researcher(an educator) could walk around the class and listen to the groups discussions. This enabled the researcher to better understand what the learners saw in the manipulative and the question. While making use of manipulatives the researcher could assess learners' abilities by listening to their discussions, observing them and analyzing their solutions and was able to see how they were linking the concrete to the abstract. In addition, the researcher was able to provide scaffolding and to pick up misconceptions, reminding learners that assumptions cannot be made in solutions.

2.21 Lesh's Translation Model

Lesh's model (1979: 34) depicts the importance of using multiple models for teaching and learning in mathematics. Figure 2.7 is a diagrammatic representation of the model and shows the possibilities for translation between real-life situations and geometric and trigonometric models. The various arrows indicate the variety of translations that are possible among the nodes that are identified as the real-life situations, pictures (geometric drawings, flow charts, diagrams, tables and graphs), verbal symbols (spoken language), written symbols (written explanations or mathematical symbols) and manipulatives (mathematical instruments for construction and measurement). The horizontal arrows within each node of the model show that flexibility and translation within each node are also important for developing mathematical understanding.

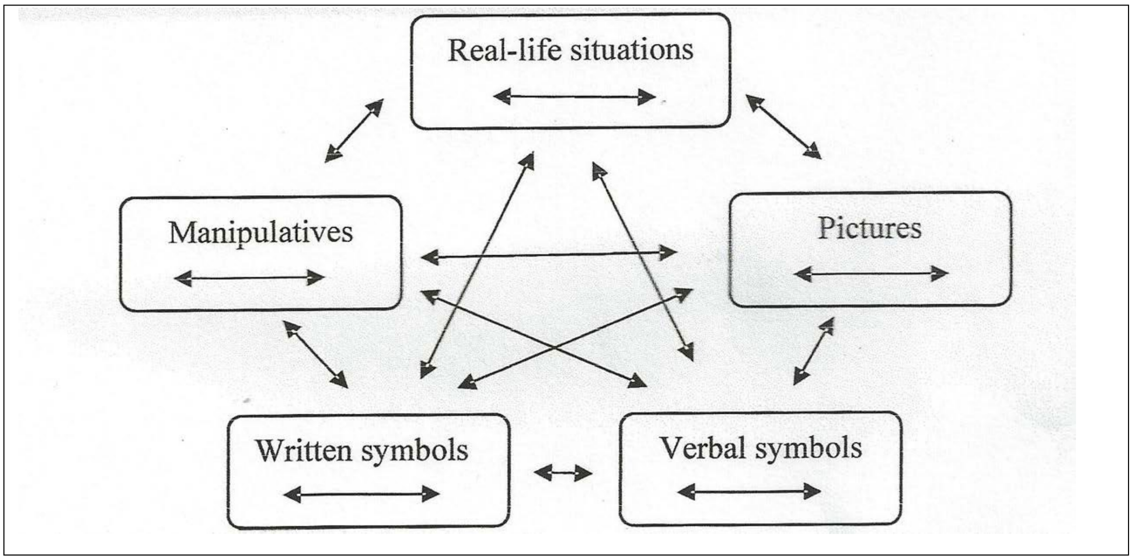


Figure 2.7: A diagrammatical representation of the Lesh (1979) model.
Source: Adapted from Lesh, Post and Behr (1979: 34)

In Mathematics the use of several different kinds of representations are valuable for concept development and problem solving abilities. It is vital to incorporate the use of manipulatives that can be translated into real life situations such as pictures in geometric diagrams, verbal symbols, written symbols like trigonometric ratios to develop concepts in trigonometry

(Lesh,1979). Brijlall & Niranjan (2015) confirms that manipulatives permit learners to relate real-world situations to mathematics symbolism.

The use of the Lesh Model applies to this study as the use of the manipulatives offers a link from real life situations for example the question 5 from the activity sheet discusses a hot - air balloon above a point and requires lengths and magnitudes of angles to be calculated which can be translated into pictures, verbal symbols and written symbols. Van Laren (2012, 210) further adds that geometric concepts are needed, as learners require the use of similar triangles to understand the basic trigonometric functions. Several models (construction, geometric and trigonometric) are encouraged when introducing trigonometry. These include the use of calculations and measurement of related values, which uses geometric shapes and is closely connected to Lesh's Translation model.

2.22 Conclusion

In this chapter the researcher discussed systemic problem areas impacting on achievement in mathematics, the state of current mathematics teaching and learning practices in South African schools, challenges experienced amongst Mathematics learners, the existence of content knowledge gaps and challenges teachers experience. Pedagogical content knowledge is addressed. Research on the factors associated to mathematics performance was explained.

Performance trends of Grade12 Mathematics, Paper 2 from 2016 to 2020 was discussed, an overview of learner performance in Paper 2 was presented and suggestions for improvement recommended. The descriptions of the different learning styles that exist and which apply to this study are discussed and the impact manipulatives have on learners' learning styles. In addition the researcher has presented an overview of literature on manipulatives, its impact on learning in the classrooms, the effect it has on learners making sense of abstract concepts and attaining a better understanding of mathematical concepts. Drawbacks and mistakes along with benefits and advantages were discussed in this chapter. Lesh's Translation Model was explained and how it fits in with this study. Chapter Three follows and focuses on the theoretical framework of this study.

CHAPTER THREE: THEORETICAL FRAMEWORK

3.1 Introduction

In Chapter Two literature relevant to the study was discussed relating it to the exploration of the use of three-dimensional manipulatives in grade 12 mathematics classrooms. In this chapter the theoretical framework within which this study was located is discussed.

3.2 The Theoretical Framework of this Study

The foundation of the theoretical framework, which grounded this study, was divided into the following major perspectives: Paiget's theory of learning, Vygotsky's sociocultural theory and Kolb's Experiential Learning Theory (ELT). In this chapter the framework used was developed to understand the ideas of Piaget and Vygotsky concerning constructivism. A description of the core of the theoretical framework used in this study, the Experiential Learning Theory is described along with an outline of previous research carried out using the ELT theory.

3.3 Constructivism as a Learning Theory

Constructivism as a paradigm or worldview suggests that learning is an active constructive process. The principle behind constructivist philosophy is such that knowledge is constructed rather that directly perceived through the senses. The learner is an information constructor and constantly builds or creates their own subjective representations of objective reality. The newly acquired information is linked to prior knowledge. Knowledge is constructed based on personal experiences and hypotheses of the environment. The researcher felt that the use of manipulatives in this research allowed for the exploration, discussion and use of the trigonometric ratios, sine rule, cosine rule and area rule to calculate unknowns and solve 3D trigonometry problems which contained horizontal, plane, vertical plane and slant plane. Ojose (2008: 26) adds that constructivism is a learning theory that is found in psychology, which explains how individuals might obtain knowledge and learn. Thus constructivism has a direct application on education. The theory suggests that humans construct knowledge and meaning from their personal experiences and environment. Constructivism is not merely a specific pedagogy. Piaget's theory of constructivist learning has had a major impact on learning theories and teaching methods in education throughout the world and is an underlying theme of several education reform movements. Research support for constructivist teaching techniques has been met with varied feelings, with some research supporting these techniques and other research not supporting this teaching technique.

3.4 Piaget's Theory of Learning

Constructivist learning theory was createdby cognitive scientists such as Jean Piaget and Vygotsky. Constructivist teaching is grounded on the idea that learning happens when learners are actively involved in the process of meaning and the building of knowledge as opposed to merely passively receiving information with little or no understanding. Through the interaction with the 3D mathematical models, learners accumulated physical experiences and were better able toconceptualize the abstract mathematical concepts. In addition learners were able to think logically and critically. These grade 12 learners developed skills to handle abstract problems with more confidence and zest. According to this theory learners are the builders of their own knowledge and its meaning and itapplies to this study.

The idea that young learners learn mathematical concepts better through concrete objects originates from the learning theory of Jean Piaget.
Piaget's theory of learning is based on his personal views of constructivism. Piaget believed that acquiring knowledge was a process of continuous self-construction. The next section explicates the developmental stages of the child as posited by Piaget.

3.5 Piagetian Theory- Stages of Cognitive Development

The concept that young children are better able to learn mathematical concepts through the interaction of concrete objects originates from the learning theory of Piaget. Jean Piaget was a Swiss psychologist who had commenced with his work on how children develop and learn before the Second World War. Unlike behaviorists, who developed theories using laboratory experiments and seldom viewed the real life behaviour of children, Piaget's theories were developed from the observation of children. Piaget (2001) assets that a major influence on children's cognitive development can be termed maturation which means the unfolding of biological changes that are genetically programmed into us at birth. The second factor is activity. Increasing maturation advances to an increase in children's ability to act on their environment and to learn from their actions. This learning has a ripple effect on an alteration of children's thought processes. The third factor in development is social transmission, which means learning from others.

Piaget (1970) studied various stages of the cognitive development of children from birth to maturity. According to Piaget, understanding occurs from actions carried out by an individual in response to the individual's environment. These actions change as time passes, from the physical actions to partially internalized actions that can be carried out with symbols. According to Piaget's theory, this can be described as a continuous process of accommodation to and assimilation of the individual environment. Cognitive development begins with the use of physical actions to form schemas, which are then followed by use of symbols.

Piaget emphasizes that learning involves both physical actions and symbols that represent previously performed actions. Learning environments therefore should include both concrete and symbolic models of the ideas that are to be learnt. Piaget's study (1972) was founded on careful and thorough detailed observation of children in natural settings and used repeated naturalistic observations. This careful examination of the functioning of intelligence in children led to Piaget discovering that at certain ages children have difficulty in understanding easy ideas. He then further investigated the thinking patterns of children from birth through to adulthood and learned that consistent systems existed within certain broad age ranges.

According To Piaget (2001) learning occurs in four stages of the cognitive development namely:

1. Sensorimotor stage (birth to age 2)
2. Pre-operational stage (ages 2 to 7)
3. Concrete operations stage (ages 7 to 11)
4. Formal operations stage (age 11 onwards)

Each major stage is a system of thinking that is qualitatively different from the preceding stage. A child must proceed through each stage in order and each stage must be mastered before proceeding to the next stage.

Sensory – motor stage (0-2 years). This stage is seen as being pre-symbolic and pre-verbal. Children obtain experience through their senses and the most important intellectual activity at this is stage is the interaction between the environment and child's senses. Activities are practical. Children become aware of surroundings. They are able to feel and see what is happening around them however; they are unable to sort their experience. At this stage children develop the concept of the permanence of objects and begin to develop basic relations between similar objects. The rich sensory environment permits for movement to the next stage.

Pre-operational Stage (2-7 years). The child utilizes language and symbols, which have letters and numbers. There is evidence of egocentrism. Conservation marks the end of the preoperational stage and the beginning of concrete operations. In this stage objects and events begin to take on symbolic meaning. There is rapid language development. The natural speech of children is dominated by monologues. This stage is mostly intuitive. Children at this stage enjoy imitating sounds and trying out different words and are not too worried with accuracy. Children show an increased ability to learn more complex concepts from experience if provided with familiar situations that have common properties that were discovered at the previous stage. There is an increase in the child's capacity to retain images. In this stage thought processes are based on perceptual cues and children are unconscious of contradictory statements.

Concrete operations stage: (7 to 11 years). At this stage the child demonstrates conservation, reversibility, serial ordering and a mature understanding of cause- and- effect relationships. Thinking is still concrete. Here the child starts to arrange data into logical relationships and begins to manipulate data in problem-solving scenarios. This learning situation will only happen provided concrete objects are presented. The child at this stage has the ability to make judgments in terms of reciprocal and reversible relations. At this stage the child has developed a logical and systematic way of thinking, which is still attached to physical reality. Overcoming this is the task of the next phase.

Formal operations stage: At this stage the child develops formal patterns of thinking and is capable of developing logical, rational and abstract strategies. The child can understand symbolic meanings and similes. Cognitive growth improves when the symbolic process becomes more active. The child (adult/learner) is now able to formulate hypotheses and deduce possible results from them, form theories and arrive at conclusions in the absence of a direct experience in the subject. Learning now is reliant upon the individual's intellectual potential and environment experiences.

The studies of Bruner support Piaget's findings. Bruner (1996) explains the three ways of knowing, namely enactive, iconic and symbolic. He states that a developing human being acts towards its environment through direct actions, imagery and language. A child starts to play with objects by touching, smelling and tasting them, which results in experiencing characteristics the objects possess. Later, mental images are developed by the child and names are attached to objects. Bruner (1996) states that after children learn to distinguish objects by shape colour and size, they start mastering the concept of numbers. Later on, in school, children learn new mathematical concepts and need to proceed in the same sequence from concrete objects to pictorial and then to abstract symbols.

Schunk (2012: 239) states that teachers will benefit when they understand at what levels their students are functioning. All students in a class should not be expected to operate at the same level. Teachers can try to ascertain levels and gear their teaching accordingly. Schunk (2012: 239) further asserts that students need to be kept active and that Piaget decried passive learning. Children therefore require rich environments that allow for active exploration and hands-on activities to take place. The use of manipulatives in this study allowed for learners to proceed from concrete objects to pictorial and then to abstract symbols and application of trigonometric ratios, the sine rule, the cosine rule and the area rule.

3.6 The Principles of Constructivist Theory and How It Applies to this Study

Constructivist teaching surely fosters critical thinking and produces learners that are motivated, independent thinkers and are self - driven to persevere. In this study, the use of the mathematical models showed that learners improved in becoming individual and independent thinkers and were motivated to solve the complex abstract problems. Through the use of group work, they offered support to each other to persevere and arrive at solutions.

Muijs and Reynolds (2005) claim that since learners are active knowledge constructors there exist several consequences. Learning is considered as being an active process where the learner builds and constructs her learning from various inputs. This translates to effectively learning when the learner is active. Learning is not merely arriving at a correct answer but about helping learners construct their own meaning. Learning to a constructivist is a search for meaning. It is important that educators construct learning activities. Learning does not occur alone but is socially constructed through the interactions with educators, peers and parents. Group work and discussion is therefore encouraged to build social learning situations. Learning is contextualized. Facts are not learnt in an abstract way but in relation to what we know. Real in-depth learning occurs by constructing knowledge, by exploration and revisiting material rather than rushing from topic to topic. Learners are only able to construct meaning if they are able to view the whole and not just parts of the whole.
Teaching is based on empowering learners and allowing them to reflect and discover realistic experiences. Von Glasersfeld (1989) concurs that this will lead to authentic learning and deeper understanding instead of surface memorization that often associates with other teaching approaches. This ultimately leads constructivists into thinking that hands- on and real materials are far superior to the use of textbooks.

In this study, information processing constructivism was used. Learners interacted with the artefacts designed by the researcher. These artefacts were designed considering familiar situations in learners' surroundings. These included a birthday card placed on a table, Pyramid at Giza in Egypt, goal post in Moses Mabhida soccer stadium, a triangular prism and a hot air balloon above the ground. The social interaction, discussion and group activity allowed for the mathematical concept to be changed into reality. This process allowed for the construction of more meaningful understanding of the trigonometry concepts. Research conducted by Kontas (2016:17) reveals the idea that the use of manipulatives is beneficial for concretizing abstract topics, an increased usage of manipulatives in teaching – learning processes can be suggested. Mathematical concepts are understood better when manipulatives are used to help learners incorporate their knowledge and link them with their thoughts.

3.7 Vygotsky 's Theory and the Zone of Proximal Development

The work of Vygotsky concentrated on the role of social and cultural influences of learning. He advocated that social interaction was crucial in the learning process. Arnold and Yeomans (2005) claim that Vygotsky viewed social interactions as being important to the learning process and that these interactions greatly influence cognitive development.

Muijs and Reynolds (2005) explain that Vygotsky did not agree on the role of maturation because he did not consider maturation as the sole factor that aided learners to achieve advanced thinking skills. Arnold and Yeomans (2005) assert that Vygotsky proposed his idea called the 'zone of proximal development'. He argued that children are inclined to new knowledge at any given time, irrespective of the development stage in which they are in . The child must achieve this new learning with the help of a teacher or a more capable peer to arrive at new levels of knowledge within a specific given domain. De Corte and Weinert (1996) describe the zone of proximal development as the distance or "opening that occurs between the learner's potential level of development and the actual level of development". When the learner is able to solve problems independently then, the actual level of development is established. When the learner can solve the same problems under adult supervision or with the help of more capable peers then, the potential level of development is determined. Ultimately all teachers would want their learners to function at this higher level.

3.8 Vygotsky's Sociocultural Theory

Vygotsky (1978) claims that social constructivism is knowledge construction that is a shared experience rather than merely an individual experience. Through the process of sharing individual perspectives, learners create understanding. Tudge & Scrimsher, (2003) state that like Piaget's theory, Vygotsky's is also a constructivist theory. However, Vygotsky's puts more emphasis on the social environment as a facilitator of development and learning. To Vygotsky social relationships form an important component to learning. He asserts, "all higher mental functions are internalized social relationships" (Vygotsky, 1981, cited in Wertsch and Stone, 1986:166). Vygotsky explains that the direct of learning transmits from the social to the individual. At first learning is social, which is dominant, and the individual follows later. Noddings (1990) states that constructivists maintain that learning is a process that requires interaction and a process that requires active participation of the learner. Brijlall, Maharaj & Jojo (2006) argue that learning without participation is a contradiction with the recommendations of constructivists such as Von Glaserfeld, who stresses that reflective ability is a major source of knowledge at all levels of mathematics and it is important that learners communicate their thoughts to the teacher and to each other. Wertsch & Stone (1986) further add that for learning to happen, the learner must reconstruct and convert external, social activity into internal individual activity through a process of internalisation. Such a creation of consciousness relies on social interaction.

Muijs and Reynolds (2001) explain that a process of 'scaffolding' occurs in the zone of proximal development. Here teachers, other learners and adults offer learners' thoughts with these scaffolds as a learner progresses fro a lower level to a higher level. There is no need for the 'scaffolds' once the learner attains the higher level. Learners are different and in the zone of proximal development, not all learn the same way. Some learn more whilst others require additional scaffolds than other learners.

Vygotsky's social-cultural theory is relevant to the present study as it clearly involves social interaction between learners and teacher and the exchange of thoughts. Zone of proximal development applies to the current study. The social interaction between learners and teacher allows for learners to interact with more capable peers and thus function at a higher level. Working with peers allowed for discussions to take place and this resulted in the provision of scaffolding to learners, which aided learners to apply the sine rule, cosine rule and area rule to solve complex 3D trigonometric problems which require multiple steps to arrive at answer. It is clear that both Piagetian and Vygotskian views of constructivism have major implications for learning and therefore inextricably influences teaching and pedagogy.

3.9 Experiential Learning Theory

This theory offers a holistic model of the learning process and can be described as a multi- linear model. This theory reiterates the central role that experience plays in the learning process. Kolb (1984: 38) states that learning is a process where knowledge is created through the transformation of experience. The theory offers a structuring and sequencing of the curriculum and shows how a course or topics may be taught to improve learning of the learner. It claims that the learning is cyclical and has four stages, referred to as feeling, watching/reflecting, thinking and doing (Fielding, 1994).

The theory possesses an important feature, this being that the different stages are related with distinct learning styles. Gibbs (1998) explains that learners prefer different learning styles and being able to see this is the first stage in making learners aware that alternate approaches are existent and helping them become more flexible in satisfying the varied demands of learning situations. Fielding(1994) argues that learning may suffer when there is a great disparity between the approach of the educator and the learner's style.

3.9.1 Kolb's Learning Cycle

Kolb's learning cycle is built on John Dewy's claim which states that learning must be a grounded experience, Kurt Lewis's ideas of the significance of active learning and Jean Piaget's emphasis on the interaction between a person and the environment on intelligence.

Kolb's Learning Cycle

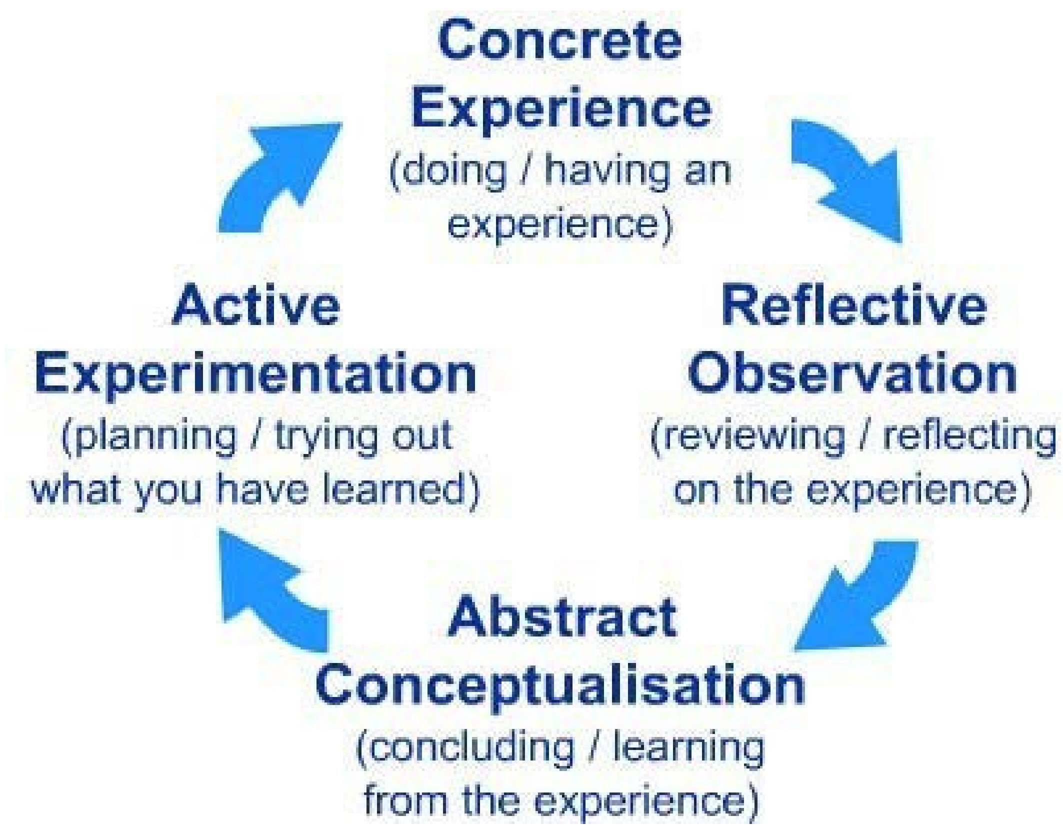


Figure 3.1: Kolb's' Learning Cycle
Source (Healy, 2000)

3.9.1.1 Concrete Experience Stage

The diagram represents the four stages, which Kolb states must be completed in order for learning to occur. The cycle begins with the concrete experience. This stage starts with doing something in which the individual has a task to do. Aiming to learn requires active involvement. In Kolb's model one cannot learn by merely reading or watching. For effective learning to take place, the individual must actively do and get involved. In this study learners are exposed to a concrete experience, this being the interaction with the manipulative, namely a 3D mathematics model.

3.9.1.2 Reflective Observation Stage

Reflective observation stage is the second stage of the cycle, which involves reflective observation. This denotes taking time out from "doing and stepping back from the task in order to review what has been completed and experienced". Several questions are asked at this stage and other members or participants of a team or group communicate with each other. Important discussion takes place, which requires rich vocabulary to verbalize with others in the team or group.

In this study the learners engage with the manipulative and discuss the different planes in the manipulatives and analyze the given information. They further discuss which formulae will be selected based on the data given. For several learners this is where the metamorphosis from seeing and doing to reflecting can embed the learning into real-time absorption of materials and mathematics. It could possibly be where learners are shown how to fulfill goals and how to apply them in various circumstances.

3.9.1.3 Abstract Conceptualization Stage

This stage of the cycle is the process of making sense of what has occurred and involves interpreting events and making sense of relationships that exist. The learner makes comparisons between what they have completed and what they know. Learners may draw upon theory from textbooks, models, and ideas from others, previous observations or other knowledge they have acquired. In this study learners form new ideas or revise current abstract ideas based on the reflections that emerge from the reflective observation stage. The learners now have the opportunity to see how the ideas and concepts learned previously can be utilized in the real world.

3.9.1.4 Active Experimentation Stage

This is the last stage of the learning cycle and where learners think about how they are going to apply their newly acquired knowledge. Planning allows taking the new understanding and translates it into predictions or if refined or revised actions should be carried out. If the learning is to be useful, the learner needs to place it in a relevant context else the learning is likely to be forgotten. This is where the learner applies new ideas to the surrounding to determine if any modifications are required in the next appearance of the experience. In this study learners apply their knowledge to determine if there are alternate solutions or if there is a shorter and quicker way of solving the problem.

3.10 Applying the theory of Kolb learning in a mathematical context

Healey (2000: 5) maintains that as educators when we hear the central ideas of Kolb’s theory, it appears to have an intuitive appeal for it connects to, even legitimatise, what we already do as educators. As researchers we see that it in part parallels the (scientific) research method of observation, hypothesis building, theory and testing. Gibbs (1988) expertly connects Kolb’s cycle to educational practice by linking teaching methods to four common experiential methods found in the sequence of Kolb’s model. These include planning for experience, increasing awareness, reviewing and reflecting on experience and offering substitute experiences. Gibb (1998: 9) strongly contends the pedagogic implication of Kolb's theory:

"*It is not enough just to do, and neither is it enough just to think. Nor is it enough simply to do and think. Learning from experience must involve linking the doing and the thinking."*

Much research has been carried out and the learning model most applicable to learning mathematics is Kolb's model of experiential learning which has been utilized extensively to enhance and evaluate teaching (Jensen and Wood, 2000; Evans, Forney,& Guido- DiBrito,1998; Pavan; Stice, 1987).

In Kolb's model, the learners learning style is established by two factors: does a learner favour the concrete to abstract or the active experimentation to reflective observation. These preferences form a classification scheme consisting of four learning styles(Hartman, 1995) .

Concrete, reflective- learners who construct on previous experiences.

Concrete, active- learners who learn by trial and error.

Abstract, reflective- learners who learn from detailed explanations.

Abstract, active- Learners who learn by developing individual strategies.

Kolb's learning styles can be understood as mathematical learning styles. Having several years of experience, observation, and learner interaction Knisley(2001) explains the four learning styles in a mathematical context:

- Allegorizers- learners believe new ideas to be reformulations of known ideas. Problems are solved by using known techniques.

- Integrators- learners strongly depend on comparisons of new ideas to known ideas. They rely on common sense when working with problems. They compare the problem to problems they can solve.

- Analyzers- learners long for logical explanations and algorithms. Problems are solved logically in a step – by – step process.

- Synthesizers- learners view concepts as tools for constructing new ideas and approaches. Problems are solved by developing new allegories and individual strategies.

The table in 3.2 shows the correspondence between Kolb's Learning Styles in a mathematical context

KOLB'S LEARNING STYLES	EQUIVALENT MATHEMATICAL STYLE
Concrete, Reflective	Allegorizer
Concrete, Active	Integrator
Abstract, Reflective	Analyzer
Abstract, Active	Synthesizer

Table 3.2 Kolb's Learning Styles in a mathematical context (Adapted from Knisley,2001)

Healey(2000: 16) points out that Kolb's theory contains several strengths. The theory gives ready pointers to application; directs educators to ensure that a variety of teaching methods are used; provides a theoretical rationale for what many teachers already do and then informs teachers on how to improve on teaching practice that ensures links between theory and application; makes explicit the importance of encouraging learners to reflect on their work and ideas and gives immediate feedback to reinforce their learning; can be applied to different subjects/disciplines; can be used by groups or individuals; supports educators in developing a diverse aware classroom and makes educators conscious of how the different learning styles have to be combined for effective learning to take place.

The flow chart figure 3.3 depicts the process that ensued. Learners interacted and engaged with the manipulative. They observed the scenario and could better visualize the horizontal plane, verticalplane and slant plane. Thereafter, they discussed and exchanged ideas. Learners then planned and solved the problem and became aware thatalternate solutions existed.

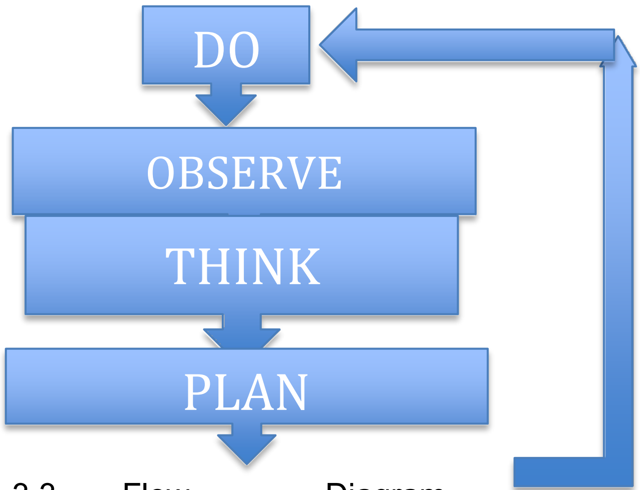


Figure 3.3 Flow Diagram :Source: Adapted from Jenkins (1998: 43)

3.11 Conclusion

In this chapter the theories of Piaget and Vygotsky and Kolb's Experiential Learning Theory (ELT) have underpinned this research. In this study, information processing constructivism was used. Constructivism as a paradigm suggests that learning is an active process. Knowledge is constructed and not directly perceived through the senses. In this study it is acknowledged that using manipulatives in the research, allowed for learners to construct new knowledge to previous knowledge and for social interaction and exchange of ideas to take place through group discussions.

Piaget believed that obtaining knowledge was a process of ongoing self-construction. He advocated that there exists four stages in the cognitive development process, namely: sensorimotor stage, pre-operational stage, concrete operations stage and formal operations stage. The knowledge of Piaget's stages assist educators to better understand the cognitive development of the learner as the educator plans stage appropriate activities to ensure that learners are kept active.

Vygotsky focused on the role of social and cultural influences of learning. He strongly believed that crucial to the learning process, was social interaction. Learners create understanding through the process of sharing individual perspectives.

The experiential learning theory emphasized the central role that experience plays in the learning process. Kolb's learning cycle is made up of four stages consisting of concrete experience stage, reflective observation stage, abstract conceptualization stage and the active experimentation stage. In the next chapter the methodology used in this study, the participants information and the limitations of the study are discussed.

CHAPTER FOUR: RESEARCH METHODOLOGY AND PROCEDURE

4.1 Introduction

Chapter three provided an overview of the theoretical framework utilized in this study. This chapter describes the methodology and procedures of this research design to investigate the effect of the use of mathematical manipulatives on grade 12 learners in developing problem solving skills in solving three dimensional Trigonometry problems in the Further Education andTraining phase (FET) 3D trigonometric problem solving.

The interpretive paradigm was identified for the framework of this study. In addition, this chapter discusses the research methodologies, the design applied in this study, instruments, data collection and analysis methods.

The research design for this study was a constructive and interpretive case study that was analysed via qualitative methods. Participant observation, semi-structured interviews, video recordings, activity sheets and a pilot study were used as data collection methods. Furthermore, justification for the use ofthe data collection methods used in this study was discussed. Finally validity, reliability and ethical issues applied in this study are discussed.

4.2 Research problem

Grade 12 learners continue to perform poorly in solving three-dimensional trigonometric problems. This study provides an alternative method to teaching three-dimensional trigonometry by developing and utilising mathematical 3D models to enhance learner knowledge in grade 12, 3D trigonometry. The study addresses the National Senior Certificate Diagnostic reports (2016, 2017 & 2018) findings which have called for better techniques to assist learners learn mathematical concepts.

4.3 Critical Research questions

Leedy and Ormrod (2010) argue that to answer research questions, one cannot skim the surface. Researches must probe deeply to get a complete understanding of the phenomenon being studied. In qualitative research, one digs deeply. In qualitative research numerous forms of data are collected and examined from different angles to construct a rich and meaningful picture of a complex and multifaceted situation.

This study made use of artefacts (manipualtives) to determine if grade 12 mathematics learners developed problem solving skills in the application of 3-dimensional trigonometric problems.

The research question addressed by this study is:

How do Mathematical models help learners to adopt more active approaches towards learning of three-dimensional trigonometric problemsamongst Grade 12 learners?

Aim

The aim of this study was to find out whether the use of three-dimensional trigonometric models would develop skills in trigonometry problem sloving among grade 12 learners.

The objectives of this study were to:

1. evaluate the learners' knowledge on application of the area rule, sine rule and cosine rule together with the use of three-dimensional models to solve three-dimensional problems;
2. establish the impact of concept images in assisting students make the necessary mental construction that will lead to conceptual understanding of trigonometric concepts;
3. identify the difficulties that learners encountered in learning the concepts, which were the barrier in making the required mental constructions.

In order to achieve these objectives, the following was done:

- a pilot study was conducted where the three dimensional models were constructed based on activities contained in the activity sheet. This was analysed for any shortcomings, which was addressed for the main study;
- learners responses on the activity sheets were analysed;
- learners were observed and video record to evaluate their interaction with the models and understanding of the three dimensional trigonometric problems;
- a semi structured interview questionnaire with the learners were carried out to establish if they were able to identify the different triangles and to place sides and angles in perspective.

4.4 Qualitative Research Methods

Kamal (2019, 1386) states that the development of qualitative study stems from a few noteworthy contributors namely; Barney Glaser and Anselm Strauss who published *The Discovery of Grounded Theory: Strategies for Qualitative Research* in 1967 and Egon Guba who published *Toward a Methodology of Naturalistic Inquiry* in 1978. The book by Glaser and Strauss (1967) focused on building the theory (inductive method) on a social phenomenon under investigation, rather than testing the theory (deductive theory). The manuscript written by Guba (1978) highlighted what happens and what is discovered in a real world context without controlling and manipulating what is being explored. Subsequent to these publications Merriam & Tisdell (2016) state that qualitative research or qualitative inquiry has been vastly used in the anthropology and sociology fields including education.

According to Merriam & Tisdell (2016) the qualitative study approach is made up of many important characteristics. Firstly, the aim of the study is to comprehend the experiences that people encounter. Secondly, the researcher is considered to be the main instrument for data collection, data analysis and thirdly, data from the study is analyzed inductively, meaning that the researcher will produce explanations in the form of concepts, hypotheses and theories from the data collected. Rich data is finally generated from several sources of data such as filed notes, interviews and documents.

Creswell (2014: 32) claims that qualitative research can be interpreted as “an approach for exploring and understanding the meaning individuals or groups ascribe to a social or human problem’. Merriam and Tisdell (2016) suggest that the aims of qualitative research are to show the meaning of an occurrence for people who are involved in the research process.
Tracy (2020) describes qualitative research as putting oneself in a scene and attempting to make sense of the situation, whether in a community festival, company meeting or during an interview. Qualitative researchers intentionally scrutinize and make note of small cues so as to ascertain how to behave, make sense of content and expand on knowledge claims about the greater picture. The key to success is that qualitative researchers pay close attention.

Qualitative research method is applied to gather the in-depth details of a specific topic. This method or approach assumes a single person represents the group feelings and emotions of a person are equally important to interpret which are normally neglected by the quantitative method. Creswell (2003) states that qualitative research method is applied when a researcher plans observing or interpretingan environment with the aim of expounding a theory.

In this study the researcher was the main instrument in the data collection and data analysis process. The rich data that was collected provided a detailed and clear explanation on the learners’ understandings and conceptualization of the 3D problems. The study provided an understanding of the learners’ experiences with the mathematics models/manipulatives. In this study individuals made use of the manipulatives to deconstruct the vertical plane, horizontal plane and the slant plane. Learners began building mental images of the triangles. The action process of gathering concrete objects is interiorized and was represented by images and drawings as seen in the learners' written responses in Chapter Six.

Marguerite, Dean & Katherine (2006: 21) present the characteristics of qualitative research.

Characteristics of qualitative research.

	How this impacts on this study?
Hypotheses are formed after the researcher commences data collection and are changed throughout the duration of the study as new data are gathered and analysed.	The hypothesis was formed after the pre study was conducted.
Data collection techniques include observation and interviewing that allows the researcher to come into close contact with the participants.	The data collection methods employed in this study also included observation and semi structured interviews which generated rich data
There is a great likelihood that that the researcher takes an interactive role to get to know the participants.	The researcher was in contact with the participants. Thisallowed for the participants and researcher to interact closely.
Studies are conducted in a naturalistic setting.	The setting of this study can certainly be described as being naturalistic as participants were in their familiar surroundings and settings, this being their classroom.
Participants are chosen through non - random methods based on whether	The participants in this study were randomly chosen. The

individuals posses information that is important to the questions being asked.	criteria that the participants had to fulfill were that they had to be grade 12 mathematics learners.
Researchers probe by asking broad questions to interpret, explore or understand the social context.	***Some Standard Probes used during the interview process*** **<u>FOR CLARITY/SPECIFICITY</u> Can you be more specific? Can you tell me more about that? <u>FOR COMPLETENESS</u>:• Anything else? • Tell me more** **<u>OTHER PROBING TECHNIQUES:</u>** **Repeat the question ,Echo their response Pause a second ,Baiting** **<u>OTHER PROBING TECHNIQUES</u> :Which would be closer? Which answer comes closest to how you feel/ think? If you had to pick one answer, what would you choose? What do you think?**

Table 4.1 Marguerite, Dean & Katherine (2006: 21) present the characteristics of qualitative research

Nind & Todd, (2011), Willis, (2007), Thomas, (2003), McQueen, (2002) and Silverman, (2000) are researchers who powerfully believe that the interpretive/ constructivist paradigm predominately makes use of qualitative methods.Willis (2007) explains that interpretivists gravitate towards qualitative methods,which could be case studies and ethnography. Thomas (2003) adds that qualitative methods are often used by interpretivists as it would appear that the interpretive paradigm depicts a world where reality is socially constructive, is continually changing and is multifaceted. Thanh and Thanh (2015) claim that the characteristics of interpretivism, with regards to qualitative methods to approach reality are completely different to that of the positivist paradigm. Glesne and Peshkin (1992) argue that the positivist paradigm frequently makes use of quantitative methods as the world is viewed through measurable and observable facts.

McQueen (2002) elaborates the use of qualitative method in the interpretive paradigm by explaining that interpretivist researchers search for methods that aid them to deeply understand relationships of human beings to the environment the occupy and the part these people play in building and shaping the society they emerge from. Interpretivists steer away from methods that provide precise or objective information rather they opt to see the world through a series of "individual eyes" and choose participants who have their own meanings of reality to assemble the worldview. Willis (2007) further explains that frequently qualitative approaches produce reports that are rich and that are needed for interpretivists to obtain an absolute understanding of situations and environments.

Creswell (2009) asserts that qualitative research can be seen as a way of exploring, understanding and interpreting groups or individuals to a social or human challenge. Creswell (2009) goes on to add that in educational research, if aresearcher wants to understand the experiences of a group of learners then, he recommends that qualitative methods are favoured to be the most suitable methods. In the interpretive paradigm researchers seek insight and in-depth information. He argues that using quantitative research describes the world in numerical values and measures instead of using words is not likely to be constructive. He claims that statistics that are often used in qualitative methods is highly unlikely to generate 'depth' and 'insight'.

Consistent with the above Punch (2009) states that researchers frequently gather data through the process of deep attentiveness of empathetic understanding and this results in qualitative data being rich and in-depth.

Table 4.2 Important differences between quantitative and qualitative research

	Quantitative	Qualitative
Principal orientation to the role of theory in relation to research	Deductive; testing theory	Inductive; generation of theory
Epistemological orientation	Natural science model, in particular positivism	Interpretivism
Ontological orientation	Objectivism	Constructivism

Source: Adapted from Bryman and Bell (2011: 27).

By its nature qualitative research methodology allows for several research strategies to be applied to gather data. The voice of the participant is heard. Teherani, Martimianakis, Stenfors-Hayes, Wadhwa & Varpio (2015) describe qualitative research as the systematic inquiry into social phenomena in natural settings. These phenomena can include and is not restricted to: how individuals experience events in their lives, how groups or individuals behave, how organisations function and also how these interactions mold and form relationships. The researcher is the main data collection instrument in qualitative research. Strauss (2015) and Bogdan & Biklen (2006) add that the researcher investigates why events happen, what occurs and interprets the meaning of these events and the links to the participants.

Austin and Sutton (2005) claim that qualitative research assists researchers to access research participants' feelings and thoughts, which aid in the development of understanding that participants attach to their experiences. Leung (2015) reiterates that the essence of qualitative research is to make sense of and recognize patterns among words in order to construct a meaningful picture without compromising its richness and dimensionality.

Qualitative research works with numerical coded information, non-numerical information and phenomenological interpretation, which inextricably links, to the human sensesand subjectivity.
Qualitative research focuses on the events that emerge and on outcomes of those events from the participants involved in the research.

Fouche and Delport (2001) describe qualitative research as research that elicits participants' accounts of meaning, produces descriptive data, permits diversity in responses as well as the ability to incorporate new issues and developments of research. They further add that data collected can include field notes, interviews, group discussions, observation and reflection, various texts and pictures.
Leedy and Ormrod (2010) state that qualitative research has many approaches to carry out research and differ from each other however they have two aspects in common. Firstly, qualitative research focus on phenomena that happen in natural settings in the "real world" and secondly, qualitative research involves the study of phenomena with all its complexities.

The above characteristics of qualitative research were suitable for this study, to establish if the group of grade 12 mathematics learners developed problem solving skills to solve three dimensional trigonometry problems. In this study the researcher used qualitative research to understand the impact of the use of mathematical models in teaching 3 dimensional trigonometric problems to grade 12 learners and the learning experiences of the grade 12 mathematics learners. The qualitative research approach allowed the researcher to carry out research in the natural setting ofa classroom. Data collection methods included observation, semi structured interviews and group discussions whilst completing the activity sheets and interacting with the mathematics models/manipulatives.

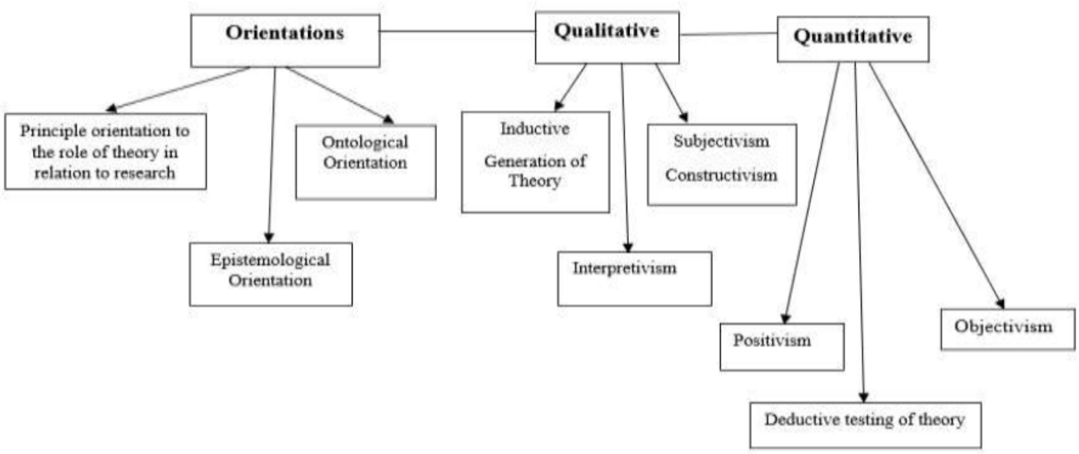


Figure 4.1 Differences between Qualitative and Quantitative Research Strategies (Mehrad & Tahriri, 2019)

Quantitative	**Qualitative**
Researcher defines the truth	Reality is defined by the contributors
Researcher self determining	Investigator as a communicating observer
Ideas reduced to statistics	Holistic viewpoint
Determination is hypothesis confirmation	Purpose is hypothesis generation
Deductive reasoning (general to specific)	Inductive reasoning
Statistical manipulation is required	Statistical testing is not compulsory
Research design is fixed	Research design is dynamic

Table 4.3 Assessment of Quantitative and Qualitative Research Methods (Mehrad & Tahriri, 2019: 4)

	Quantitative Research	Qualitative Research
Reality	exists independently from the researcher	is influenced by the researcher
Interaction with research object is	low, objective	high, subjective
Research questions aim on	quantitative answers numerical change relationships between variables testing hypotheses	in-depht information developing hypotheses and theories meaning of situations
Research design	descriptive or causal	exploratory, descriptive or causal
Data collected	numerical	non-numerical (opinions, feelings, words)
Data analysis	mathematically based (statistics)	Abstracting, comparing coding memoing content analysis grounded theory hermeneutics
Interpretation/ conclusions	deductive	inductive

Table 4.4 Characteristics of Qualitative and Quantitative Research Source: Adapted from Guba and Lincoln, 1994, Mayring, 2002, Punch, 2013

In this thesis the researcher adopted a qualitative research approach with the aim to determine the impact the use of Mathematical models had on grade 12 learners to help adopt more active approaches towards learning of three-dimensional trigonometric problems and to discover how the use of models improved the learning of mathematics. The researcher was the main instrument and collected and analysed data from observation, semi-structured interviews and documents (activity sheets) to gather information from grade 12 mathematics learners. These methods were chosen to ascertain the participants' attitudes, thoughts and feelings which assisted in providing a rich description of the study carried out. The findings of this study were analyzed inductively and many themes appeared and were discussed. This being a qualitative study did not produce numerical data such as frequency and percentage, which are often generated from questionnaires or surveys from a quantitative study approach.

4.5 The research Paradigm

The term paradigm is derived from Greek, which means pattern and has been greatly defined by several academics (Kivunja & Kuyini, 2017). Huges (2010) explains that a paradigm can be perceived as a way of viewing the world that frames the research topic and influences the way researchers think about research topics.

Research or enquiry is guided by a set of beliefs and this set of beliefs or worldview is known as a paradigm (Killan, 2013, Huges, 2010). She further adds that the term paradigm emanates from the Greek word paradeigma, which means "pattern". A paradigm is basically a way of thinking about or viewing the world. Paradigms are seen as frameworks which researchers use as a basis for everything else that is done. Killan (2013) describes theparadigm as being like a lens on a pair of glasses. She uses the analogy of looking through coloured glasses. If one puts on red glasses, everything looksred. If one wears pink glasses, everything appears pink. If one wears yellow glasses, everything around looks yellow. The lens (paradigm) that researchersselect, influences the way they view the world. The paradigm therefore directsand guides everything a researcher sees and does.

Creswell (2003) describes a "paradigm" as being an important collection of beliefs that are shared by scientists, a set of agreements on how problems should be understood, how the world is viewed by us and how we proceed in carrying out research. Guba and Lincoln (2005) add that paradigms are made up of a basic set of beliefs or assumptions that guide our inquiries when carrying out specific research. Meyers and Avison (2002) assert that when defining a valid research the most appropriate method is to follow or trail the research paradigm. Rahi (2017) further adds that a research paradigm is vital as it prevents the researcher from dwelling in his or her personal philosophy.

In another understanding, Mackenzi and Knipe (2006) categorize variable theoretical paradigms as positivists (post positivist), constructivist, interpretivist, emancipatory, transformative, interpretivist, critical, pragmatism and deconstructivist. Creswell (2003) explains philosophy in the postpositive paradigm is decided by the concept cause and effect whereas according to Cohen and Manion (1994) in contrast, the interpretivist researcher understands the world of human experiences. Various researchers such as Yanow & Schwartz-Shea (2011) and Creswell (2003), are in agreement with Cohen & Manion's view and further add that the interpretivist researcher discovers reality via their personal experiences, background and through the participant's view.

In essence, paradigms represent the researcher's beliefs and values about the world, the way they define the world and the way they work within the world. In relation to research, the researcher's thoughts and beliefs about any issues explored would subsequently guide their actions (Kamal, 2019: 1389). Kivunja & Kuyini (2017) asserts that the paradigm has important implications for every decision made in the research process.

This research study will be based on cognitive and social constructivism of knowledge and follow an interpretivist paradigm. Angen (2000: 384) states the following as characteristics of the interpretive paradigm:

a) Interpretive approaches rely strongly on naturalistic methods such as interviews, observations and analysis of existing texts.

b) These methods ensure an adequate dialogue between the researchers and those with whom they interact so as to collaboratively build meaningful reality.

c) Normally meanings emerge from the research process.

d) Qualitative methods are used.

This paradigm fits with this study as the data collection methods include observation, semi-structured interviews, video recordings, analysis of learner's written responses to activities and a pilot study. According to Cohen, Manion & Morrison (2011: 17), the interpretive approach depends heavily on interviewing, observation and analysis of existing texts.

4.5.1 The Interpretive Paradigm

Taylor & Medina (2013) describe the interpretive as a humanistic paradigm, which made its appearance in educational research in the 1970's and was strongly influenced by anthropology, which aims to understand other cultures, from the inside. This is to understand the culturally different 'other' by learning to 'stand in their shoes', 'look through their eyes' and 'feel their pain or pleasure'. They claim that the epistemology of the interpretive paradigm is inter-subjective knowledge construction. Interpretive knowledge of the other is obtained through an extended process of interaction undertaken by ethnographers who put themselves within the culture that they are studying. Utilizing ethnographic methods such as informal interviewing, participant observation and creating ethically sound relationships, the interpretive researchers are able to build trustworthy and authentic accounts of the cultural other. Used in educational research, this paradigm allows researchers to construct rich local understandings of the life-world experiences of students and teachers and of the cultures of classrooms, schools and communities they work in.

There are various quality standards that regulate interpretive knowledge construction but one of the most well known are those of Guba & Lincoln (1989) who established standards of trustworthiness and authenticity that are clearly different but 'parallel to ' the validity, objectivity and reliability standards of positivism. The criteria of the trustworthiness comprises of credibility (did the researcher undertake prolonged involvement in the field, check interpretation with the informants and show process of learning), dependability (did the researcher engage in open-ended or emergent inquiry?), transferability (is there enough rich description to the reader to compare her or her own social context with the social setting of the research and confirmability (can the data be traced to its source?).

Cresswell (2003) and Neuman (2000) describe the paradigm as having three major dimensions namely epistemology, ontology and methodology.

The constructivist paradigm is alternatively known as a naturalistic (Guba and Lincoln, 1989) and interpretive paradigm (Merriam & Tisdell, 2016; Guba & Lincoln, 1989). The ontology, epistemology and methodology are connected to the research paradigm. Lincoln & Guba (2013) explains that the methodological explanation is further compelled by the epistemological and ontological explanation of a research. Cohen, Manion and Morrison (2018) further explain that such an explanation suggests that different ontologies and epistemologies that a researcher adopts, needs different kinds of methodology.

A paradigm is essentially a way of viewing the world or thinking about a concept. Paradigms can be considered as being frameworks, which form the basis for everything else that researchers perform. The American philosopher and physicist, Thomas Kuhn is recognized with giving the term paradigm it's contemporary meaning and offering scientists a convenient model to apply to problems and obtain solution (Killam, 2013).

4.5.1.1 Axiology

Killam (2013) states that ***axiology*** addresses nature of ethical behaviour. The term originated from the Greek word *axios,* which means value. In philosophy axiology is a term that deals with ethics, religion and aesthetics. Axiology is what researchers believe to be valuable and ethical. In a research paradigm, beliefs of what is ethical can be obtained and this aids in guiding researchers' decisions.

4.5.1.2 Ontology

Killam (2013) states that ***ontology*** refers to the researcher's belief about the nature of reality. The term ontology in philosophy refers to the study of our existence and the important nature of reality or being of existence. Beliefs about what is real or true determine what can be known as real.

Ontology is concerned and interested with the nature of existence (Crotty, 1998), or reality (Hammersley, 1992) or social entities (Bryman, 2012). Cohen *et al*. (2000: 5) asks the following question for us to think about, "Is social reality external to individuals - imposing itself on their consciousness from without – or is it the product of individual consciousness?" Lincoln and Guba (2013: 39) state that ontology deals with the questions, "What is there that can be known?" or, " What is the nature of reality?" Antwi & Hamza (2015) offer another explanation of ontology and describe ontology as 'the way the investigator define the truth and reality". The real world setting was depicted in the study in the form of a real time class setting.

4.5.1.3 Epistemology

According to Killam (2013) ***epistemology*** explores the relationship between knowledge and the researcher during discovery. It therefore, refers to how we come to know what we know. In epistemology researchers consider "How is knowledge acquired? How do we know what we know?" Gray (2014) adds that epistemology deals with the sufficient and valid kinds of knowledge. Kivunja & Kuyini (2017: 27) state the questions related to epistemology are, " Is knowledge something which can be acquired on the one hand, or is it something which has to be personally experienced?" Guba (1990) further adds that epistemology as being the process by which the investigator comes to know the truth and reality or put in another way, how do we know what we know? Epistemology therefore examines the relationship between the enquirer and the enquired. I as the facilitator and researcher, designed and constructed the models to direct participants to construct their own knowledge.

4.5.1.4 Research methods

Killam (2013) adds that ***methodology*** is concerned with the way one goes about discovering knowledge in a systematic way. It is described as being more specific and practice based than epistemology. A methodology is steered by the researcher's ontological and epistemological beliefs. The methods for data collection used within different methodologies have changing degrees of objectivity. According to Antwi and Hamza (2015) methodology can be defined as the method used in conducting an investigation. Lincoln and Guba (2013) claim that an important question related to methodology is "How does one go about acquiring knowledge?" Methodology has been described as, "How should we study the world?" (Kawulich, 2012: 51). Kamal (2019) explains that the methodological aspect of a research must agree with the ontological and epistemological stances of the research. A systematic approach that followed logical order and sequence in the study was applied.

Table 4.5 adapted from Carolin, (2015) includes the positive, post-positive, critical theorist, scientific realist, constructivist and interpretivist paradigms and compares the ontology, epistemology, methodology and consequence

	Positivist	Post-positivist	Critical theorist	Scientific realist/	Constructivist	Interpretivist
Ontology	Real world exists	Real world exists, but cannot truly be perceived	Real world exists, but cannot truly be perceived	Real world is independent of human thought, but meaning or knowledge is always a human construction	Real world is independent of human thought, but meaning or knowledge is always a human construction	Real World can only ever be perceived and its working out(social reality) is a human construction
Epistemology	Inquirer separate from the phenomenon under consideration Objective	Inquirer strives to be as neutral as possible. Acknowledgement of position	Value inherent. Bounded by a particular ideology. Aimed at changing social structure. Social produced 'facts' are central	Inquirer strives to adopt a contemporary scientific perspective. Universalistic in scope but particular in interpretation.	Realities exist as multiple mental constructions. Socially and experientially based. Inquirer and the phenomenon under consideration interact to literally 'create' the findings.	Realities exist as multiple mental constructions. Inquirer is embedded and influences the construction of the shared reality. Context-dependent construction.
Methodology	Empirical Perceptual Precision, control and manipulation. Verification Single method Etic	Empirical Perceptual Precision, control and manipulation. Triangulation Seeks 'Understanding' Falsification and discovery Etic	To capture the social reality. Social and activist Transformative Single method Emic	Empirical Seeks "understanding" Multi-methodology	Case Ethno-methodology Multi-methodology Emic	"Lived experience" Ethno-methodology Multi-methodology Emic
	Experiment Observation Survey Deductive Inductive	Experiment Observation Survey Deductive Inductive Abductive	Observation Activism Historical Dialectic	Experiment, Observation Survey, Deductive, Inductive, Abductive Hermeneutic Dialectic	Hermeneutic Dialectic Ideographic	Observation Hermeneutic Dialectic
Consequences ,	Correspondence theory of truth Laws	Knowledge remains tentative Strives for balance	Change Ideologically based conclusions	Context and goal dependent Science is driven by questions not methods Methods must follow the questions	Empathetic Specific Rich Thick descriptions	Everything is interpreted Centrality of the symbolic and cultural Rich Empathetic

Table 4.5 Positive, post-positive, critical theorist, scientific realist, constructivist and interpretivist paradigms Adapted from Carolin, 2015.

4.6 Constructivism

Rahi (2017) claims that the supporter of the interpretive paradigm is intensely interested in the deep understanding of the world in which they live. They develop subjective meanings of the experiences or towards things and objectives. This paradigm is also called constructivism, Social Constructivism or Qualitative Research paradigm. The follower of the interpretive paradigm strongly believes that access to true knowledge can only be attained by deep interpretation of the subject.

Research Paradigm- Constructivism:

- Understanding
- Social and historical construction
- Multiple participant meanings
- Theory generation

Table 4.6 shows key functionalities of Constructivist paradigm as postulated by Creswell (2003).

4.7 The research Process

Figure 4.3 describes the research process. In this research under step one an interpretivist /constructivist paradigm suited this research. In step two the area of investigation was solving three-dimensional trigonometric problems at grade 12 level using mathematical models/artefacts. Step three included a case study approach. Step four included recent literature review to address the research problem. Step five included qualitative data that was collected in the research. Step six included the data collection tools, which in this study comprised of activity worksheets, interviews, questionnaires and observations. Step seven included source of data and these were grade 12 mathematics learners. Step eight included obtaining ethical clearance. Step nine comprised of data collection followed by step ten, which was data analysis and thematic analysis. Finally, step eleven concluded with the writing up of the findings and conclusions.

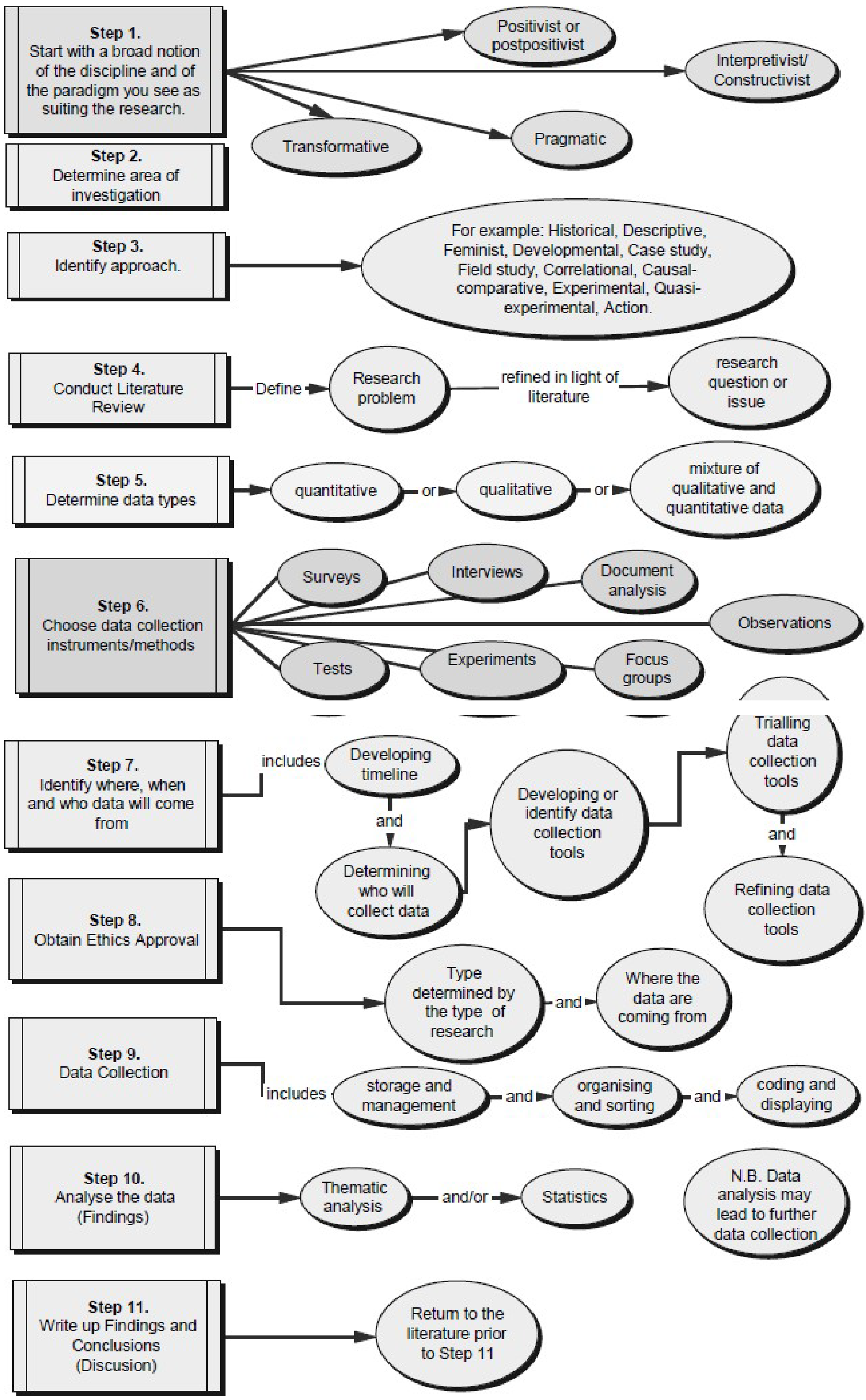


Figure 4.3 Research Process

4.8 Research Methods

A qualitative approach and a case study were employed in this research study

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4.8.1 Qualitative

The study followed a qualitative approach. The approach was considered to be a more appropriate method as it provided a greater understanding and justification of the use of models in teaching trigonometry. The models based on past year NSC examination questions were designed and constructed by the researcher and present to the learners.

Qualitative research was carried out closely with learners to document cognitive learning experiences and understanding of models and its use to solving three-dimensional trigonometric problems. Six models were used in this study. According to Leedy and Ormrod (2010: 137) and Welman, Kruger and Mitchell (2009: 193) qualitative field studies can be used successfully in the groups. Qualitative research methodology naturally allows the implementation of several research strategies to gather data. It is to ensure that the participants voice be heard. Fouche' & Delport (2002: 79) delineate the following qualities of this type of research as follows:
(1) Qualitative research elicits participants accounts of meaning, experience or perceptions about a concept, (2) it produces descriptive data, (3) qualitative approaches allow for more diversity in responses as well as the capacity to adapt to new development or issues, (4) in qualitative methods, forms of the data collected, include interviews and group discussions, observation and reflection field notes, various texts, pictures, and other materials.

The research Questions are:

- Q1 How will Mathematical models enhance learning of three-dimensional trigonometric problems amongst Grade 12 learners?
- Q2 How will the use of models improve the learning of Mathematics?

Data was collected to:

- Ascertain how the application of mathematical models improve learning in trigonometry amongst Grade 12 learners.
- Determine how the use of mathematical models will improve learning and understanding.

4.8.2 A case study design

The term "case study" has several meanings. It can be used to describe a unit analysis such as a case study of a particular organization or to describe a specific research method. A case study research could be positivist, interpretive or critical, depending upon the triggering philosophical assumptions (Maree, 2012). In this case study the research was considered to be interpretive as the approaches concur with that of Angen (2000) being that it relies heavily on naturalistic methods such as observations, interviews and analyzing of texts.

The Case study method was employed in this study. This method is suitable for individual studies as it allows in-depth study of a problem in minimum time (Koparan, 2017). According to Cepni (2008) the most important advantage of this study is that it allows for the focus on a specific case. This method allowed for an in-depth study into the reasons and understanding of the poor response in the trigonometry section on 3D problem solving in the NSC exams. This method focuses on an aspect of a case and makes it possible to juxtapose various data collection methods

Maree (2012: 75) states that in literature there exist several definitions and understandings of case study research. The case study research method has been applied by researchers for several years across a variety of disciplines to answer "how" and "why" questions. Case studies provide a multi-perspective analysis in which the researcher considers not just the voice and perspective of one or two participants in a situation but also the views of other relevant groups of actors and the interaction between them. It opens the possibility of giving a voice to the powerless and voiceless, such as children and marginalized groups. This is vital and necessary for researchers to arrive at a deeper understanding of the dynamics of the situation and this aspect is a salient feature of many case studies. The use of the mathematics models/manipulatives in solving 3D trigonometry problems allowed for a deeper understanding of the learners' thought processes. They were comfortable in their surroundings and participated in discussions. They were given a voice in this process.

De Vos, Strydom, Fouché & Delport (2005: 272) state that the case being studied may refer to a process, activity, event, programme or individual or multiple individuals. The exploration and description of the case takes place through detailed, in-depth data collection methods, involving multiple sources of information that are rich in context. These may include interviews, documents, observations or archival records. As such the researcher requires access to and the confidence of participants. This research is an in-depth description of a case. Maree (2012) claims that one of the advantages of using the case study method is the use of several sources and techniques that are used in the data collecting process. In advance the researcher is able to decide what evidence to collect and also the analysis techniques to be used with the data in order to answer the research question. Data is mainly qualitative but can also include quantitative data. In the case study research the tools used to collect data can include interviews, surveys, documentation review, observation and collection of physical artefacts (Yin, 1994). Several methods of data collection were used in this case study. They included observation, video recording, activity sheets and semi-structured interviews.

Maree (2012: 75) explains that from an interpretivist perspective, the typical characteristics of case studies is that they strive towards a comprehensive and holistic understanding of how participants relate and interact with each other in a specific situation and how they derive meaning of a phenomenon under study. Mark (1996: 219) describes three types of case study, all with different purposes namely the intrinsic case study, instrumental case study and the collective case study. The collective case study furthers the understanding of the researcher about a social issue or population being studied. The interest in the individual case is secondary to the researcher's interest in a group of cases. Cases are chosen so that comparisons can be made between cases and concepts and so that theories can be extended and validated. This research therefore followed a collective case study where the population being studied was grade 12 mathematics learners. The use of the mathematical models/ manipulatives aided the researcher in understanding the influence and the consequences the mathematical models/ manipulatives had on the understanding and application of the trigonometry concepts to solve 3D-trigonometric problems among the grade 12 mathematics learners.

According to Maree (2012) the unit of analysis is a critical factor in case study research. It is frequently focused on a system of action instead of a group of individuals or an individual. Case studies can also appear to be choosy, focusing on one or two issues that are important to the understanding of the system. Different contemporary reports in education (Stake, 1978), in psychology (Bromley, 1991) and sociology (Yin, 1994; Creswell, 1997) have examined the individual as the unit of analysis and have applied the case study method to generate rich and detailed understandings about people. This study generated rich and detailed data on the use of the mathematical models/ manipulatives and concurred with the findings of the various contemporary reports in education discussed above.

Case study design is criticized for its dependability on a single case andas a result claimed that case study research is incapable of offering a generalizing conclusion. Maree (2012) argues that this is not the purpose or intent of a case study design. The aim of a case study design is directed at obtaining greater insight and understanding of the dynamics of a specific situation. Hamel, Dufour & Fortin (1993) describe singularity as a concentration of the global in the local. The aim of this case study was not to provide a generalizing conclusion but rather to obtain a better understanding of whether the use of models in teaching 3 dimensional trigonometry problems improved the cognitive understanding in learners.

4.9 Sampling of the Study

Purposive sampling was employed in this study. Etikan, Musa & Alkassim (2016) state that the purposive sampling technique is also known as judgment sampling. Purpose sampling is the deliberate choice of participants based on the qualities the participant's poses. This is a nonrandom technique and does not need any specific theories or a specific number of participants. Bernard (2017) further adds that the researcher is the one who decides what is required and finds the individuals who can and are willing to participate in the research process by providing information in the form of experiences or knowledge.

Purposive sampling is frequently used in qualitative research to recognize and choose information rich cases to ensure that the available resources are effectively used (Patton, 2002). According to Bernard (2002) data gathering is critical in the research process, reason being that data is meant to add to a better and clearer understanding of the theoretical framework. Tongco (2007) concurs with Bernard by adding that the data gathering process then becomes important when selecting the way data is obtained and from whom the data will be retrieved from so as to make the best possible decision since no unlimited amount of analysis can compensate for incorrectly collected data. Maree (2007: 178) describes purposive sampling as being a method ofsampling that is applied in special situations where the sampling is done with a specific purpose in mind.

Grade 12 learners who were studying mathematics as one of their content subjects were used in this study. Fifteen grade 12 learners formed the data resources that were accessed, observed and interviewed at a selected public secondary school in the Pinetown District. The participants were not 18 years and below and therefore had no implications on ethical considerations.

This is a public school situated on the North coast in a suburb called Phoenix, Durban that is a suburb in Kwa-Zulu Natal, a province of South Africa in a continent of Africa. It comprises of over one thousand learners of mixed races, which is inclusive of African, Indian and Coloured. This school services a middle socio economic community. Learners attending this school come from both lower and middle socio economic classes.

4.10 Data Collection Instruments

Data collection instruments included observation, video recording, semi-structured interviews and learners' written responses to activities. A pilot study was conducted at the school. In this study triangulation of data ensured the validity of data. Observation, video recording, semi-structured interviews and learners' written responses to activities were anal to show triangulation ofdata. The sample of this study was therefore purposive and consisted of 15 grade 12 learners currently studying Mathematics. Data was collected via three lessons and four follow up lessons with fifteen grade 12 mathematics learners in lessons after school. These lessons where conducted with the researcher. During the three 55-minute lessons the learners were observed and a follow up lesson was done after every lesson. There was one final follow up lesson.

4.10.1 Observations

Observation is a way to collect primary data. Observation is systematic, purposeful and is a selective way of listening and watching to a phenomenon or an interaction as it unfolds (Kumar,2014).
Kumar (2014:173) claims that there are several situations that exist in which observation is the most appropriate method of data collected, for example when one wishes to learn about the interaction in a group. He further adds that

observation is appropriate in situations where accurate information is difficult to be extracted by the process of questioning, reason being participants are not co-operative or are unaware of the answers because it is difficult for them to detach themselves from the interaction. Kumar (2014) claims that observation is the best approach to gathering information when subjects become so involved in the interaction that they no longer are able to provide objective information.

Two types of observation exists, namely, participant observation and non-participant observation. In this study non-participant observation was carried out. Kumar (2014) describes non-participant observation as when the researcher does not participate in the activities of the group, instead remains a passive observer by watching and listening to the activities the participants are engaged in and drawing conclusions from this.

During classroom activities data was collected through observation notes made by the researcher. The learners' interactions with the models were observed and detailed notes werecompiled on learners' behaviour and actions. Marshall and Rossman (1995: 234) claim that observation is not merely looking but as looking systematically and noting systematically people, events, behaviour, settings, artefacts and routines. Observation studies are superior to experiments and surveys when data are being collected on non-verbal behaviour. Observation studies allowsfor the researcher to discern continuous behaviour as it occurs thus enabling the researcher to compose notes about important and relevant features(Bailey, 1994:112).

4.10.2 Written Responses- Activity Sheets

A lesson revising prerequisites to the new mathematical content was delivered. Trigonometric ratios, the sine rule, cosine rule, theorem of Pythagoras, area rule and algebraic manipulations of equations were revised. Learners were given an activity worksheet (annexure one) which comprised of six trigonometric questions based on three dimensional problems involving the sine rule, cosine rule and area rule that was sourced from previous years National Senior Certificate Examination papers. According to Brijlall & Niranjan (2015:376) findings in their case study verify that manipulatives played an important role are in the growth of both conceptual and procedural understanding of mathematical ideas.

4.10.3 Semi-structured Interviews

Pathak & Intratat (2012) state that the interview technique provides rich data for a qualitative study provided it is used carefully. They discovered in their study that interviewees experienced greater flexibility and freedom. In addition the claim that semi structured interviews offer a flexible technique for small-scale research. This study involved retrieving information from fifteen grade 12 mathematics learners. Semi structured interviews were conducted to establish what effect the use of models has on the learning of trigonometry 3 - D problems among grade 12 learners. The interviews conducted provided a clear picture of the students' thoughts and perceptions.

Within the social sciences according to Bradford & Cullen (2012), qualitative semi-structured interviews are considered to be one of the most domineering and extensively employed methods of gathering data. Flick(2006) adds that semi-structured interviews are considered valuable as it permits researchers to explore subjective viewpoints and to obtain in-depth accounts of individual experiences. Choak (2012) explains that interview schedules are used to assist researchers to address specific topics while making provisions for the respondent to answer in their own terms and to discuss issues and topics that are important to them. The schedule must allow for the interview to be guided and in addition allow other relevant themes to form throughout the interview process. In other words the interview should be a “flowing conversation” (Choak, 2012, Rubin & Rubin, 2005)). Evans (2018) claims that the popularity of semi-structured interviews within the social sciences partially reflects their independence from a single theoretical framework or epistemological position.

Qualitative semi-structured interviews can be unlimitedly used to consider experience, meanings and the “reality” of participant’s experiences as they can be utilized to explore how these experiences, “realities” and meaning could be informed by discourses, assumptions or ideas which exist it the greater society (Braun & Clarke, 2006). De Vos, Strydom & Delport (2005: 287) state that in qualitative research interviewing is the main mode of data collection. Maree (2007: 87) adds that the aim of qualitative interviews is to see the world through the participant’s eyes. Semi-structured interviews necessitates the participants to answer a predetermined set of questions which in addition allows for further enquiring and clarification of answers.

The interview schedule (annexure two) was designed with key questions. For this study the researcher-interviewed participants from the selected schools. Interviewing was chosen for the current study reasons being that it allowed for the generation of rich data; language used by the participants was used to obtain insight into their perceptions and values; contextual and relational features have been observed as important to understanding others perceptions.

4.10.4. Video recordings

Data was collected through observations and analysis of video recordings. While learners were engaged on their activity sheets they were recorded.
This was mostly performed so as to capture all the non-observant activities among students. Flick (2006: 233) explains that videos are a valuable tool in collecting data because they catch facts and processes that are too rapid or too intricate for the human eye. Video recordings assisted in analyzing the learners’ behaviour and actions as well as to determine the impact the models had on their thought processes and solutions to the 3D trigonometric problems.

The benefits and drawbacks of video method and traditional observation method.

	Pros	Cons
Traditional observational method	Enables rich data	Researcher may be intrusive
	Can capture events before and after the interactions	Aspects of interactions may be overlooked
	Makes provision for researcher to ask follow up questions during the observation	Does not permit for data validation through cross-coding
	The researcher has the ability to view all spaces in the room	Difficult to capture non verbal cues during the encounter
	Provides the opportunity to concentrate on one individual all the time	Unable to capture all interactions in a complex environment
Video method observation	Less intrusive method for data collection (avoid the observer effect)	The reviewing and coding of video data becomes labor intensive
	Offers sufficient detail to analyse the work environment and human interactions.	Participant's awareness of being recorded may influence behaviour.
	Permits researchers to analyse events retrospectively	Raises concerns about the confidentiality and discoverability of participants in the research.

Table 4.6 illustrates the pros and cons of traditional human observationmethod and video recording by cameras (Asan & Montague, 2014).

Pringle & Stewart- Evans (1990) claim that the video method limits the Hawthorne effect, which refers to the inclination of some participants to work harder and perform better when they are being observed as part of an experiment or research study, since video cameras have been shown to have less influence on the behaviour of the participant as compared to that of a human observer. Furthermore, some participants may be reluctant to be videotaped as opposed to live observation and they feel anxious as more risk is involved in video data due to various reasons which include: over a period of time several people may view the recordings; if the video data is incorrectly stored then unauthorized persons may obtain access; the participants identity can be easily determined from the video recording (Asan & Montague, 2014).

In this study some learners clearly displayed the Hawthorne effect as they wanted to give me correct answers and became overly excited and were not displaying their normal behaviour. The learners appeared to be providing the researcher with the desired outcome. In addition some learners became shy and blocked their faces, as they felt uncomfortable when being video taped. The majority of the learners preferred to see the researcher walking around and making observation notes as they found this less intrusive. This resulted in the researcher stopping the video recorder so as to resume to a normal setting in the classroom.

4.10.5 Pilot Study

A pilot study was carried out to have a trial run so that adjustments could be made to the study for the main survey to be tested. Bless and Higson-Smith (2000: 155) define a pilot study as being small study that is administered prior to a larger piece of research to determine whether the methodology, sampling, instruments and analysis are suitable and sufficient.

Figure 4.4 Steps of pilot study plan. (Adopted from Ismail, Kinchin, & Edwards, 2018)

The diagram shows the suggested steps in the pilot study, beginning with the determination of the pilot study plan including all aspects of the method and methodology plan.

4.11 Data Analysis Strategy/ Data Processing in Qualitative Studies

Maree(2012: 37) states that researchers in the positivist paradigm prefer deductive data analysis strategy whereas researchers in the interpretive (naturalistic) paradigm often prefer inductive data analysis, which is more likely to aid them in identifying the multiple realities potentially present in the data. Interpretivism is based on the assumption that there is not one reality but many and interpretivist researchers, therefore, conduct their studies in natural context to obtain the best possible understanding.

Kumar (2014: 317) explains that three ways exist in which one can write about one's findings in qualitative research. These are:

- Developing a narrative to describe a situation, episode, event or instance;
- Identifying the main themes that emerge from the filed notes or transcriptions of in-depth interviews and writing about them, quoting extensively verbatim; and
- In addition to the point above, also quantifying, by indicating their frequency of occurrence, the main themes in order to provide their prevalence.

4.12 Validity and Reliability: Criteria for good measurements in Research

Reliability and validity are considered to be the two most important features in the evaluation of any measurement instrument or tool for a good research. Validity is concerned with what an instrument measures and how effective or how well it performs. Reliability is concerned with the faith that one has in the data obtained from using an instrument (Mohajan, 2017). Altheide and Johnson (1994) add that reliability refers to the stability of findings and validity is represented by the truthfulness of findings. Singh (2104) explains that validity and reliability in qualitative research ensures increased transparency and decreases opportunities to add bias from the researcher.

Qualitative research is often criticized for lacking scientific rigour with poor justification of the methods used, lack of transparency in the analytical procedures and the findings just being a collection of personal opinions subject to research bias (Rolfe, 2006; Sandelowski, 1993). Tests and measures that are used to establish validity and reliability of quantitative cannot be used in qualitative research. There is continuous debate about whether terms such as reliability and validity and generalizability are appropriate to evaluate qualitative research (Rolfe, 2006; Long & Johnson, 2000). In a larger context, the term validity refers to the integrity and the application of the methods carried out and the precision in which the findings accurately reflect the data and reliability describes consistency within the analytical procedures used.

Brink (1993) adds that validity and reliability are key aspects to all research. Meticulous attention to these two aspects can ensure the difference between good research and poor research. This would assist and assure that fellow scientists accept and acknowledge findings as trustworthy and credible. In quantitative research this is important as a researcher's subjectivity can easily cloud the interpretation of the data. The scientific community often questions or looks with skepticism at the research findings in qualitative studies.

4.13 Reliability and Validity in qualitative research

In qualitative research, reliability refers to exact replicability of the processes and the results. In qualitative research with diverse paradigms, such definition of reliability is challenging and epistemologically counter-intuitive (Leung, 2015). Cohen, Manion and Morrison (2011:199) state that reliability is essentially a synonym for dependability, consistency and replicability. Blumberg, Cooper & Schindler (2005) describes reliability as a measurement that provides consistent results with equal values. Chakrabartty (2013) adds that reliability measures consistency, repeatability, precision and trustworthiness of a research. It is the degree to which an assessment tool generates stable and consistent results that are error free (Mohajan, 2017). Twycross and Shields (2004) claim that in qualitative research, reliability is referred to when a researcher's approach is consistent across different projects and different researchers.

Leung (2015) states that validity in qualitative research means "appropriateness" of the tools, data and processes. Whether the research question is valid for the desired outcome, the choice of methodology is appropriate for answering the research question, the design is valid for the methodology, the sampling and data analysis is appropriate and finally the results and conclusions are valid for the sample and context. Leedy & Ormrod

(2010) claim that the validity of a measurement instrument can be described as the extent to which the instrument measures what it was intended to measure.

Kumar (2014) agrees that validity in a broader context refers to the ability of a research instrument to show that it is finding out what you designed it to and reliability refers to the consistency in its findings when used time after time. He further argues that in qualitative research, as answers to research questions are explored in several methods and procedures which are flexible and evolving, standardization of research tools and processes is made difficult. He questions how these concepts can be used in qualitative research when it does not make use of standardized and structured methods and procedures that form the bases of testing validity and reliability as defined in quantitative research. Guba and Lincoln (1994) have recommended a framework that comprises of four criteria as a part of the constructivist paradigm paralleling validity and reliability in quantitative research. Guba and Lincoln explain that trustworthiness in a qualitative study is decided by four indicators, which are closely connected to validity and reliability namely: ***credibility*** (paralleling internal validity), ***transferability*** (paralleling external validity), ***dependability*** (paralleling reliability) and ***confirmability*** (paralleling objectivity).

Traditional criteria for judging quantitative research	**Alternative criteria for judging qualitative research**
Internal Validity	Credibility
External Validity	Transferability
Reliability	Dependability
Objectivity	Comfirmability

Table 4.7 Four criteria provided by Guba and Lincoln with reference to validity and reliability (Adapted from Trochim and Donnelly, 2007)

In Table 4.7 Trochim and Donnelly (2007) compare the four criteria provided by Guba and Lincoln with reference to validity and reliability as contained and explained in quantitative research.

4.13.1 Credibility

Trochim and Donenelly (2007: 149) explain that credibility entails confirming that the results of qualitative research are credible or believable from the view of the participant in the research process. Since qualitative research studies explore people's perceptions, feelings, experiences and beliefs, it is thought that the respondents are the most suitable judge to determine whether or not research findings have been able to reflect their opinions and feelings accurately. Hence, credibility is synonymous with validity in quantitative research, and is judged by the extent of respondent concordance when you take your findings to those who participated in the research for confirmation, congruence, validation and approval. It would appear that the higher the agreement of the respondents with the findings, the higher the validity of the study.

4.13.2 Transferability

According to Trochim and Donenelly (2007: 149) tranferability refers to the degree to which the results of qualitative research can be generalized or transferred to other contexts or settings. It is difficult to establish transferability, the reason being that because of the approach one adopts in qualitative research, to some extent this can be achieved if one extensively and thoroughly describes the process one has adopted for others to follow and replicate.

4.13.3 Dependability

Dependability is concerned with whether the same result would be obtained if we observed the same thing twice. Qualitative research advocates flexibility and freedom and it may be challenging unless one keeps a detailed and extensive record of the processes for others to replicate to determine the level of dependability (Trochim and Donenelly, 2007: 149)

4.13.4 Confirmability

Trochim and Donenelly (2007: 149) explain that confirmability refers to the degree to which the results can be confirmed or corroborated by others. Comformability is similar to reliability in quantitative research. Conformability is attainable if researchers follow the process in an identical manner for results to be compared.

4.14 Ethical issues

The research proposal for this study was presented in October 2015 and approval for the study was granted by the Durban University of Technology Institutional Research Ethics Committee on 29 October 2021. The ethical component of the study was approved by the Institutional Research Ethics committee(IREC) with ethical clearance number 124/21

The Gatekeeper's letter was obtained form the KwaZulu Natal, Department of Basic Education. (Appendix I). Consent letters to parents and participants were given out. Participation was completely voluntary.

4.15 Conclusion

This detailed chapter began with a discussion of the research methodology employed. Methodological issues relevant to this study were considered. The critical research questions and research instruments were explained to be in keeping with the dictates of some experts in the field of educational research. The interpretive paradigm was explained in detail and proven to coinside with the theoretical framework adopted for this study. The data capture strategies were aligned to a qualitative approach used in this study. In addition reliability and validity were discussed. The next chapter discusses the pre study conducted.

CHAPTER FIVE:PILOT STUDY: TESTINGOF RESEARCH INSTRUMENTS

5.1 Introduction

This chapter provides details of a pilot study undertaken by the researcher in a school in the Pinetown District, in search of validation of the research instruments used in this research and thus placing it in context. The pilot study made use of mathematical models/manipulatives to teach grade 12 learners to solve three-dimensional trigonometric problems. In this chapter, discussions and results on the pilot study are reported.

In the pilot study qualitative methods were employed and data collected through the use of mathematical models, activity worksheets, semi structured interview schedules, observation and video/audio recordings that were administered to grade 12 mathematics learners (n=9). The semi-structured interview schedule was designed to give an insight into the learners' experiences, knowledge of working with mathematical models and their cognitive processes when solving the three-dimensional trigonometric problems and to obtain insight into their knowledge and application of the area rule, sine rule, cosine, trigonometric ratios and the Theorem of Pythagoras. Modifications to the interview schedule are discussed and findings and recommendations of the pilot study are presented.

5.2 The Concept "Pilot Study"

In (2017) claims that a pilot study asks whether something can be done, can researchers go ahead with it and how can they proceed with it. For this study the pilot study is referred as a pilot study as the focus was on testing the conformity of the research instruments. In (2017) further claims that the pilot study has a specific design feature and is conducted on a smaller scale than that of the main or full-scale study. The pilot study is important for the improvement and efficiency of the main study (Fraser, Fahlman, Arscott & Guillot, 2018;In, 2017). Arnold, Burns, Adhikari, Kho, Meade and Cook, (2009) and Thabane, Ma, Chu, Cheng, Ismaila, Rios, Robson, Thabane, Giangregorio & Goldsmith (2010) concur with In and Fraser *et al*. and state that a pilot study can be considered to be the first step of the whole research protocol and is often a smaller sized study assisting in planning and modification of the main study to be conducted. Before the commencement of the pilot study, researchers need to completely understand not only the clear purpose and question of the study but also the experimental methods and schedule. Researchers thus become conscious of the procedures involved in the main study through the pilot study, which helps in choosing the best research method most suitable for answering the research question in the main study.

A pilot study study was carried out so that adjustments could be made to the study for the main survey to be tested. Bless and Higson-Smith (2000: 155) define a pilot study as being a small study that is administered prior to a larger piece of research to determine whether the methodology, sampling, instruments and analysis are suitable and sufficient. Ismail, Kinchin & Edwards (2018) concur with Bless and Higson and add that a pilot study is a small -scale research project conducted before the final full-scale study. The pilot study assists researchers to test in reality the likelihood of the research process to work in order to decide how best to carry out the final main study. In the process of piloting a study, a researcher is able to identify or refine a research question and to decide on the best methods to use in the main study, to estimate time required and to determine resources that would be required to complete the main study.

Figure 5.1 Steps of Pilot Study Plan.
Source Ismail, Kinchin & Edwards (2018)

Figure 5.1 illustrates the steps to be followed in a pilot study plan. The pilot study plan begins with the applied method and methodology, followed by assessment tools and the feasibility. Thereafter, adjustments to the tools may be made if required. At the revisiting stage there is another opportunity to examine the study before commencing with the final data collecting process. Finally, a reflection on the entire pilot study is completed to allow for it to be used in future research to prevent a repeat of similar mistakes or to refine similar pilot studies. The above steps in the pilot study plan (figure 5.1) were followed in this pilot study. In this study models were designed based on three-dimensional questions examined in past year NSC exam papers. A sample for the pre –study was chosen randomly from grade 12 mathematics classes. Data collection tools were activity worksheets, semi-structured interviews and observation and video/voice recordings. Once data was collected an assessment was performed to evaluate feasibility of the pilot study. Thereafter, adjustments were made to the data collection tools. The interview schedule had to be revisited and changes were made. The model for the activity sheet, question four had to be restructuredand designed such that it did not look like a ninety-degree angle. The model had to be redesigned because in the pilot study a learner indicated that the angle looked like a ninety-degree angle. Finally, a reflection of the entire study was completed to preventa repeat of mistakes.

5.3 Sample Size and Selection of Participants for the Pre- Study

In general, sample size calculations may not be required for some pilot studies. It is important that the sample for a pilot be representative of the target study population. It should be based on the same inclusion/exclusion criteria as the main study. As a rule of thumb, a pilot study should be large enough to provide useful information about the aspects that are being assessed for feasibility (Thabane *et al*,. 2010: 5).

De Vos, Strydom, Fouche and Delport (2005: 212) state that a sample size is drawn according to a certain sampling frame. The pilot study allows the researcher the opportunity for testing the effectiveness of the frame.

In (2017: 604) states that the primary purpose of pilot studies is not hypothesis testing and, therefore, sample size is often not calculated. An appropriate sample size needs to be determined, not for providing appropriate power for hypothesis testing but to understand the feasibility of participant recruitment or study design. An important point is that a sample in the pilot study needs to be identical to that of the main study, thus the inclusion and exclusion criteria should the same (Thabane, Ma, Chu, Cheng, Ismaila & Rios, 2010). A randomly selected sample of grade 12 learners was chosen to participate in the pilot study. These learners had to be studying mathematics as a subject in grade 12. The pilot study was made up of three groups comprising of three learners each. The pilot study was carried out with a different group of learners from the same school, who were not part of the main study.

The main study also comprised of grade 12 mathematics learners.

5.4 Aspects of a Pilot Study

Janghorban, Roudsari & Taghipour (2013:1) claims that the general application of pilot studies can be summarized in four areas:

- 1) finding problems and barriers related to participants' recruitment;
- 2) being engaged in research as a qualitative researcher;
- 3) assessing the acceptability of observation or interview protocol;

- 4) determining epistemology and methodology of research.

A pilot study was carried out to have a trial run so that adjustments could be made to the study for the main survey to be tested. Bless and Higson-Smith (2000: 155) define a pilot study as being small study that is administered prior to a larger piece of research to determine whether the methodology, sampling, instruments and analysis are suitable and sufficient. Additionally Nunes, Martins, Zhou,Alajamy & Al-Mamari (2010) claim that pilot studies offer unique opportunities to improve skills of the qualitative researcher in carrying out semi-structured interview including dealing with participants, selecting an appropriate venue to conduct interviews, conducting the in-depth and detailed interview and finally grabbing opportunities for probing emerging topics in the interview process. Six models/manipulatives (artifacts) were constructed based on the questions drawn from past National Senior Certificate (NSC) papers. The pilot study was carried out with nine grade 12 learners which formed three groups comprising of three grade 12 Mathematics learners each. The researcher had to revise the area rule, sine rule, cosine rule, basic trigonometric ratios and the Theorem of Pythagoras. The six problems contained in the activity sheet were answered in three lessons and learners simultaneously completed the interview schedule. The nine learners were interviewed separately. The fourth lesson was a follow up lesson.

Figure 5.2 The researcher explains to the subjects the procedure to be followed.

A pilot study acts as a tool of contextual information management, including testing and developing the adequacy of the data collection and the processes of analysis. In qualitative research it appears that designing and conducting a pilot study with clear objectives enhances the rigour and validity. In addition, it offers a broader view of structure of meaning of the phenomenon for a researcher who is a novice so as to avoid experiencing unmanageable problems in gathering data and carrying out the data analysis and interpretation process (Janghorban, Roudsari & Taghipour, 2013). In this study the pilot study allowed for the instruments to be tested to determine if the instruments needed refinement.

5.4.1 Observation

During classroom activities data was collected through observations. The learners' interactions with the models were observed and detailed notes were made on learner's behaviour and actions. Marshall and Rossman (1995: 234) claim that observation is not merely looking but as looking systematically and noting systematically people, events, behaviour, settings, artefacts and routines. Bailey (1994:112) adds that observation studies are superior to experiments and surveys when data are being collected on non-verbal behaviour. Observation studies allow for the researcher to discern continuous behaviour as it occurs thus enabling the researcher to compose notes about important and relevant features. In this study the researcher was able document detailed observations of the learners interactions with the models and with each other in the groups, conversions and discussions that learners had when discussing the possible solution to the three-dimensional trigonometry problem.

5.4.2 Written Responses

The activity worksheet comprised of past year question adapted from the National Senior Certificate Examinations. Trigonometric activity worksheets consisting of six trigonometric questions based on three-dimensional problems and involving the use of sine rule, cosine rule, area rule and trigonometric ratios was given to the learners. The written responses of the learners were collected and analysed. The detailed analysis of the questions is discussed later.

Brijlall & Niranjan (2015) state that manipulatives played an important role are in the growth of both conceptual and procedural understanding of mathematical ideas.

5.4.3 Video recording

Data was also collected through observations and analysis of video recordings. Recording of learners took place as they engaged in their activities. This is mostly performed so as to capture all the non-observant activities among students. Flick (2006: 233) explains that videos are a valuabletool in collecting data because they catch facts and processes that are too rapid or too intricate for the human eye. Also, when participants interact with the models, they were video recorded. This assisted in analysing their behaviour and actions and the effect the model had on their thought processesand solution to the 3D trigonometric problems.

5.4.4 Semi-Structured Interviews

Semi-structured interviews were conducted to establish what effect the use of models had on the learning of trigonometry 3 D problems among grade 12 learners. The interviews conducted provided a clear picture of the students' thoughts and perceptions. De Vos, Strydom & Delport (2005: 287) state that in qualitative research interviewing is the main mode of data collection. Maree (2007: 87) adds that the aim of qualitative interviews is to see the world through the participant’s eyes. Semi-structured interviews necessitates the participants to answer predetermined set of questions which in addition allows for further enquiring and clarification of answers. For this study participants were interviewed from the school.

Interviewing was chosen for the current study for the following reasons:

- It allowed for the generation of rich data;
- Language used by participants was used in gaining insight into their perceptions and values;
- Contextual and relational features was observed as important to understanding others ‘perceptions.

5.5 Value of a Pilot Study

De Vos, Strydom, Fouche and Delport (2005: 210) explain that since the purpose of the pilot study is to improve the success and effectiveness of the investigation, space must be allocated on the questionnaire, during the interviews or whatever data collection methods are used, to allow for criticisms or comments by the respondents. The researcher must then carefully consider these comments during the main investigation.
De Vos, Strydom, Fouche and Delport (2005) add that the main value of the pilot study is that provision for modifications to the questionnaire and that measuring instruments can be made after the pilot study and before the commencement of the main study or investigation.

5.5.1 Suitability of the Interview Schedule or Questionnaire

This aspect according to Moser and Kalton (1973) can be considered to be the most beneficial function of the pilot study. The pilot study allows for the opportunity to test the interview schedule before it can be applied to the main investigation. Babbie (2001) claims that even though the data collection instrument may be designed well, there is always the possibility of error and the way to be guarded against such error is by pretesting the instrument.

The pilot study allowed for the questionnaire to be fine-tuned. Questions were reworded and new questions where added. Questions one to five on the questionnaire were added so as to elicit more information on the use of artifacts and models to learn trigonometry. It each comprised of four options namely A-Never, B - Sometimes, C – Usually and D-Always. These questionswere asked to gather information about what they felt about the use of manipulatives. The questionnaire had been refined and changed from twenty questions to thirty-seven questions. Testing on a small sample permits the evaluator to identify any challenges and difficulties that may occur with the procedure or materials and in addition to establish accuracy, appropriatenessand reliability of the methods or instruments to be applied in the study(Neuman, 2000).

5.5.2 Testing and adapting the measuring instrument

De Vos, Strydom, Fouche and Delport (2005) claim that if specific instruments such as own scales, assessment scales and standardized scales have been thoroughly tested during the pilot study, then there should be no problems encountered during the main study. In this pilot study the 3D models/manipulatives, interview schedules, activity – sheets video recordings and observations were tested. Modifications to the interview schedules were effected. This was discussed in 5.5.1. above. During the pilot study one of the learners assumed that the model looked as though there was a right angle. Slight modifications were applied on the specific model. The activity sheets remained unchanged.

5.6 Implementing Educational Applications with the tangible mathematical models: The pilot study in this Research Project

Six mathematical models were designed by the researcher to aid learners solve three-dimensional trigonometric problems. The pilot study was organized in three fifty-five minute sessions, followed by a follow up session. The first lesson was an introduction to the mathematical models and a review of prerequisites which included area rule, cosine rule, sine rule, Theorem of

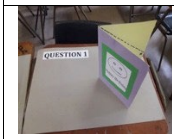
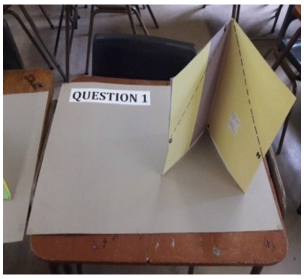
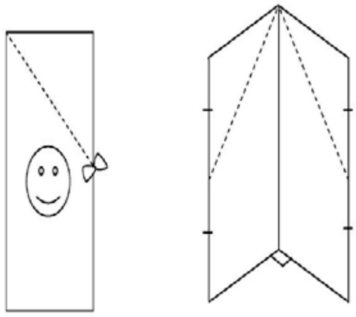
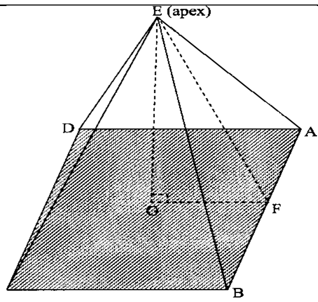
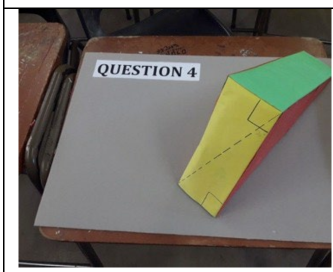
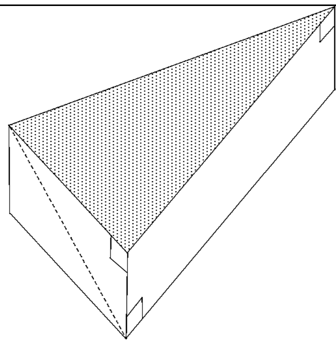
Pythagoras and basic trigonometric rules. The activities were aimed to solve three-dimensional trigonometric problems, with the main objective of developing spatial vision and taking advantage of the real life three-dimensional models. In contrast to the traditional methodology the mathematical learners were put in groups of three and allowed to interact with the tangible models and with each other. In total there were 9 learners who participated in the pilot study. There were three groups of three learners each. For each group the students were chosen randomly. The sessions were conducted on learners that were in grade 12 and who were studying Mathematics as a subject. During a session a group was given time tocomplete two questions from the activity sheet. In addition the semi-structuredinterview schedule/questionnaire was being completed. The models were rotated with other groups once they had completed their question. Learners were observed. They no longer sat passively and listened to the teacher. Theywere now active participants engaging with the tangible models and constructing their own knowledge. In order to evaluate the study, the followingwere used:

- A semi-structured questionnaire/interview schedule
- Activity worksheet
- Observation
- Video recording

Table 5.1 Result of survey about use of Mathematical models (9 students)

Kind of question	**Question about the use of 3D Mathematical Models**	**No**	**Yes**	**Unanswered**
Previous knowledge	Have you ever used artifacts/models before to learn trigonometry?	5 (56%)	3 (33%)	1 (11%)
Easy to use and Interactivity	Was it easy to solve the problem when using models/artifacts?	2 (22%)	2 (22%)	5 (56%)
Useful to learn Mathematics and Trigonometry	Did the models/artifacts help you learn during the lesson?	1 (11%)	3 (33%)	5 (56%)

From table 5.1 it is clear that 56% of the learners had not interaction with mathematical models. Both 22% said it was easy to solve problems using models and 22% said the opposite. Surprisingly 56% could not decide if using a model made problem solving easier or not. Of the learners 33% agreed that models aided them while 11% said models did not help. Once again 56% of the learners could not commit to a decision as to whether or not models helped them learn.

3D Models Designed by researcher	Learners interacting with the models	Diagrams as appeared on NSC question paper2. Refer to appendix L for complete question
 		
		
		

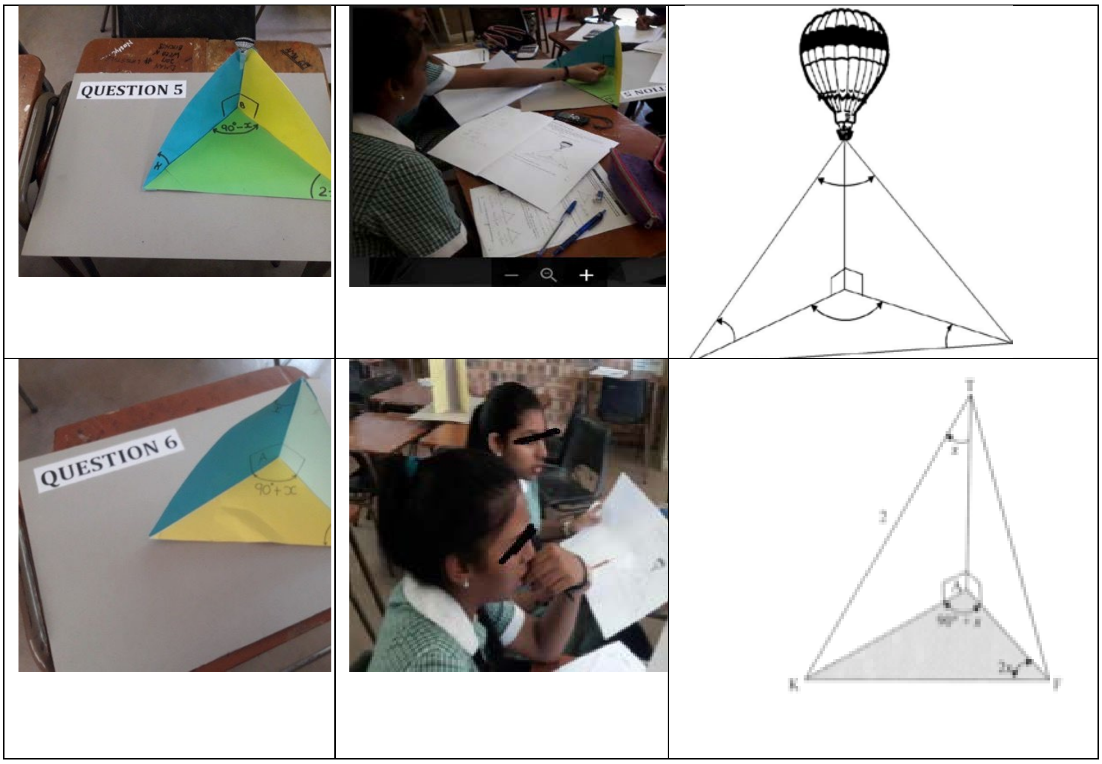


Figure 5.3 Illustration of 3D models, pictures of learner engagement and diagrams in NSC paper 2

Figure 5.3 illustrates the models designed by the researcher, pictures of learners actively engaging with the models and the diagrams as appeared in the NSC paper 2. Learners actively engaged with the models and showed immense interest in the lesson.

5.7 Modifications to the data collection tools after completion of Pre-study

The findings and modifications to the data collection tools are discussed below.

5.7.1 Semi-Structured Interview Schedule

On completion of the pilot study the interview schedule had to be modified. The entire interview schedule changed from twenty questions to thirty-nine questions in total.

The table below 5.2 lists the initial planned interview questions, including the changes made after conducting the pilot study. The reasons for the individual additions, changes and deletions of questions are explained fully in 5.7.1.1.

Table 5.2 Initial Interview questions

1. **Changes: Question 1 was added.** Manipulatives help me understand mathematics better
2. **Changes: Question 2 was added.** Manipulatives help me finish my work quicker
3. **Changes: Question 3 was added.** I enjoy using manipulatives in Mathematics lessons
4. **Changes: Question 4 was added.** Would you like your teacher to use manipulatives?
5. Changes: Question 5 was added. Manipulatives make learning meaningful, a link from concrete to abstract
6. **Changes: Question 1 changed to 6.**Have you ever used artifacts/models before to learn trigonometry? Explain .
7. **Changes: Question 2 changed to 7.** What trigonometric concepts/rules are needed in order to solve the QUESTION 1 in the activity?
8. **Changes: Question 3 changed to 8.**Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 1 ?
9. **Changes: Question 4 changed to 9.** Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?

10. **Changes: Question 10 was added.** were you able to draw the triangles in the different planes?
11. **Changes: Question 11 was added.** What were some of the challenges you experienced when answering this question one?
12. **Changes: Question 5 changed to 12.** What trigonometric concepts/rules are needed in order to solve the QUESTION 2 in the activity?
13. **Changes: Question 6 changed to 13.** Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 2 ?
14. **Changes: Question 7 changed to 14.** Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?
15. **Changes: Question 15 was added.** What were some of the challenges you experienced when answering question two?
16. **Changes: Question 8 changed to 16.** What trigonometric concepts/rules are needed in order to solve the QUESTION 3 in the activity?
17. **Changes: Question 9 changed to 17.** Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 3 ?
18. **Changes: Question 10 changed to 18.**Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?
19. **Changes: Question 19 was added.** What were some of the challenges you experienced when answering question three?
20. **Changes:Question 11 changed to 20**What trigonometric concepts/rules are needed in order to solve the QUESTION 4 in the activity?
21. **Changes: Question 12 changed to 21.**Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 4 ?
22. **Changes: Question 19 was added.** What were some of the challenges you experienced when answering question four?

23. **Changes: Question 13 changed to 23.**What trigonometric concepts/rules are needed in order to solve the QUESTION 5 in the activity?
24. **Changes: Question 14 changed to 25.**Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 5?
25. **Changes: Question 15 changed to 25.**Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?
26. **Changes: Question 26 was added.** What were some of the challenges you experienced when answering question five?
27. **Changes: Question 16 changed to 27.**What trigonometric concepts/rules are needed in order to solve the QUESTION 6 in the activity?
28. **Changes: Question 17 changed to 28.**Why did you decide to use the trigonometric concept/rules you wrote down QUESTION 6?
29. **Changes: Question 18 changed to 29.**Could you explain how the sides and angles in your trigonometric concept/rules could be found in the artifact?
30. **Changes: Question 30 was added.** What were some of the challenges you experienced when answering question six?
31. **Changes: Question 19 changed to 31.**Was it easy to solve the problem when using models/artifacts ?
32. **Changes: Question 32 was added.** What do you think of Mathematics, particularly three -dimensional problems? Explain
33. Describe how the models/artifacts helped you learn during the lesson.

5.7.1.1 Reasons for Individual Additions, Changes and Deletions of Questions

Questions 1 to 5: Question one to five had to be added so as to reveal more information on the use of artifacts/manipulatives to learn trigonometry. The Likert Scale was adapted and these questions each comprised of four options namely –never, B- sometimes, C- usually and D-always. These questions were asked to gather information about what learners felt about the use of artifacts/ manipulatives.

Questions 10

Question 10 was added **to** determine if learners were able to draw triangles in the different planes.

Questions 11

Question 11 was added so as to ask the learner to explain challenges confronted in answering question one.

Questions 15

Question 15 was added so as to ask the learner to explain challenges experienced in answering question two.

Questions 19

Question 19 was added so as to ask the learner to explain challenges experienced in answering question three.

Questions 22

Question 22 was added so as to ask the learner to explain challenges experienced in answering question four.

Questions 26

Question 26 was added so as to ask the learner to explain challenges experienced in answering question five.

Questions 30

Question 30 was added so as to ask the learner to explain challenges confronted in solving question six.

Questions 32

Question 32 was inserted in the interview schedule so as to obtain the learners' perceptions of mathematics three-dimensional problems. The learners' thoughts and challenges experienced.

5.7.2 Mathematical Models/ Manipulative

Five of the six models remained unchanged. In the pilot study it was brought to my attention that in a question, a vertex looked like a right angle. Theparticipant suggested when constructing the model to ensure it did not look like a 90-degree angle as she had assumed it was 90 degrees. During the group discussion learners were cautioned that in problem solving assumptionsshould not be made. This was the only model that needed reconstruction. Theother models were unchanged.

5.7.3 Video Recording

Video recording posed a challenge, as the participants were hesitant and uncomfortable to share information in the group discussion. Some learners had become shy and reserved. A voice recorder was then used. Learners were morecomfortable and felt more relaxed in their classroom environment. More audiorecords were used in the main study.

5.7.4 Activity worksheet

Before learners attempted the answering of the activities, a recapitulation of the area rule, sine rule, cosine rule, basic trigonometry ratios and theorem of Pythagoras was executed to refresh the learners' memories. On completion of the pilot study it was discovered that a recapitulation of the rules proved to be insufficient and that in the main study, a revision exercise would be done in order to recapitulate and consolidate the area rule, cosine rule, sine rule, theorem of Pythagoras .The content of the activity worksheet remained unchanged.

5.7.5 Observation

Movement by the researcher was reduced who then positioned herself at strategic places in the classroom from where she observed the participants interactions with the mathematics models, activity worksheets and the group.

5.8 Analysis of questions in the activity sheet Tool

Table 5.3 Analysis of questions in the activity sheet Tool **(n=9)**

Question number	**Completed and correct**	**Completed and Incorrect**	**Partially completed**	**Not attempted**
1.	3 (33%)	3 (33%)	2 (22%)	1 (11%)
2.1	8 (89%)			1 (11%)
2.2	4 (44%)	4 (44%)		1 (11%)
3.1	9 (100%)			
3.2	1 (11%)	8 (89%)		
3.3	7 (78%)	1 (11%)		1 (11%)
4.1	9 (100%)			
4.2	8 (89%)	1 (11%)		
4.3	8 (89%)	1 (11%)		
5.1	6 (67%)			3 (33%)
5.2	4 (44%)	2 (22%)		3 (33%)
5.3	5 (56%)	1 (11%)		3 (33%)
6.1.1	5 (56%)	1 (11%)		3 (33%)
6.1.2	5 (56%)	1 (11%)		3 (33%)

Table 5.3 reveals that 33% of the learners did not attempt questions five and six. A possible reason could be that the allocated time was insufficient. The findings suggest that in the main study the time allocated would have to be increased.

5.9 Evaluation and Recommendations

Positive feedback was obtained relating to the influence and impact the six mathematical models had on learners' understanding and problem solving abilities in three-dimensional trigonometric problems. Areas that were identified to be refined were the semi-structured interview and more time needed to be allocated to solving the problems and completing the activity sheet, as some learners did not attempt question five and six in the activity sheet. The pilot study was valuable since it allowed for the refining of wording, order, layout and filtering in order to aid the pruning of the semi-structured interview schedule. It was recommended that the researcher not merely recapitulate the prerequisites needed to solve the trigonometric problems but rather provide a revision exercise on the application of the sine rule, area rule, cosine rule, trigonometric ratios and the Theorem of Pythagoras. This exercise would provide and better equip the learners in solving the three-dimensional problems.

5.10 Conclusion

This chapter has provided the preliminary results on the pilot study study carried out to validate the research instruments used. A proper analysis of the procedure and results from the pilot study facilitates the identification of weaknesses that may be addressed. A carefully organized and managed pilot study has the potential to increase the quality of the research as results from such studies can inform subsequent parts of the research process (Malmqvist, Hellberg, Mollas, Rose & Shevlin, 2019: 1). The pilot study was the first step towards the data collection stage and was made up of a small sized study (of nine grade 12 mathematics learners) that assisted in the planning and modification of data collection instruments. The interest and motivation shown by the learners when interacting with the mathematical models was motivating and encouraging. There was no need for major revisions to the pilot study and it was proven to be feasible and informed the researcher that she could proceed with the main study after modifying the interview schedule in the study design. The next chapter discusses the analysis of the main study.

CHAPTER SIX: MAIN STUDY: PRESENTATION, ANALYSIS AND DISCUSSION OFDATA

6.1 Introduction

In the previous chapter details of the pilot study undertaken by the researcher were discussed. Comprehensive descriptions of each tool of enquiry and data sources were provided. The pilot study was conducted to validate the research instruments.

This chapter discusses the analysis of the data. Qualitative methods were applied in this study and the data was retrieved through activity sheets, semi-structured interviews and observations. The learner responses in the activity sheets were analysed to ascertain how the purposely designed manipulative, enhanced the learning of trigonometry among grade 12 learners and how the use of manipulatives impacted the learners' understanding and learning. The semi -structured interview was designed to acquire an in-depth insight into the learners' knowledge of and skills in solving three-dimensional trigonometric problems. While learners were interacting and engaging with the manipulative, the researcher observed the learners.

6.2 Detailed Analysis of Cognitive Levels of questions in the activity sheet

In South Africa the Subject Assessment Guidelines for Mathematics (SAGM) taxonomy is used to assess the alignment of the South African matriculation mathematics examination (Berger, Bowie and Nyaumwe, 2010). Learners leaving the schooling system write a common examination known as the

National Senior Certificate (NSC) examination, commonly known as the 'Matric Exam'.

The SAGM Taxonomy comprises of the following:

Knowledge
☐Knowledge and use of formulae or algorithms.
Routine procedure
☐Problems are not necessarily unfamiliar and can involve integration of different Learning Outcomes. ☐Well-known procedure. ☐Simple applications and calculation with many steps and which may require interpretation from given information. ☐Identifying and manipulating formulae.
Complex procedures
☐Mainly unfamiliar and involve integration of different Learning Outcomes. ☐No direct route to solution but involve higher-level calculation skills and/or reasoning. ☐May be abstract and require fairly complex procedures.
Solving problems
☐Non-routine, unseen. ☐Interpreting and extrapolating from solutions obtained by solving problems based in unfamiliar contexts. ☐Using higher-level cognitive skills and reasoning to solve non-routine problems ☐Breaking down problem into constituent parts. ☐Non-routine in real context.

(Adapted from Berger, Bowie and Nyaumwe 2010:39).

Cognitive Levels	Description of skills to be demonstrated
Knowledge 20%	✓ Straight recall ✓ Identification of correct formula on the information sheet (no changing of the subject) Use of mathematical facts ✓ Appropriate use of mathematical vocabulary
Routine Procedures 35%	✓ Estimation and appropriate rounding of numbers ✓ Proofs of prescribed theorems and derivation of formulae ✓ Identification and direct use of correct formula on the information sheet (no changing of the subject) ✓ Perform well known procedures ✓ Simple applications and calculations which might involve few steps ✓ Derivation from given information may be involved ✓ Identification and use (after changing the subject) of correct formula ✓ Generally similar to those encountered in class
Complex Procedures 30%	✓ Problems involve complex calculations and/or higher order reasoning ✓ There is often not an obvious route to the solution ✓ Problems need not be based on a real world context ✓ Could involve making significant connections between different representations ✓ Require conceptual understanding
Problem Solving 15%	✓ Non-routine problems (which are not necessarily difficult) ✓ Higher order reasoning and processes are involved ✓ Might require the ability to break the problem down into its constituent parts

TABLE 6.1 Weighting of cognitive levels Source CAPS Document , 53

Table 6.1 provides the weighting of the cognitive levels and the detailed description of skills that are to be demonstrated by learners. The questions contained in the activity sheet have been extracted from past year NSC papers. The questions selected contain various cognitive levels. This is discussed further in detail later in the chapter.

6.3 Analysis and discussion of interviews

In addition to written responses to activities and observation, semi- structured interviews were conducted with sixteen participants (grade 12 mathematics learners): The main objectives of the interviews were to: gain clarity on the written responses that appeared on the activity sheets, determine if the use of manipulatives had an impact on the learners understanding in solving the given tasks/questions and to verify understanding and application of trigonometric ratios, sine rule, area rule cosine rule and Theorem of Pythagoras in solving 3D trigonometric problems.

Furthermore, the learners were asked to respond to open-ended questions contained in the semi-structured interview. This was done to justify the learners' responses to particular questions in the research instrument, to explain the mathematics applied to solve the problems, to describe the impact the manipulative had on their understanding and to provide an explanation as to the Mathematics that they applied in solving the problem. Questions 1 to 5 in the interview schedule are analysed and the data obtained from the learners' responses are displayed in the graphs below.

6.3.1 Analysis of Question One of the semi-structured interview

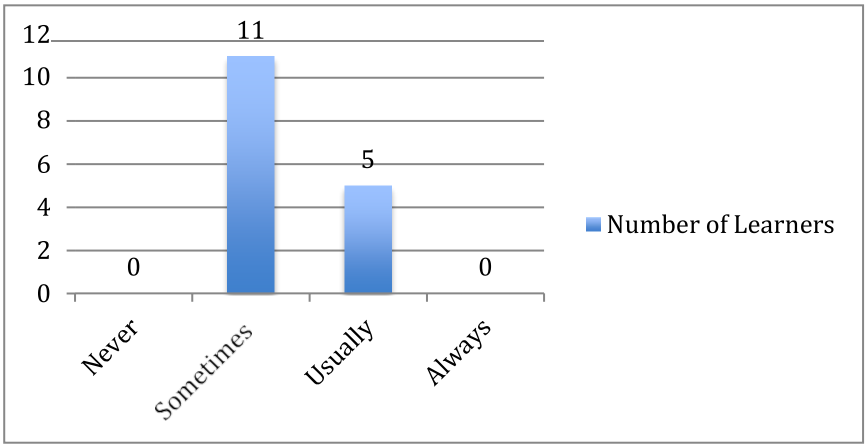


Figure 6.1: Manipulatives help me understand mathematics better.

Many learners, eleven (69 %) reported that manipulatives sometimes helped and had a positive impact on their understanding Mathematics better, with five (31%) learners agreeing in some form that the use of manipulatives helped them realize that it aided them in their understanding. A possible explanation to this could be that learners are rarely or never exposed to mathematical manipulatives. They are clueless as to how to maximize the use of a manipulatives. The findings in this research concurs with that of McNeil & Jarvin (2007) (refer to chapter 2) who claim that the use of manipulatives assist learners to make the link with real-world knowledge and aid in understanding and increase memory.

Even though manipulatives prove to have several benefits, there is no guarantee of success if teachers use them incorrectly. Studies against the use of manipulatives show that teachers appear to see activities that use manipulatives as playtime (Green, Piel & Flowers, 2008). The study carried out by Moyer (2001) which comprised of ten school teachers revealed that teachers discovered the use of manipulatives to be rewarding and enjoyable with learners but they could not identify the value of the use of manipulatives as aids for learning Mathematics.

6.3.2 Analysis of Question Two of the semi-structured interview

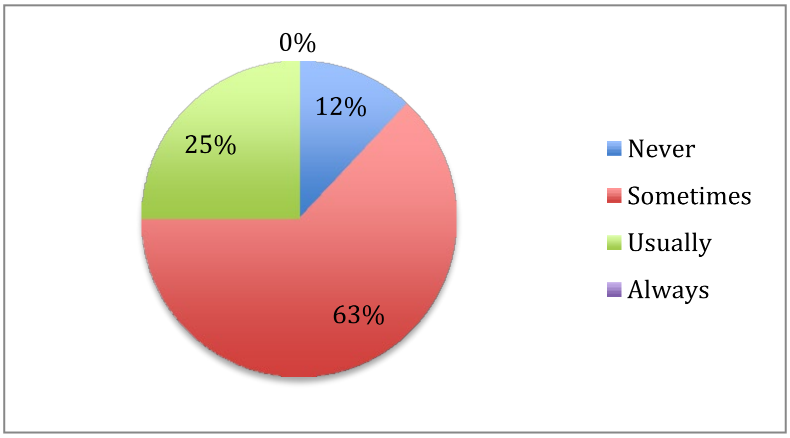


Figure 6.2: Manipulatives help me finish my work quicker.

Of the learners, 25% who participated felt that manipulatives usually aided them in completing a task faster, whilst a 63% indicated that the manipulatives sometimes aided them in completing a task faster. Furthermore, 12 % of the learners had indicated that the manipulative had no impact on the speed of completing a task.

The findings in this study relate to studies conducted by Ndlovu (2019) and that of Johnson, O' Meara & Leavy (2021). Ndlovu (2019) states that teaching mathematics while using manipulatives assists learners conceptualize mathematical concepts and therefore there is a great need for educators' competencies in using manipulatives to assist learners. Additionally Johnson, O' Meara & Leavy (2021) claim that manipulatives provide learners with an additional lens through which to view the mathematical concepts and offers an additional resource to assist in the development of their understanding.

6.3.3 Analysis of Question Three of the semi-structured interview

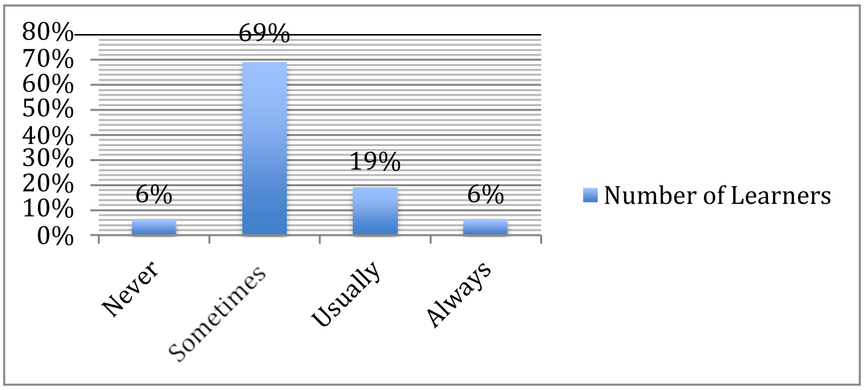


Figure 6.3: I enjoy using manipulatives in mathematics lessons

While 94% (69% + 19% + 6%) of the learners reported being somewhat satisfied with the use of manipulatives, 6% of the learners indicated that they had experienced no joy in the use of manipulatives in the mathematics classroom (Fig 3). The application in the use of manipulatives in the mathematics classroom context needs to be more structured to ensure that teachers and learners have adequate background and previous knowledge to make the most of the experiences and interactions of the mathematics manipulatives.

The findings in this study are similar to the study done by Enki (2014) where the participants indicated that as opposed to previous traditional instruction methods, learning through activities that made used of manipulatives provided them with pleasure and increased their motivation and allowed them to have fun while learning. Strom (2009), Reimer and Moyer (2005), Day and Hurell (2019) and Stiegelmeier and Moore (2019) claim that research has shown that learners of varying ages obtain enjoyment when taught Mathematics through participatory and interactive methods that allow for the utilization of manipulatives. Larbi & Mavis (2016) add that manipulatives are beneficial to learners since each learner has a unique way of learning and senses are introduced in learning. When manipulatives are utilized, they act as visual representations of mathematical concepts. Apart from satisfying the needs of learners who enjoy learning this way, teachers are given innovative ways of introducing mathematical topics. The research findings by Hidayah, Dwijanto & Istiandaru (2018) verified that the use of manipulatives offered opportunities to learners to be attentive and to observe the teacher's questions and statements. By using manipulatives learners were aided to think and to easily recall the concepts. They discovered that learners were excited and glad to join in the learning process, which made use of solid figure manipulatives, worksheet and statements and questions.

Learners indicated that they had a better understanding and enjoyed Mathematics. Several studies that have been conducted concur with these findings. Cockett (2015, 48) states that manipulatives can be an important tool to assist learners to think and reason in a more meaningful manner. Manipulatives help students develop conceptual understanding of mathematical ideas by representing ideas in several ways Shaw (2002,1). Similarly Marasigan *et al*. (2019) adds that the use of manipulatives in teaching mathematics permits learners to build their own cognitive models for abstract mathematical ideas and processes. Cockette (2015: 49) explains that using mathematical manipulatives is an important tool for aiding learners to develop Mathematics ideas. The benefits include increased learner interaction and enjoyment, which results in increased efficiency and understanding.

The findings of this study concurs with the findings of Bernard, and Ulyani & Qohar, 2021; Sumarna, Rolina & Akbar, 2019; Prabowo, Usodo & Pambudi, Ibrahim & Ilyas, 2016 and Jorda & De los Santos, 2015; Yusha'u, 2013. Ulyani & Qohar (2021) claim that learning through manipulative materials allows for the learning of trigonometry to become easier to understand. Further more learning that includes elements of play, allows for improvement of students' understanding in trigonometry (Sumarna, Rolina & Akbar, 2019; Prabowo, Usodo & Pambudi, 2019; Prabowo, Anggoro, Adiyanto & Rahmawati, 2018;

Ibrahim & Ilyas, 2016; Bernard, Jorda & De los Santos, 2015 and Yusha'u, 2013).

6.3.4 Analysis of Question Four of the semi-structured interview

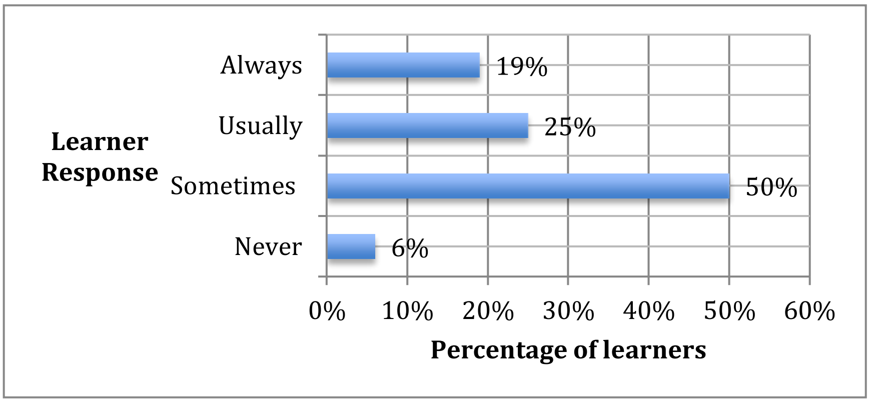


Figure 6.4:Would you like your teacher to use manipulatives?

Learners were asked if they would like their teacher to use manipulatives,. Only 94%(19% always + 25% usually + 50%) of the learners agreed in someform fig (4) that they would be pleased if the teacher made use of manipulatives more often in the classroom. The findings may reveal that teachers require training in the use of manipulatives and need be encouraged to use manipulatives more often in the mathematics lessons. The study of Merrill, Devine, and Brown (2010: 16) affirm that improving and enhancing content knowledge requires mathematics teachers to use three-dimensional (3D) solid modeling in mathematics classrooms to improve learners' understanding of mathematical concepts and principles. When students visualize, they see the relevance and this promotes rigour. Weng (2011: 52) reiterates that visible and visual 3D dynamic design has the potential to increase learning interests in Mathematics, by improving learner's interdisciplinary and multimedia design abilities.

6.3.5 Analysis of Question Five of the semi-structured interview

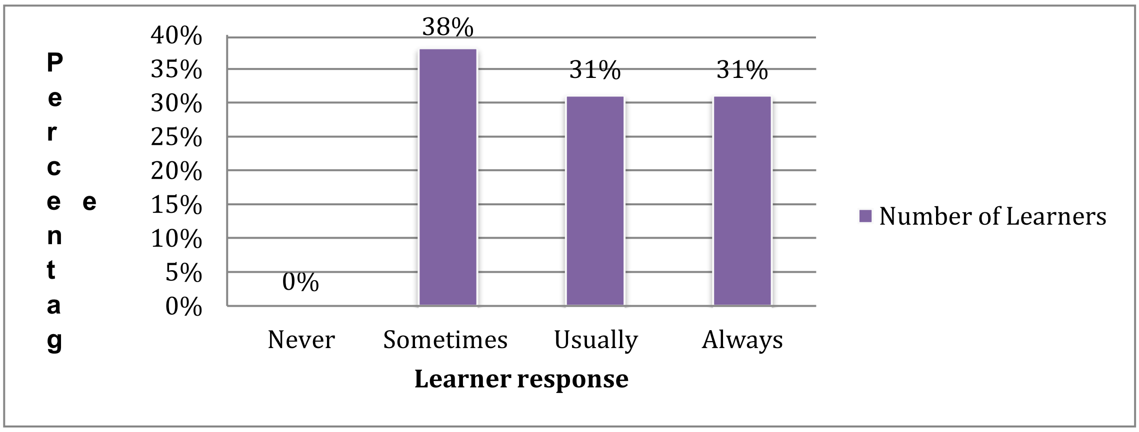


Figure 6.5: Manipulatives make learning meaningful, a link from concrete to abstract

Learners were asked for their perceptions of manipulatives and if they found them meaningful and whether they provided a link from concrete to abstract. It was found that all 100% (38% + 31% = 31%) of the learners had indicated in some form (Fig 5) that manipulatives did make learning Mathematics meaningful and provided a link from concrete to abstract. Findings of this study are similar to research that that has been conducted. Manipulatives allow for learners to construct their own cognitive models for mathematical ideas and processes that are abstract. In addition, manipulatives offer a common language to communicate these models to other learners and educators (Marasigan *et al.*, 2019). As discussed in Chapter Two, it was Bruner who also recommended the use of concrete materials when he had hypothesized that learning occurs by transitioning through the three stages of representation namely enactive (concrete), iconic (representation) and symbolic (abstract) (Bruner, 1966). The use of physical objects to discover and explore a mathematical concept aligns with the first stage of Bruner' s three-stage model, this being the enactive stage which involves learners interacting with a hands –on, tactile exploration of the topic before proceeding towards a more abstract understanding of a concept. Piaget (1970) likewise recommended the use of physical objects in his theory of cognitive development. This is the third stage of his theory and is the concrete operational stage that involves learners applying logic to solve problems that apply to concrete objects.

6.4 Learner responses to selected questions in the interview schedule

Question one to five of the semi-structured interview is discussed above. Excerpts from the semi-structured interview schedule are given below. These questions were selected as they generated rich data to determine:
(1) If learners had encountered the use of manipulatives in the classroom, (2) did the use of manipulatives aid in problem solving, (3) their attitude towards manipulatives and did it help them, (4) learners' opinion of teachers using manipulatives and if they would like the use of manipulatives in the future lessons and (5) did their confidence to write the NSC examination increase?

6.4 .1 Response to question six on Interview schedule

Learners were asked the following question contained in the interview schedule. Have you ever used artifacts/models to learn trigonometry? Explain. Comments included:

Learner 1: " No"

Learner 2: " No"

Learner 3: "No. Was never exposed to this before. Until now."

Learner 4: "No"

Learner 5: "No. I never used it before because I didn't think it would make understanding the problem easier. Now that I have used it, I am aware that it does help and I will use artifacts/models to answer problems in the future."

Learner 6: "No"

Learner 7: "No, never needed to"

Learner 8: "No"

Learner 9: "No! Never thought to use it. Always used diagrams or big charts"

Learner 10: "No. We only used diagrams."

Learner 11: "No, we only used diagrams."

Learner 12: “No”
Learner 13: “No. I did not understand how to sum up the model which is why I did not use it.”
Learner 14: “No. I have never used it before because I did not think it would make me understand the problem easier. I will use artifacts/ models more often to answer problems in the future.”
Learner 15: “ Yes in tutorials our teacher used different models to explain eg. A door or a flexible ruler, blank page. However, in class discussions models were not used to explain trigonometric problems.”
Learner 16: “ No, teachers didn’t use as trig always left for last section thus time didn’t permit.”

It is evident that these learners did not have the opportunity to use manipulatives in their classrooms. Learner three remarks that she/he was never exposed to mathematics manipulatives until now. Learner five and fourteen after being exposed to manipulatives, have a different view on manipulatives and would use it in the future. It is interesting to note that Learner thirteen did not know what to do with the manipulative. This once again shows that learners exposed to manipulatives was limited or non-existent. Koparan (2017) advocates that using materials in mathematics teaching is an important determinant of understanding mathematics topics. Furner & Worrell (2017) claim that while there are benefits to using manipulatives, evidence has shown that teachers’ conception of the use of manipulatives in the classroom instruction is limited. Although the reasons for the lack of effective use of manipulatives has not been extensively researched, the literature shows that the lack of teacher knowledge or expertise in a particular dimension is the main reason why certain resources or topics are not taught well (Ndlovu & Chiromo, 2019: 2).

Puchner, Taylor, O Donnel & Fick (2010) claim that teachers experienced problems in using manipulatives when teaching mathematical concepts in the delivery of their lessons. Since the use of manipulatives has been identified as a resource that aids learners' understanding of abstract mathematics, it is vital that pre- service teachers acquire the expertise and knowledge to use manipulatives and effectively use them in their teachings.

6.4.2 Response to question Thirty- one on Interview schedule

Learners were asked the following question contained in the interview schedule: Was it easy to solve the problems when using models /artifacts? Explain. Comments included:

Learner 1: "Yes. They helped me to conceptualise the problem and it made it easier to answer these questions."
Learner 2:"Yes. Helped me to get a proper visual and interpret the question better."
Learner 3:"Yes. It made understanding clearer."
Learner 4: "Yes. Simple to understand visualize."
Learner 5: "It was completely easy, however it has made understanding three dimensional problems much better."

Learner 6: "No! I felt it was the same, the only requirement is to separate and draw the different triangles/planes."
Learner 7:" I did not need it."
Learner 8:'I did not need it."
Learner 9: "Yes! Reason being it helps me identify each triangle or shape with their own angles individually. So when I see the model I am able to separate and identify and solve."
Learner 10: "Yes. I was able to see what I needed to do."
Learner 11: "Yes. I viewed what I was supposed to do."
Learner 12:"Yes. We had a clear model of each angle and side."
Learner 13:"Yes, because having a 3D object makes me understand things better."

Learner 14: No response

Learner 15: No response

Learner 16: "Yes. Equation weren't the longest."

From the above responses several learners agreed that the manipulative did assist them in solving the problem. Manipulaives helped them visualize. Learner nine clearly explains the manipulative helped her identify and separate each triangle or shape with its own angles individually. She was then able to solve the problem.

Learners six, seven and eight are exceptions to the norm and stated that they did not require a manipulative to solve the problem. Learner six added that she could draw the 2D triangles and solve. This could imply that these learners are functioning at a higher cognitive level and could internalize the problem mentally.

Learners fourteen and fifteen were confused and could not commit to an answer. They were unsure of what impact the manipulatives had on them and therefore remained silent when asked the question. Manipulatives in the environment influenced students' mathematical understanding (Gulkilik, Ugurlu & Yuruk, 2015). As discussed in chapter one, the National Senior Certificate Diagnostic report 2017 stated that in trigonometry 3D problem solving, the learners experienced difficulty in visualising the different planes in the sketch. Learners were unsuccessful in linking the two right-angled triangles. In addition learners were unable to state correct trigonometric ratios in the triangles. Learners continuously mixed up angles and sides. Several learners were confused with the orientation of 3-D shape and were unable to use the correct formula. When students visualize, they see the relevance and this stimulates rigour. Herbst and Chazan (2011: 1) corroborate this view by stating that artifacts can be used in activity systems for learners to interact with the object and other learners and the teacher thus enriches the lesson.

6.4.3 Response to question Thirty- two on Interview schedule

Learners were asked the following question contained in the interview schedule: What do you think of mathematics particularly three-dimensional problems? Explain. Comments included:

Learner 1: “They can be tricky. They aren’t extremely difficult but they require a good understanding of the trigonometric principals.”

Learner 2:”Diagrams make it easier to conceptualise the problems because they are tricky.”

Learner 3:”It is complex, but can be simplified to understand.”

Learner 4: “Very interesting.”

Learner 5: “It is tricky but, but if you isolate the triangles, it is easier to understand.”

Learner 6: “It is difficult but with practice it becomes easier. It is hard to determine the angles and separate the triangles. If you determine the angle wrongly, the rest of the question becomes wrong.

Learner 7: ”It is tricky but if you break it down it is simple.”

Learner 8:”It is tricky unless you break it down.”

Learner 9: “It is tricky and hard to analyse. If you carry the wrong answer the whole answer will end up wrong and more marks will be lost.”

Learner 10: “It is interesting as it forces you to see things differently.”

Learner 11: “Its interesting. It allows you to see things differently.”

Learner 12:”It is a bit difficult working with the rules”

Learner 13:”It is interesting as it forces you to see things differently.”

Learner 14: No response

Learner 15: No response

Learner 16: “it is hard but have to see different triangles and shapes.”

Several learners indicated that 3D trigonometry problems were tricky and difficult. Learners four, ten, eleven and thirteen explained that they found 3 D trigonometric problems interesting. Learners three, six, twelve and sixteen indicated that they experienced difficulty. Learners fourteen and fifteen had not responded to this question and remained silent. Learner sixteen explains that although it is hard one has to see the different triangles and shapes in the horizontal, vertical and slant planes. Studies on the use of manipulatives have revealed the importance of manipulatives in enhancing the understanding of abstract mathematical concepts (Saka & Roberts 2018; Furner & Worrell 2017; Brijlall & Niranjan 2015).

6.4.4 Response to question Thirty- three on Interview schedule

Learners were asked the following question contained in the interview schedule. Describe how models/ artifacts helped you during the lesson. Comments included:

Learner 1: "They gave a sort of scenario, which made it easier to view each triangle and the information that was present."
Learner 2:"It gave better visuals to interpret the questions. Allowed us to manipulate the objects in order to understand the question."
Learner 3:"It shows the way the shape is and helps applying those shapes properties in an easier way."
Learner 4: "Helps to visualize."
Learner 5: No response
Learner 6: "It helped me to see the different angles and shapes."
Learner 7:"I used diagrams."
Learner 8:"I used diagrams."
Learner 9: "It helped to separate each shape in order to work out answers on their own. Then put everything together and see the better picture."
Learner 10: "I was able to see what was happening in the problems."
Learner 11: "It gave me a visual representation of the diagram to understand better."
Learner 12:"We identified given angle more clearly."
Learner 13:"I saw what happened in the problem."
Learner 14: "No response."
Learner 15: "No response."

Learner 16: " It shows a bigger picture and it is understandable."

From the responses above it is evident that the mathematical manipulatives helped learners Learners one, two, three, four, six, nine, ten, eleven thirteen and sixteen visualize in some form what was required. Learner eleven explains that manipulatives gave her a visual representation of the diagram to understand better. Learner nine added that It helped her to separate each shape in order to work out answers on her own. Then put everything together and see the better picture. Learner twelve states that the manipulatives provided more clarity. Whilst manipulatives aided in visualization, Learners seven and eight indicated that they used diagrams to solve the problem. Learners fourteen and fifteen did not respond and did not provide any feedback. The findings in this study indicate that learners were able to apply spatial visualization. As mentioned in the Literature chapter two, spatial visualization is the process of constructing, maintain and manipulating 2D and 3D objects in one's mind (Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013; Cracow & Sorby, 2008). Mental rotation is described as the rotation of mental representations of 2D or 3D objects to determine their images from various viewing angles (Ha & Fang, 2016). Johnson, O' Meara & Leavy (2021) claim that manipulatives provide learners with an additional lens through which to view the mathematical concept and offers an additional resource to assist in the development of their understanding.

6.4.5 Response to question Thirty- four on Interview schedule

Learners were asked the following question contained in the interview schedule: What do you think of manipulatives? Comments included:

Learner 1: "They are helpful"
Learner 2:"They are helpful in helping us to interpret questions."
Learner 3:"It's helpful. Really helps understand the problem."
Learner 4: " It is useful. Really helps to understand the problem."
Learner 5: 'They make understanding three dimensional problems much easier."
Learner 6: "They are a good visual aid"
Learner 7:"They are fun."
Learner 8:"It's useful. Really helps understand the problem."
Learner 9: "They are useful and make it much better to work out the angles and sides of objects. Often helps to see the bigger picture and things such as angles become clear."
Learner 10:" They are helpful"
Learner 11: "They help a lot."
Learner 12:"It makes questions easier."
Learner 13:"They help a lot."
Learner 14: "I feel it is helpful, makes me understand better because I am able to understand the diagram better."
Learner 15: "It is very useful. Really helps to understand the problem."
Learner 16: "They help and make us understand 3 dimensional problems."

Learner seven stated that manipulatives were fun. Learners one, two, three, four, eight, nine, ten eleven, thirteen, fourteen, fifteen and sixteen claimed that they found manipulatives to be helpful in understanding and solving 3D trigonometry problems. Manipulatives have proven to be well accepted and enjoyed in mathematics lessons. Manipulatives intrigue, motivate and help students learn and solve challenging problems.

The finding of this study are parallel to that of Niranjan (2013) and Hunt, Nipper and Nash (2011) who claim that the use of manipulatives allowed for learners to emerge as active participants, to relate real world circumstances to mathematics, to engage in discussion of mathematical ideas and concepts, to aid in transforming abstract ideas to concrete. They are useful tools to solve problems, provide opportunities for learners to apply their own methods in solving mathematical problems and to create opportunities to work co-operatively. They also intrigue and motivate learners to learn and allows for the learning of mathematics to be interesting, exciting and enjoyable. Silva, Costa & Martins (2020) in their findings state that students were highly motivated and displayed strong enthusiasm in the involvement of tasks, fondness of the discovery process and collaboration learning that resulted in sharing of solutions obtained for challenges that were given to them.

6.4.6 Response to question Thirty- five on Interview schedule

Learners were asked the following question contained in the interview schedule: How have manipulatives affected your learning of mathematics? Comments included:

Learner 1: “ They have improved my understanding of 2Dand 3D problems”
Learner 2:”improved understanding of 3D problems”
Learner 3:”Made learning easier ad more understandable”
Learner 4: “Make learning easier and more understandable”
Learner 5: “My learning of mathematics has now improved with the help of manipulatives”
Learner 6: “It has helped me see the different angles and shapes.”
Learner 7:”It has not.”
Learner 8:”Made learning easier and more understandable.”
Learner 9: “In a positive perspective. Reason being it helps to solve problems. I will be sure to work at home with these manipulatives to improve on my skills.”
Learner 10: “It put things in perspective.”
Learner 11: “It gives a better perspective.”
Learner 12:”Provided a simpler way to solve problems.”
Learner 13:”It gave me a better perspective.”

Learner 14: ‘It gives me a better understanding with math as the physical part is more interactive for me. Simple explanations indicate better agreement.”
Learner 15: “Make learning easier and more understandable. Creates a wide view of possible answers.”
Learner 16: ‘It has improved my knowledge.”

Learner one, two,t hree, four, six, eight and fifteen all stated that manipulatives made learning somewhat easier and more understandable. Learner seven has indicated that the manipulatives have not affected his learning in anyway. Learner ten added that manipulatives put things in perspective while Learner fifteen claims that manipulatives creates a wide range /view of possible answers. Manipulatives are considered to be effective in fostering the development and enhancement of conceptual understanding in mathematics as they assist learners to link and relate concrete ideas to abstract ideas (Witzel and Allsopp, 2007; Uribe-Florez & Wilkins, 2010). Jones and Tiller (2017) claim that by making use of hands–on, concrete manipulatives during mathematics teaching time could result in learners having a higher retention rate and develop a more positive attitude towards their education. Smith (2009) further adds that a well designed manipulative bridges or closes the gap between formal mathematics and informal mathematics.

6.4.7 Response to question Thirty- six on Interview schedule

Learners were asked the following question contained in the interview schedule:
What is your opinion of the performance of your teacher after the use of manipulatives? Comments included:

Learner 1: “She was really helpful and she explained the manipulatives quite well.”
Learner 2:”She helped me gain a better understanding of this section and the different questions to expect.”

Learner 3:" Teachings were more effective with the manipulatives."

Learner 4: "Teachers explanation of the problem is much clearer."

Learner 5: "It is evident that she has put in a lot of effort, providing us with manipulatives in helping us to understand the problems."

Learner 6: "It has remained the same."

Learner 7:"She was very informative and visual was easy."

Learner 8:"Teachings were more effective as manipulatives increased understanding

Learner 9: "Mam has been very helpful throughout the lesson. She has assisted me when I asked for or was in need of explaining. She taught us easier methods on how to work out examples."

Learner 10:"Explanations were a lot clearer."

Learner 11: No response.

Learner 12:"She gave thorough explanations."

Learner 13:"Manipulatives are used and breaking down the question."

Learner 14: "I am able to learn and understand with her method of learning. Her performance is better."

Learner 15: "She is a very creative person for coming up with visual aid to help us understand more."

Learner 16:"She has put in a lot of effort providing us with manipulatives in helping us to understand the problems."

From the above responses there appears to be a general appreciation of the teachers efforts to make her teaching interesting and more importantly like how Learners eight **and three** indicated that the teachings were more effective as the manipulatives increased understanding. Learners fifteen and sixteen added that the creativeness and additional effort by the teacher increased there understanding.

Diagnostic report NSC paper 2 (2017:1) acknowledges that Mathematics analysis conducted reveals the weaknesses in learners' responses. The analysis of the misconceptions and error patterns exposed in learners' responses can inform instructional practice. These identified weaknesses can allow teachers to refine their teaching strategies appropriately. The Department of Education will, through interventions, continue to capacitate teachers to design responsive and suitable instructional programmes that will effectively focus on the areas of weakness identified in Mathematics. From the learner's responses in this study, the use of manipulatives is surely an effective teaching strategy, which more high school teachers should employ.

The finding in this study is consistent with other studies carried out. Learners are keen to engage with manipulatives and secondary teachers need to use manipulatives more often in the mathematics classrooms. A study by carried out by Unlu (2017) with teacher, revealed that teachers wanted to use manipulatives in their lessons as manipulatives made teaching easier and aided to concretize abstract mathematical concepts in the lesson. A study carried out by Piskin-Tunc, Durmus & Akkaya (2012) with secondary school mathematics teachers revealed that teachers believed that using concrete manipulatives and virtual learning objects in mathematics teaching would result in an increase in learners' success.

The study of Marshall and Swan (2008) revealed that teachers believed that the use of manipulatives in Mathematics lessons increases children's mathematics learning. In addition the results showed that if mathematics manipulatives are to be effective, then it is important that mathematics manipulatives be part of a carefully planned mathematics programme. In particular teachers should have adequate knowledge of mathematics, students and manipulatives.

Golafshani (2013) investigated the beliefs of teachers of mathematics on the use of manipulatives in mathematics teaching and the effects it had on student's learning. The findings revealed that teachers were eager and willing to incorporate manipulatives in the mathematics lessons and the use of manipulatives had a direct impact on student learning. Studies by Ciftci, Yildiz& Bozkurt (2015) and Kutluca & Akins (2013) reveal that teachers did not believe that concrete manipulatives could be used in all topics related to Mathematics. In addition, secondary school mathematics teachers also did not think that manipulatives could be incorporated in every subject of a Mathematics course. Kucukgencay, Acar & Peker (2020) examined teachers access statuses to manipulatives and discovered that teachers generally had a positive view of the use of manipulatives in lessons but the challenges that existed prevented them from using manipulatives. The greatest challenge experienced is that teachers have limited access to manipulatives. In addition the opportunity to access manipulatives for teachers and to design appropriate manipulatives may have an impact on the frequency with which manipulatives are used in lessons.

6.4.8 Response to question Thirty- seven on Interview schedule

Learners were asked the following question contained in the interview schedule: Are you now more confident of answering your three dimensional question in the NSC November Examination? Explain. Comments included:

Learner 1: “I am more confident than I was previously.”

Learner 2:”I am more confident in answering these questions.”

Learner 3:”I understand how to simplify complex examples and thereafter solving these become easier.”

Learner 4: “Yes! This activity was enjoyable, boosts my confidence in attacking 3D problems. I can visualize and conceptualize better.”

Learner 5: “Yes!”

Learner 6: “Not really, but the practice has definitely made me more clued up with the section.”

Learner 7:”Yes. Exposed myself to more questions.”

Learner 8:”Yes, I exposed myself to more questions.”

Learner 9: “Yes! Because the more exams you do, the better you become at that section. As you know that if you fail, then you must try and try again. Practice makes perfect, makes much easier to understand examples in paper.”

Learner 10: “Yes. Practice makes perfect.”

Learner 11: “Yes this refreshed my memory for the exam.”

Learner 12:"Yes. This exercise provided good practice in the relevant questions."
Learner 13:"Yes, this refreshed my memory for the NSC exams."
Learner 14: "Yes. With the understanding of the work done I am able to do the working better because I would be confident."
Learner 15: "Yes, if I continue using examples like this, I will be more prepared and confident to write the NSC November exam."
Learner 16: "Yes. I was able to understand the question better.

It is pleasing to note that the majority of the learners responded positively and stated that they felt more confident to answer these 3D trigonometry problems in the NSC examinations.
As discussed in chapter one it was evident from the above NSC reports that intervention strategies have to be put in place to ensure learners become more confident in solving 3D trigonometric problems. This study made use of manipulatives to aid studends to visualise the 3-D figure and extract the triangles in the horizontal, vertical and slant planes. The use of manipulatives aids learners in deeper conceptual uderstanding . The manipulatives made learning more meaningful and provided a link from concrete to abstract.
The findings is this research is similar to the findings of Hunt, Nipper & Nash (2011:1) who claim that an advantage of using manipulatives is that students feel more capable and competent because they do things on their own and discover things on their own. They feel less dependent on their teachers. Carbonneau, Wong & Borysenko (2020) state that learning with concrete manipulatives is highly dependent on learners' interpretation of the manipulatives and that collaborative learning is a strategy that can be easily included in classrooms. Silva, Costa & Martins (2020) in their findings state that students were highly motivated and displayed strong enthusiasm in the involvement of tasks, fondness of the discovery process and collaboration learning that resulted in sharing of solutions obtained for challenges that were given to them. The above finding align with the findings of Choden & Chalermnirundorn (2021) who carried out studies on the use of manipulatives

and cooperative learning. From their findings it was clear that students possessed a positive perception towards the use of manipulatives and cooperative learning. In addition the students expressed their enjoyment of the lessons when they were taught using manipulatives and using cooperative learning styles. They claimed that the use of manipulatives kept them engaged and aided them to develop an interest in Mathematics. They also added that it was easier to solve mathematical problems with the aid of manipulatives and the help from fellow classmates through the use of cooperative learning. They also stated that they had a better understanding of concepts when they learned and taught each other in groups. Learners expressed that they felt comfortable and motivated learning from their peers than through the teacher. The study carried out by Charbonneau *et al*. (2020) showed that by applying the use of manipulatives and cooperative strategies, learners were aided to work well in groups, which improved problem solving mathematical skills and stimulated a positive attitude and outlook towards mathematics learning. The use of manipulatives also assisted in improving learner performances and developed analytical and critical skills.

6.5 Detailed analysis of questions from the activity sheets

Two and three-dimensional trigonometry uses all the trigonometry knowledge that learners have thus acquired. Prerequisites of trigonometric ratios, the area, and sine and cosine rules were revised prior to the learners attempting the activities. Discussions of when would one utilize the sine rule, area rule and cosine rule had transpired. It was concluded and agreed upon that one would use the sine rule when given two sides and an angle that is not between the given sides or if two angles and a side are given. The cosine rule would be used if in a triangle all three sides or two sides and an included angle were given. In addition if a side of a triangle contains a square root in it then, one would most likely use the cosine rule.

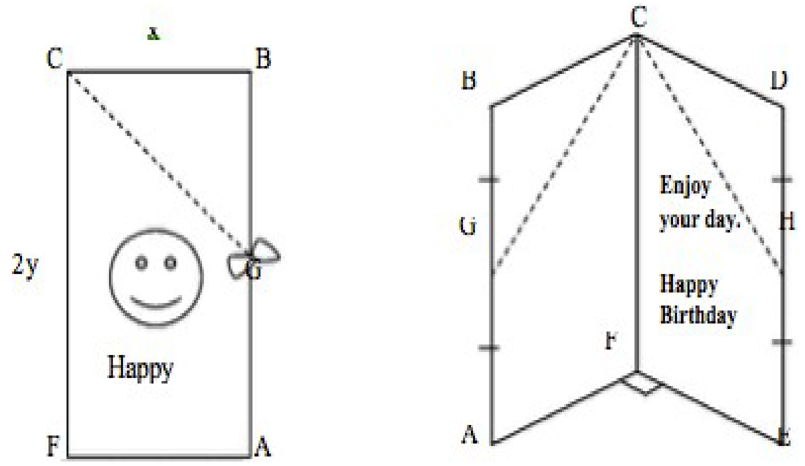
6.5.1 Analysis of question one

QUESTION 1 (NSC PAPER 2- FEB/MARCH 2012)

A rectangular birthday card was tied with a ribbon at the midpoints, G and H, of the longer sides. The card was opened to read the message inside and then placed on a table in such a way that the angle $A\hat{F}E$ between the front cover and the back cover of the card was 90°. The points G and H are joined by straight lines to the point C inside the card, as shown in the sketch.

Let the shorter side of the card, BC = x, and the longer side, CF = 2y

Prove that <GCH = $\frac{y^2}{x^2+y^2}$

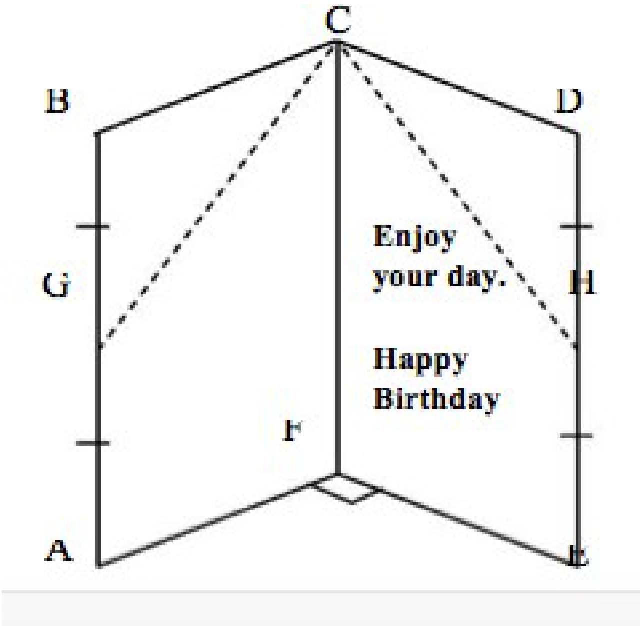


This is a multiple step question. This problem-solving question falls under cognitive level 4, which requires higher order thinking and the ability tovisualize and make significant connections and links between the various representations and information given. It is given that CF = 2y and learners would have to, therefore, make the following deductions:

- BG = y and AG = y since G was the midpoint

- Using the theorem of pythagoras side CG and side CH on the vertical plane could be calculated
- Visualise triangle CGH on the slant plane
- Link side GH on horizontal plane equal side AEon the slant plane.
- Given two sides and required to find an angle meant that the cosine rule had to be applied.
- A knowledge of surds was also required. Squaring a surd : $(\sqrt{x^2})(\sqrt{x^2}) = x^2$.

Look at Δ GBC and Δ CDH :



BC = x

GB = y

Using the theorem of Pythagoras

GC = $\sqrt{x^2 + y^2}$

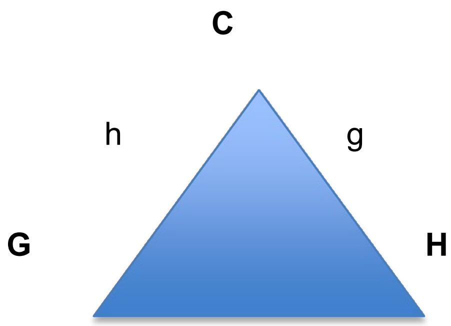
CD = x

DH = y

Using Theorem of Pythagoras, similarly

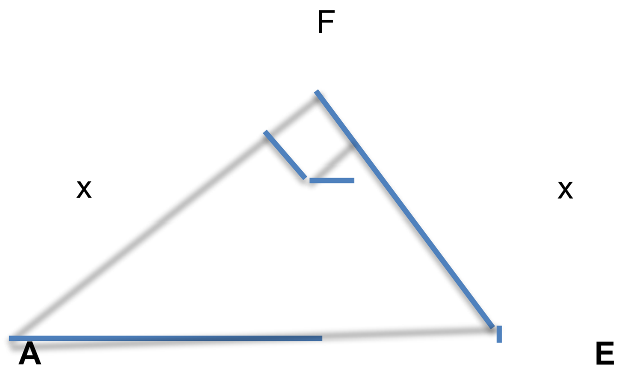
CH = $\sqrt{x^2 + y^2}$

Look at ΔCGH on the slant plane



GC = h = $\sqrt{x^2 + y^2}$ and CH = g = $\sqrt{x^2 + y^2}$ …… Pythagoras from above

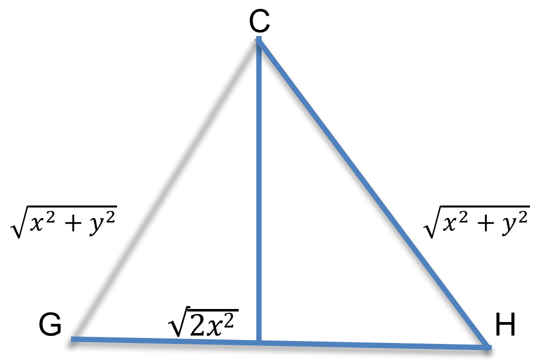
Link had to be made GH = c = AE…..from triangle on horizontal plane



AE = $\sqrt{x^2 + x^2}$ = $\sqrt{2x^2}$ ………Pythagoras

Link has to be made, side GH = c = AE

Now applying the cosine rule in ΔGCH



$c^2 = h^2 + g^2$ -2hg.cos c

$(\sqrt{2x^2}\ )^2 = (\sqrt{x^2 + y^2})^2 + (\sqrt{x^2 + y^2})^2 - 2(\sqrt{x^2 + y^2})(\sqrt{x^2 + y^2})$. cos c

cos c $2x^2$ = $x^2 + y^2 + x^2 + y^2 - 2(x^2 + y^2)$. cos c

$2x^2$ = $2x^2 + 2y^2 - 2(x^2 + y^2)$. cos c

cos c = $\frac{2x^2 - 2x^2 + 2y^2}{2(x^2+y^2)}$

$\cos c = \frac{y^2}{x^2+y^2}$ Knowledge of surds was required.

Table 6.2 Learner performances in question one

Question One	Cognitive demand level(K/RP/CP/PS)	Correctly answered	Partially correct	Incorrect	Not answered
1.	PS	4	11	0	1
		25%	**69%**	**0%**	**6%**

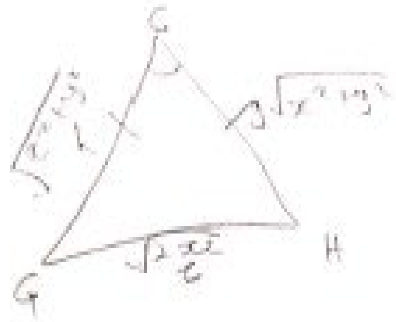
CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedures

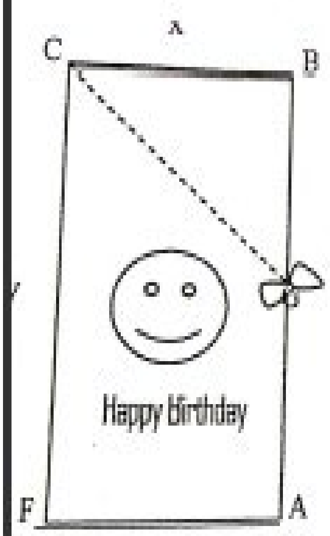
Only 4 learners (25%) of the learners were successful in applying the correct mathematical procedures and provided the correct proof. Eleven learners were categorized as partially correct and one learner did not attempt the question. The learners' errors were classified as conceptual errors or those errors that were due to non-conceptual understanding of the concept and procedural errors, which included the incorrect use of the procedure to solve the problem. Luneta & Makonye (2011) claim that procedural problems include inappropriate use of formula, application errors, which are errors that relate to misuse of rules and careless errors, which learners unknowingly make but can be corrected if learners are guided. In question one it was discovered that learners had a common misunderstanding of surds and many believed that $\sqrt{x^2+y^2} = x^2 + y^2$. In addition, the multiplication of surds in brackets proved to be a challenge and learners made another common mistake. The Theorem of Pythagoras also proved to be a challenge. The common errors where $GC^2 = x^2 + y^2$

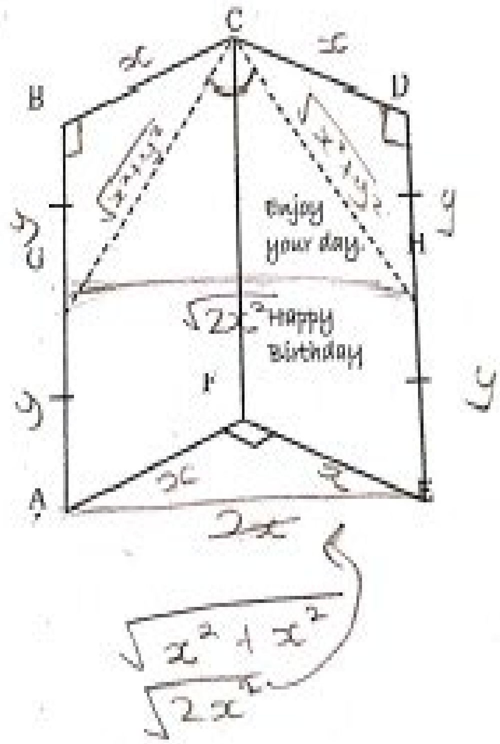
$$GC = \sqrt{x^2+y^2}$$
$$= x^2 + y^2$$

Incorrect substitution into the formula added to the error. Several learners were unable to multiply a binomial by a binomial that contained surds. It was evident that learners were unable to make cos Θ the subject of the formula. Several learners did not understand that terms in a fraction cannot be cancelled. The numerator had to be first factorized before a simplification could occur. On completion of the step that required substitution learners simply filled the gap in between and merely wrote equations that were riddled with errors and then finally wrote what was required to be proven.

Some learners were successfully able to calculate the length CG and AE and substitute in the correct formula. They correctly identified that the cosine formula had to be applied. After substitution the greatest challenge that many faced was squaring binomials that contained surds and factorizing to simplify and finally reach the desired outcome. It would appear that learners understood the trigonometry, however, fell short in their use of their algebra skills in solving and manipulating the equation.


C
G
H


C
B
F
A
Happy Birthday


C
B
D
G
H
F
A
Enjoy your day.
Happy Birthday
2x

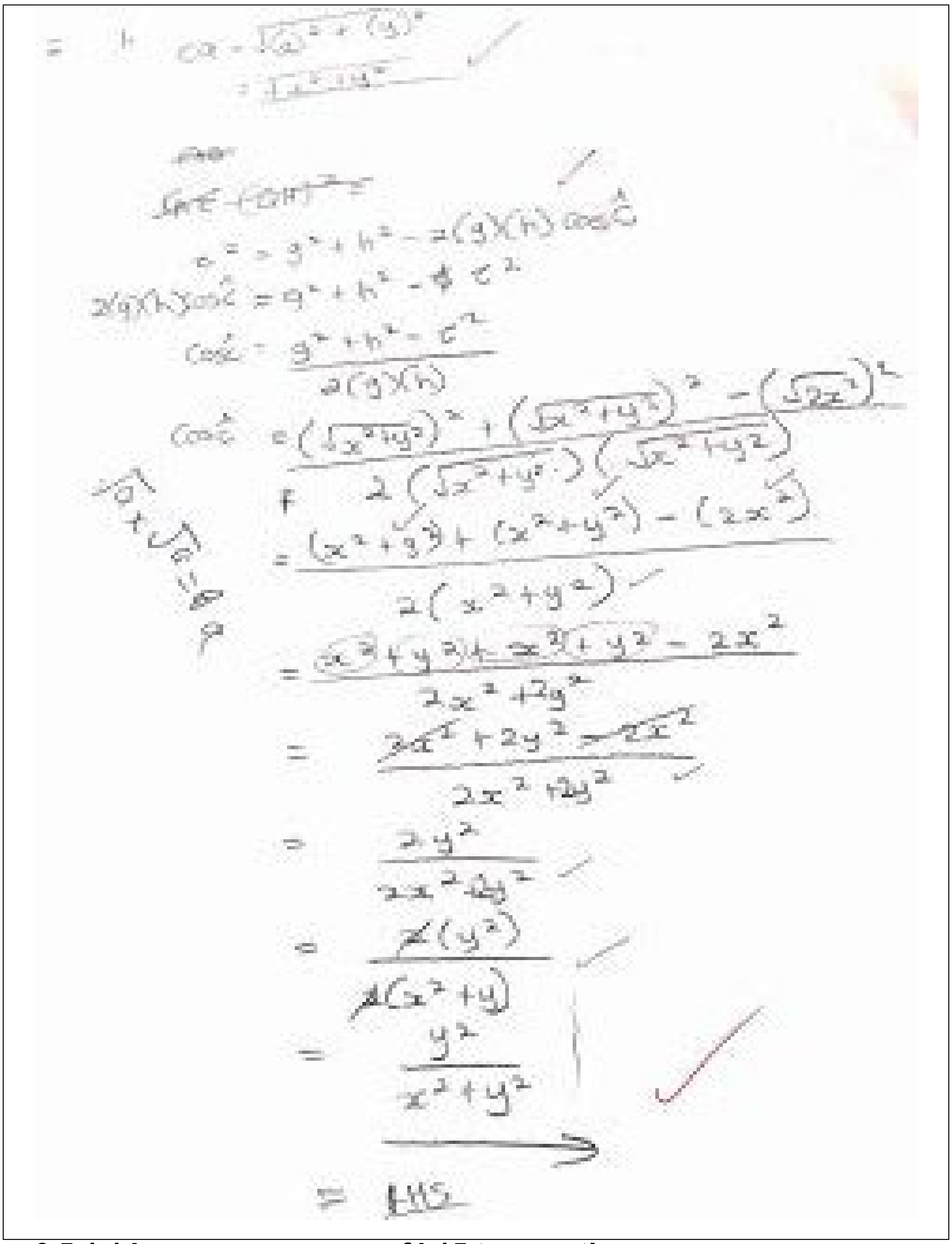


Figure 6.5.1.1 Learner response of L15 to question one

This learner displayed a good understanding of the question one as they calculated sides using Theorem of Pythagoras applied the cosine rule successfully. The learner had a good understanding of surds, algebraic skills were good and the learner was successful in the proof.

Researcher: I noticed you put additional information in the diagram, like 90^0 angles, x and y. Did the manipulatives help you?

L15: Yes the manipulatives helped me to visualize better and see things better in perspective. I was better able to see the triangles in the vertical plane, horizontal plane and the slope plane. Working in a group with my classmates was fun and enjoyable.

Researcher: Can you explain further?

L15: We put our ideas together.

Researcher: What do you mean?

L15: We could see that BC was x and CD was x and since G and H were midpoints, it implied that GB was y and DH was y too. Then by using the Theorem of Pythagoras we calculated GC and HC.

Researcher: I noticed you drew the triangle on the slant plane.

L15: Yes. After interacting with my mathematics model I drew triangle GCH from the slant plane.

Researcher: How did you get GH to be $\sqrt{2x^2}$?

L15: My group…..err and I looked carefully at the model and the diagram and….(pause) we realized there was a triangle on the horizontal plane, triangle AFE. After some time in this triangle we could see GH= AE and AE could be calculated.

Researcher: How?

L15: Err… mam since this is a birthday card BC=AF and CD=FE.

Researcher: I see you put x on AF and x on FE. You wrote 2x and then struck it of.

L15:Yes mam. I made a careless mistake and group members reminded me that I had to use the Theorem of Pythagoras. So you see mam that is how I calculated GH to be $\sqrt{2x^2}$. Does that make sense mam?

Researcher: Yes it does. Please continue and explain how you arrived at the proof.

L15: Well you see mam…. If you look at my triangle GCH, I had 3 sides and needed to find <C. I knew I had to use the cosine rule. I wrote the rule down and made the angle the subject of the formula.

Researcher: Did the surds complicate things?

L15: Yes mam…. You can see I wrote my simple rule $\sqrt{a}$ x $\sqrt{a}$ = a on the side to help me navigate around the surds.

Researcher: That is good to go back to basics. Please continue with your explanation

L15: Mam….. I then squared the surds and added like terms. Finally I factorized the numerator and the denominator and simplified the fraction to arrive at $GCH = \frac{y^2}{x^2+y^2}$

Researcher: Well done!

Learner fifteen's interview response to question one:

From the interview responses above this learner has attained mathematical procedural fluency (Kilpatrick et al., 2001) and has clearly demonstrated this in his/her solution. According to the revised Bloom's Taxonomy, Learner fifteen has reached higher order thinking. Hidayah *et al*. (2021) claims that a learner in the final stage implies that they have the skills in the preceding stages. Learner fifteen with the help of other learners had performed the steps cognitively to represent the solution and this is in keeping with Vygotsky's theory. Cognitive development begins with the use of physical actions to form schemas, which are then followed by the use of symbols. Piaget emphasized that learning involves both physical actions and symbols that represent previously performedactions.

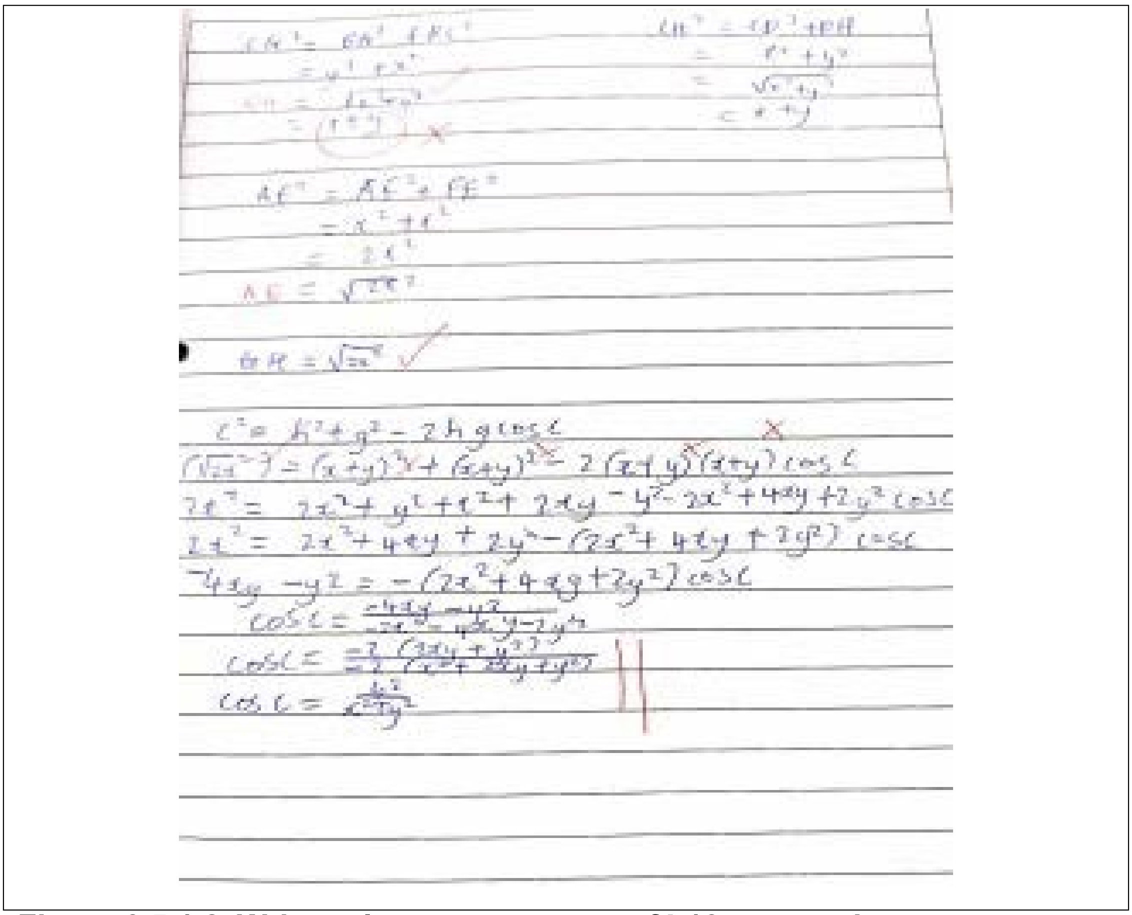


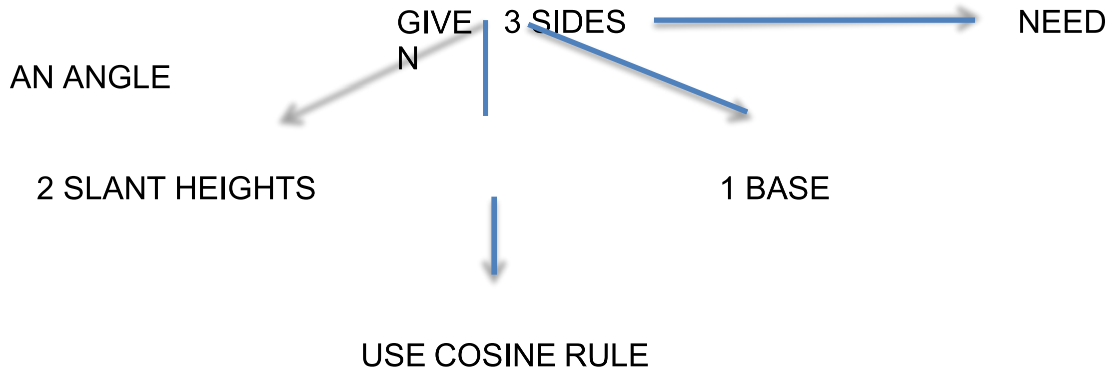
Figure 6.5.1.2 Written learner response of L13 to question one

Leaner thirteen was unable to apply the theorem of Pythagoras that contained variables. $\sqrt{x^2 + y^2} = x^2 + y^2$ was a common mistake. The learner was able tocalculate GH as this had only one term. The correct formula was selected however incorrect substitution resulted in not being able to logically prove the RHS = LHS.

There was no response that contained a totally incorrect answer. See table 6.1 above.

Observation of Group 1

Learners engaged with each other. Peer collaboration was encouraged by allowing learners to work on mathematical tasks given. There was a large amount of interaction among learners as they were encouraged to verbalise what they saw and thought. The learners were motivated to justify and provide explanations of their written answers. It was through the learners' comments, questions and answers that their mental constructions regarding three-dimensional trigonometric problems were assessed. One learner from the group queried which was the base. The other members of the group pointed out which was the base. Another learner suggested that they write down the rule. After discussions the group established what was given.

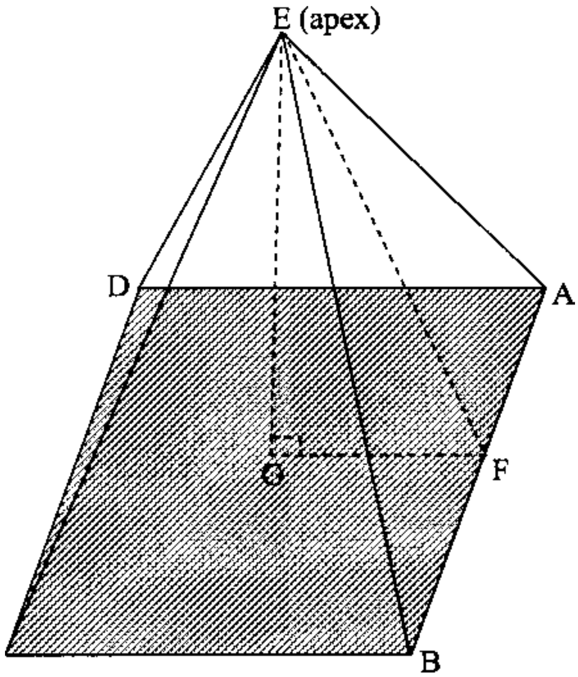


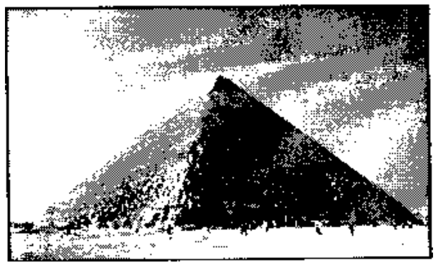
It can be concluded that in question one, learners appeared to understand the trigonometry of the question and correctly identified the rule to be applied. Unfortunately several learners were unsuccessful due to their poor algebraic skills.

6.5.2 Analysis of question two

QUESTION 2 (NSC PAPER 2-NOV 2013)

The Great Pyramid at Giza in Egypt was built around 2 500 BC. The pyramid has a square base (ABCD) with sides 232,6 metres long. The distance from each corner of the base to the apex (E) was originally 221,2 metres.





GREAT PYRAMID AT GIZA IN EGYPT

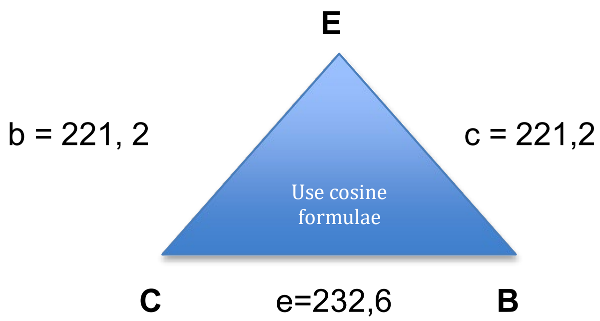
2.1 Calculate the size of the angle at the apex of a face of the pyramid (for example

$C\hat{E}B$). (3)

2.2 Calculate the angle each face makes with the base (for example $E\hat{F}G$, where

EF $\perp$ AB in ΔAEB). (6)

2.1 This is a routine procedure question. The question involved straight recall of application of cosine rule. Identification of correct formula, which was the cosine rule was applied to solve the problem. Then changing the subject of the formula to calculate the angle was required.



$e^2 = c^2 + b^2 - 2(b)(c) \cos e$

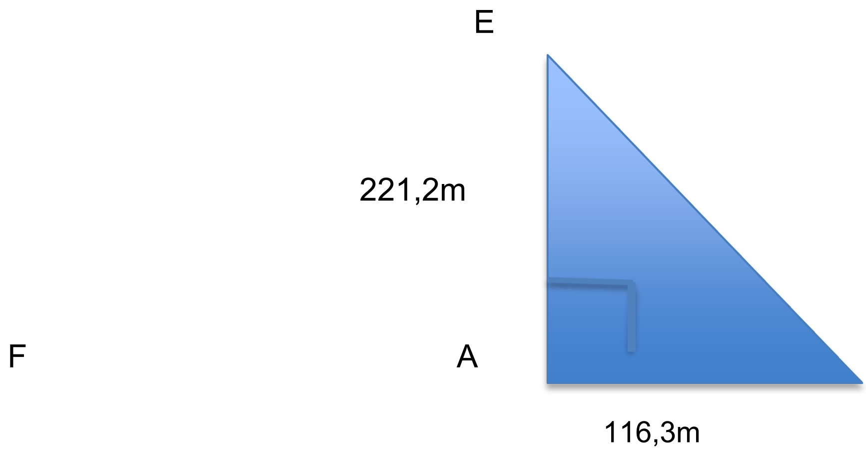
$(232,6)^2 = (221,2)^2 + (221,2)^2 - 2(221,2)(221,2) \cos e$

$\cos e = 0,45^0$

$e = 63,44$

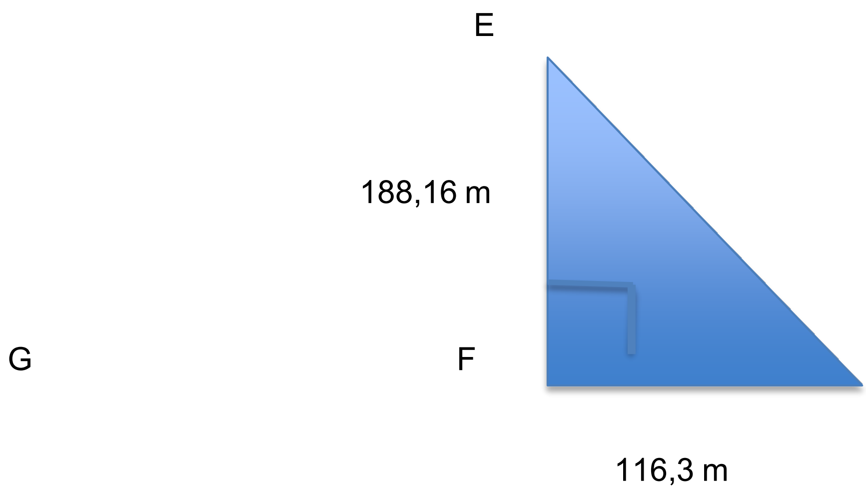
2.2 Problem solving-cognitive level 4. This is a non-routine problem. The problem is not necessarily difficult but higher order reasoning and processes are required. The problem needs to be broken down into its constituent parts. First learners would need to extract ΔFEA. In addition learners need to know that FA = $\frac{1}{2}$ (BA) = $\frac{232,6}{2}$ = 116,3 metres.

The slant height would need to be calculated using the theorem of Pythagoras.



FE = $\sqrt{221,2^2 + 116,3^2} = 188,16m$

Now learners would have to extract ΔEGF and use the value of FE that was calculated.



Since this is a right-angled triangle, trig ratios may be used.

$$\cos F = \frac{GF}{EF}$$

$$\cos F = \frac{116,3}{188,16}$$

$$< F = 51,82^0$$

Table 6.3 Learner performance in question two

Question One	Cognitive demand level(K/RP/CP/PS)	Correctly answered	Partially correct	Incorrect	Not answered
2.1	RP	15 (94%)	0 (0%)	1 (6%)	0 (0%)
2.2	PS	9 (56%)	6 (38%)	1 (6%)	0 (0%)

CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedures

From table 6.3 the results demonstrate that 94% of the learners were able to successfully answer question 2.1 which was a routine procedure question. Only one learner was unsuccessful in obtaining the correct answer to question 2.1. Question 2.2, which was a problem-solving question, was only answered correctly by 56% of the learners.

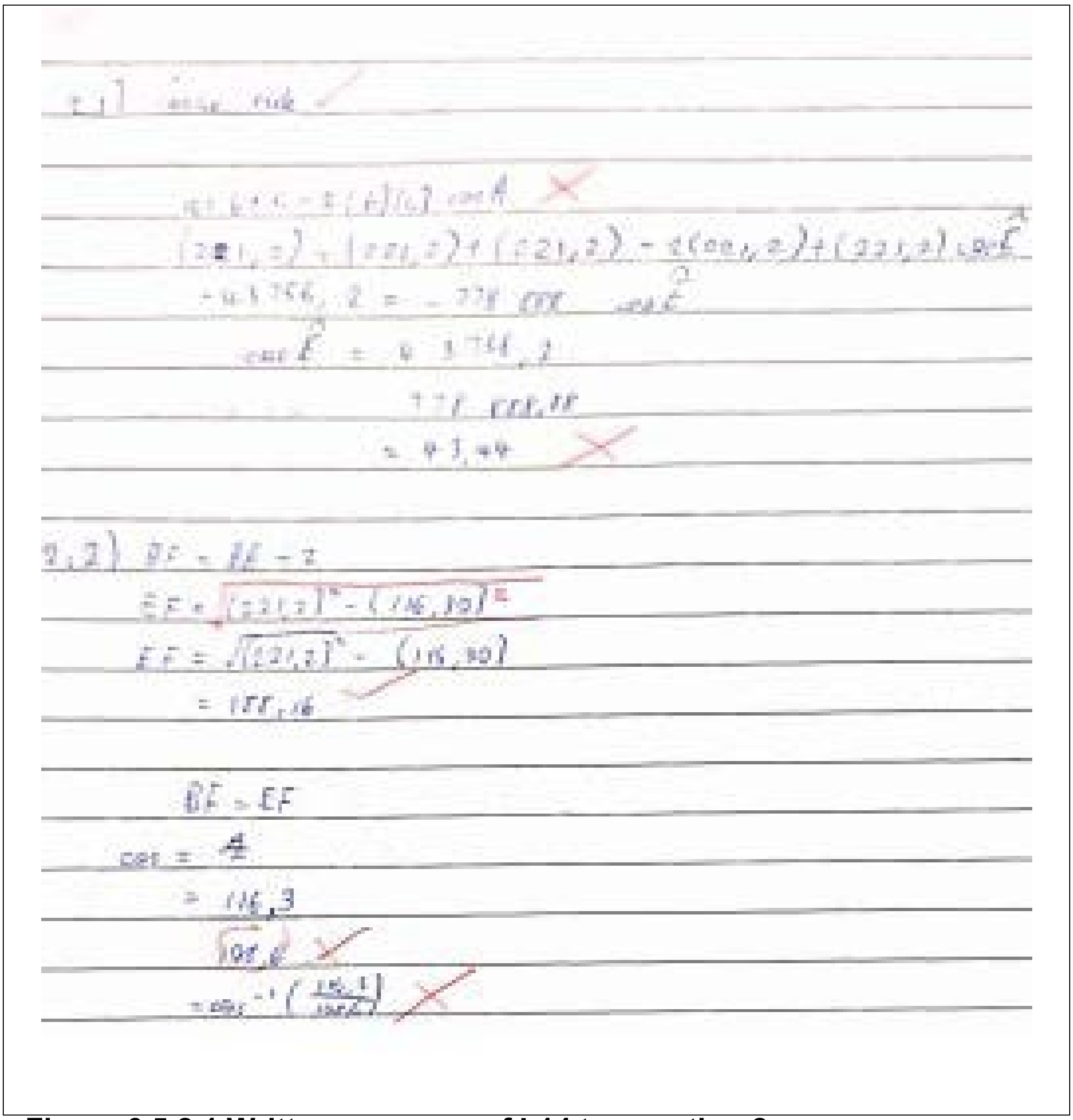


Figure 6.5.2.1 Written response of L14 to question 2

In question 2.1 Learner fourteen knew it was the cosine rule that was needed to be applied. Learner fourteen's response reveals lack of knowledge and was unable to write the cosine formula. In addition Learner fourteen substituted a wrong value for a. These errors resulted in Learner fourteen not being successful in solving this routine problem. In question 2.2 Learner fourteen disregarded the square root and squaring 116,30 but arrived at the correct answer for EF. Learner fourteen was unsuccessful in calculating an angle.

Incorrect use of trigonometric ratio resulted in incorrect answer. Learner fourteen displayed difficulty in this problem-solving question.

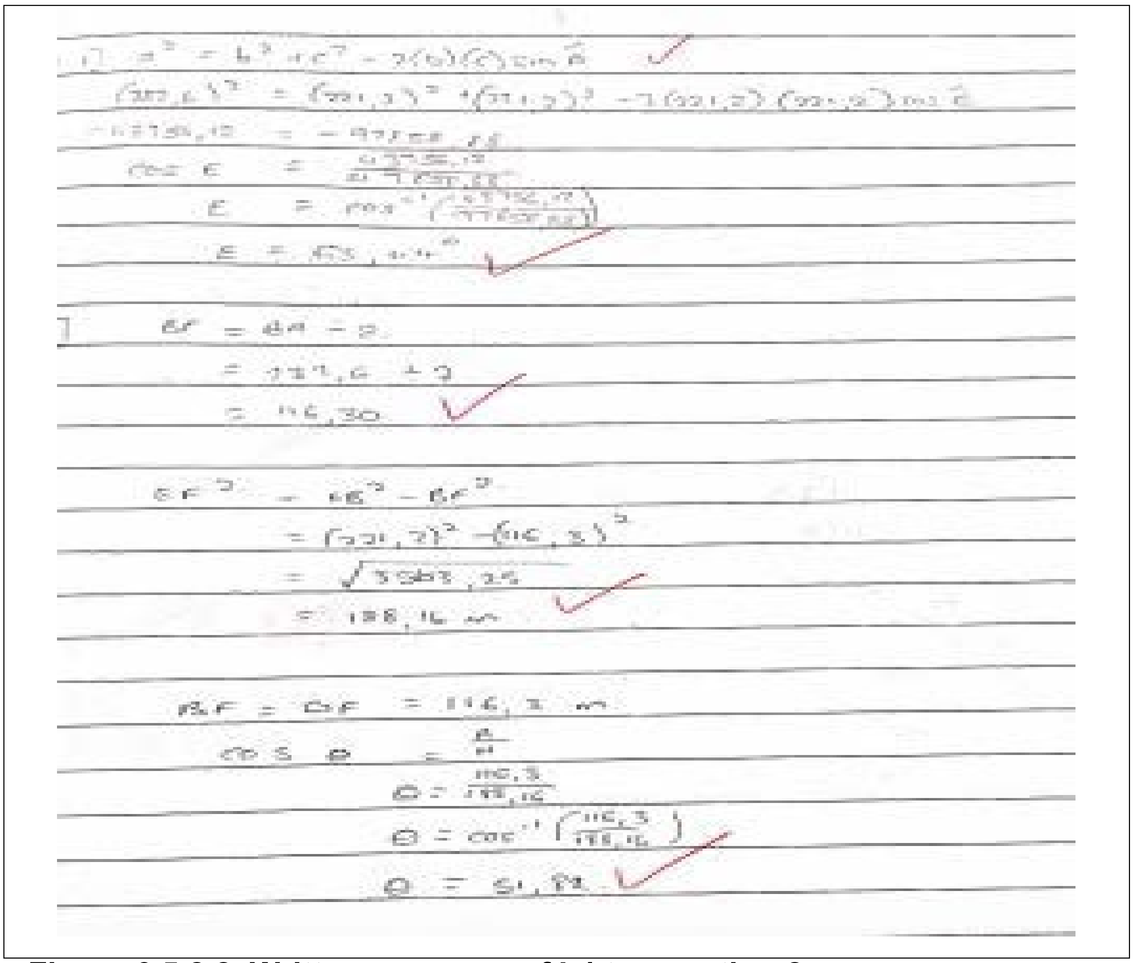


Figure 6.5.2.2 Written response of L4 to question 2

Learner four displayed an excellent knowledge of the cosine rule, substituted correctly, manipulated the equation, used the calculator correctly and was successful in obtaining the value for angle E. In question 2.2 once again Learner four written response reveals that that he/she was able to visualize and calculate sides BF and EF. Application of the correct trig ratio was used. Learner four displayed good calculator use and arrived at the correct answer. Learner four was able to answer routine procedure and problem solving type questions.

Researcher: What did you think of the problem when you first read it:

L4: Oh mam… it looked so complicated and difficult but then being in a group and having the…the… maths model, the manipulative in front of us helped.

Researcher: Please explain how the manipulative helped you to answer question 2.1.

L4: It was easier to find as visual made problem become alive and easier to see. My model helped me fill information on the diagram and it made it less confusing.

Researcher: Which trigonometric concept or rule did you decide to use and why?

L4: We decided to use the cosine rule because in our question they gave two sides of the triangle and in the rule the thing you need is either 2 or 3 sides which is

$a^2 = b^2 + c^2 - 2bc \cos a$. It's the best way to find answer. Researcher:

What were some of the challenges you experienced?

L4: When it came down to calculating my cosine rule, I was getting the wrong answer from the rest of my group. I used shift solve after and it was giving me the wrong answer. But then my team member showed me how to do it.

Researcher: How did you solve 2.2?

L4: Well … you needed to know that FA = ½ BA. EA was given. Then mam this was a right -angled triangle and I was able to use CAH, Cos θ = $\frac{A}{H}$ and solved for the angle.

Researcher: What did you enjoy the most?

L4: Mmmm ... (pause) I enjoyed working with the maths model and with my friends in the group. I was comfortable and we exchanged ideas. I was not scared to give the wrong answer. The model helped me see things better.

Learner 4 –L4 interview response to question two

The learner's strategic competence (Kilpatrick *et al.,* 2001) facilitated the decomposition of this complex multi-procedural three - dimensional trigonometric problem.

Learner four interacted with the manipulative and it is clear that Kolb's Cycle was applied. Learner four experienced concrete experience, then moved on to reflective observation followed by abstract conceptualization and finally active experimentation.

Learner four indicated that he enjoyed working in a group and this concurs with Munday (2019) who asserts that working with peers allows for discussions to occur and this results in the provision of scaffolding to learners. Learning does not occur alone (Muijs & Reynolds, 2015) but is socially constructed through interactions with parents, peers and educators. The use of manipulatives encourages group and discussion to build construct social learning situations. Vygotsky advocated that social that social interaction was crucial in the learning process. Vygotsky's social cultural theory is relevant to this study as L4 was clearly involved in social interaction between learners in the group and educator and the exchange of thoughts.

The data obtained from learner response to question two revealed that no responses where partially correct. Refer to table 6.3 above.

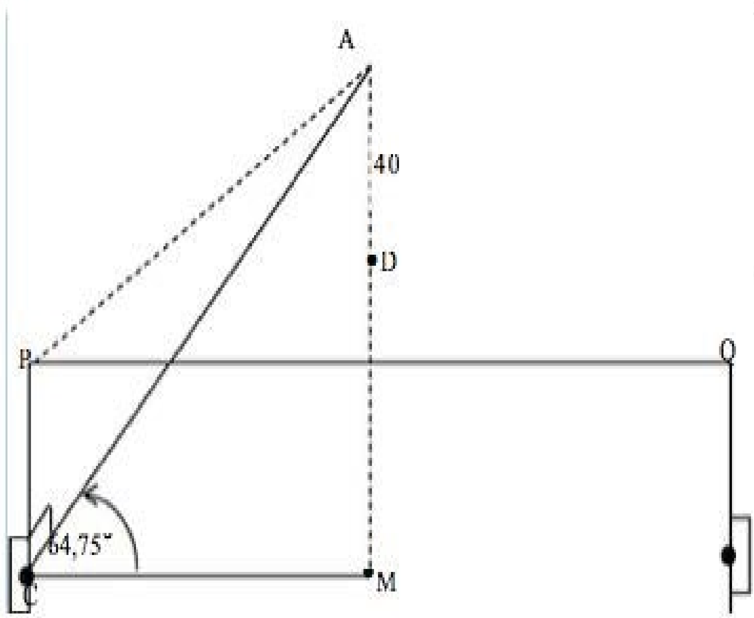
Figure 6.5.2.3 Learners engage with the manipulative and work with the activity sheet.

6.5.3 Analysis of question Three

Question three requires routine procedures to solve the question.

QUESTION 3 (NSC PAPER 2-Nov 2010)

The angle of elevation from a point C on the ground, at the centre of the goalpost, to the highest point A of the arc, directly above the centre of the Moses Mabhida soccer stadium, is 64,75°. The soccer pitch is 100 metres long and 64 metres wide as prescribed by FIFA for world cup stadiums. Also AC ⊥PC. In the figure below PQ = 100 metres and PC = 32 metres.



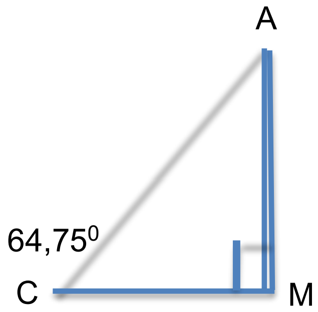
3.1 Determine AC. (3)

3.2 Calculate $P\hat{A}C$. (3)

3.3 A camera is positioned at point D, 40 metres directly below A. Calculate the distance from D to C. (4) **[10]**

Solution

3.1 Need to extract Δ CAM and calculate CM. Identify cos $\theta = \frac{adj}{hyp}$ trigonometry ratio. Perform familiar procedures. Manipulation of formula is needed



CM = $\frac{1}{2}$(100) = 50 m

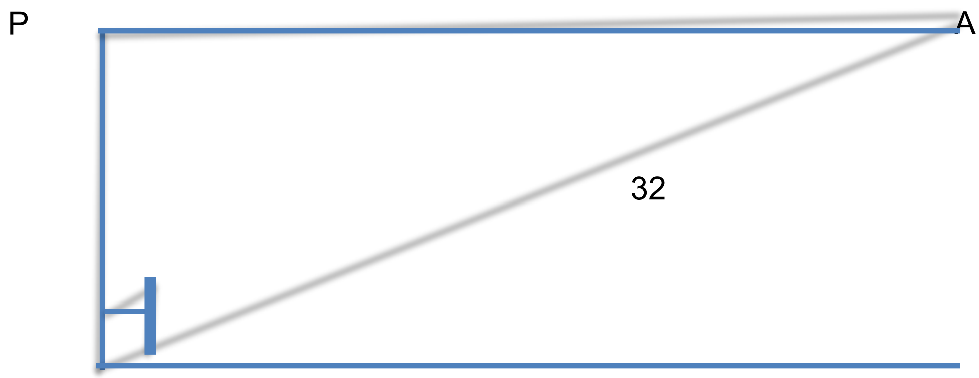
cos θ = $\frac{CM}{AC}$

cos 64,75 = $\frac{50}{AC}$

AC = $\frac{50}{\cos 64{,}75^0}$

= 117,21

3.2 Cognitive level 2. Requires extraction of triangle PAC.

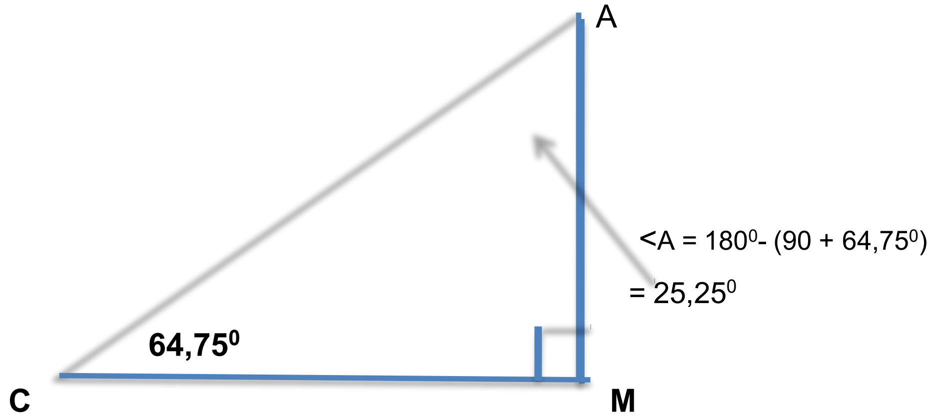


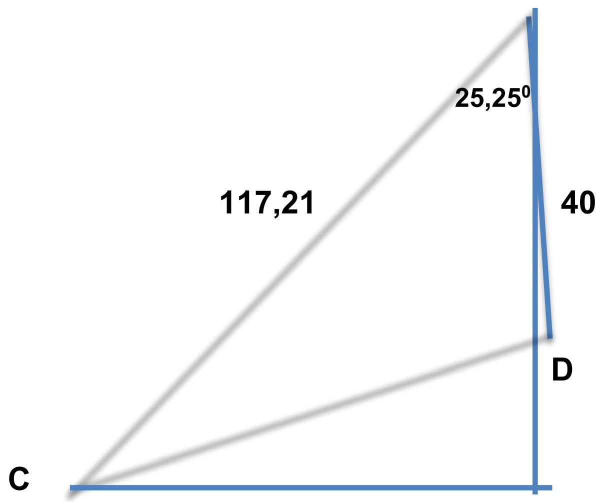
$\tan \theta = \frac{opp}{adj}$

$\tan < PAC = \frac{32}{117,21}$

$< PAC = 15,27$

3.3





Use cosine rule

$CD^2 = 117,21^2 + 40^2 - 2(117,21)(40) \cos 25,25^0$

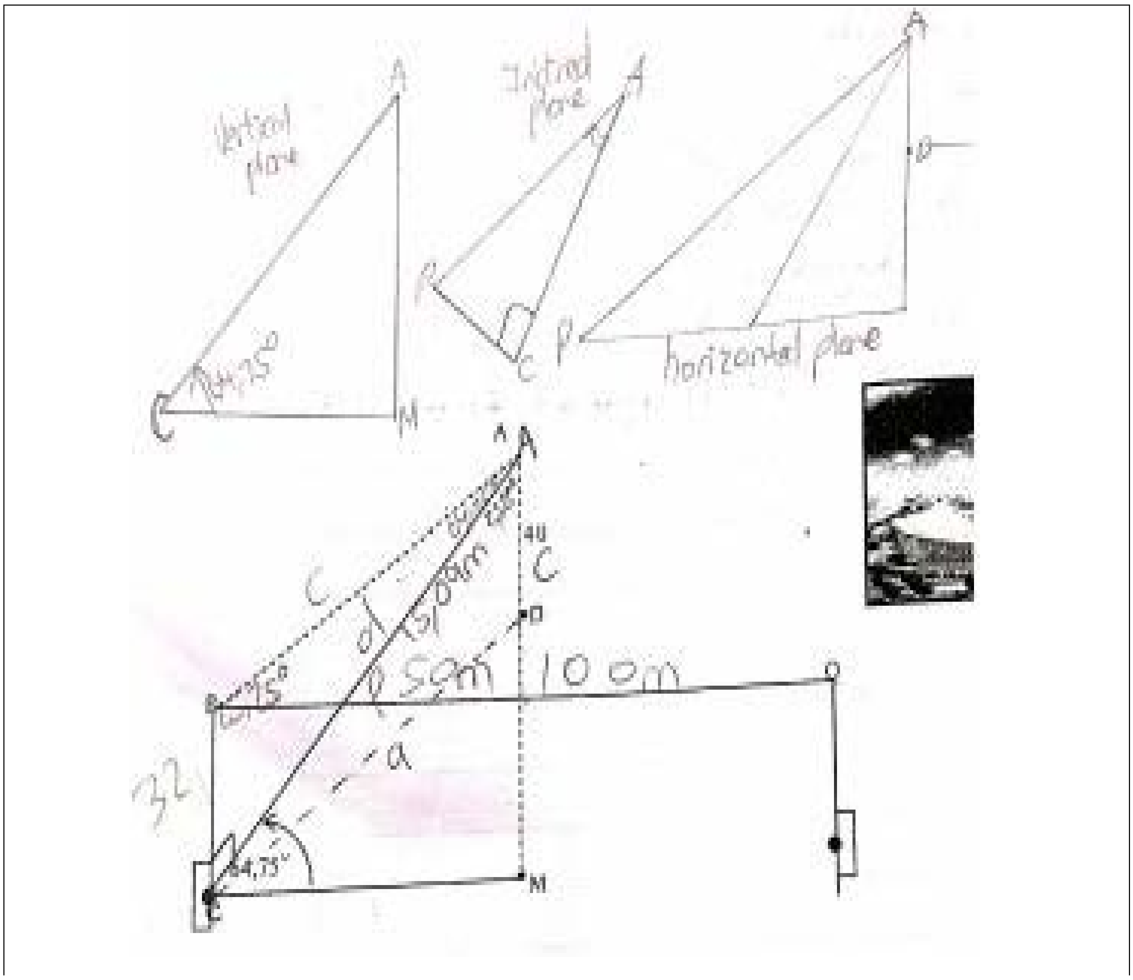
$= 6857,29$

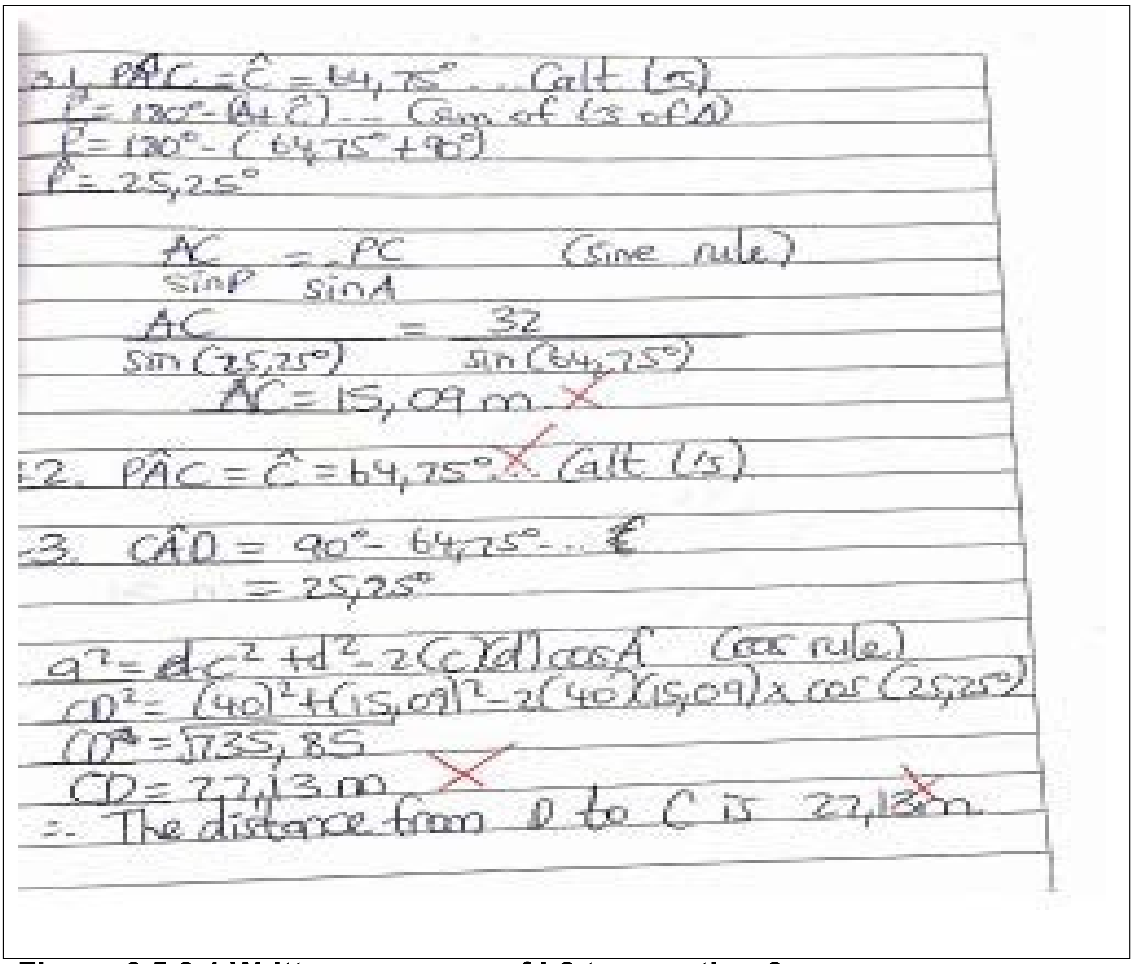
$CD = 82,86 \text{ m}$

Table 6.4 Learner performance in question three

Question One	Cognitive demand level (K/RP/CP/PS)	Correctly answered	Partially correct	Incorrect	Not answered
3.1	*****	15 (94%)	0 (0%)	1 (6%)	0 (0%)
3.2		14 (87%)	0 (0%)	2 (13%)	0 (0%)
3.3		13 (81%)	2 (13%)	1 (6%)	0 (0%)

CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedure





3.1 $P\hat{A}C = \hat{C} = 64,75°$ ... (alt ∠s)
$\hat{P} = 180° - (\hat{A} + \hat{C})$ ... (sum of ∠s of Δ)
$\hat{P} = 180° - (64,75° + 90°)$
$\hat{P} = 25,25°$

$\frac{AC}{\sin P} = \frac{PC}{\sin A}$ (sine rule)
$\frac{AC}{\sin(25,25°)} = \frac{32}{\sin(64,75°)}$
$AC = 15,09$ m

3.2 $P\hat{A}C = \hat{C} = 64,75°$ (alt ∠s)

3.3 $C\hat{A}D = 90° - 64,75°$ ... [illegible]
$= 25,25°$

$a^2 = c^2 + d^2 - 2(c)(d)\cos A$ (cos rule)
$CD^2 = (40)^2 + (15,09)^2 - 2(40)(15,09) \times \cos(25,25°)$
$CD^2 = \sqrt{735,85}$
$CD = 27,13$ m
∴ The distance from D to C is 27,13m

Figure 6.5.3.1 Written response of L8 to question 3

In question 3.1 immediately we see Learner eight making an error when she claims < PAC = 64, 75^0 and uses the reasoning of alternate angles. No parallel lines are given and the angles are on different planes. Learner eight was unable to extract Δ CAM and calculate CM. L8 failed to identify cos θ = $\frac{adj}{hyp}$, trigonometry ratio had to be applied. Learner eight also displayed the misunderstanding in question 3.2 and stated alternate angles. Required extraction of triangle PAC on the vertical plane was required. In question 3.3 L8 was correct in choosing the cosine rule, however error was carried forward from question 3.2 and 3.1.

Researcher: I noticed you made a construction CD. Then you assumed the lines were parallel I notice you used alternate angles in the reasoning.

L8: Why mam? ...I thought that was correct.

Researcher: Look carefully. You used entire angle of 64.75^0. That is not correct. Golden rule in mathematics" You cannot assume"

L8:..(learner chuckles)… I see now what I did was incorrect.

Researcher: I noticed you drew triangles in the different planes. However I do not agree with the one on the horizontal plane. Explain what you did I question 3.2.

L8: I calculated <P which I can now see it is wrong. I then used sine rule. Eish mam it was difficult to answer this problem.

Researcher: I also observe that in question 3.2 you made same error regarding alternate angles and parallel lines. I noticed you wrote 50m and 100m

L8: Yes mam. I knew you must halve 100m.

Researcher: Good thinking. You were able to select cosine rule, but there was incorrect substitution, which resulted in an incorrect answer.

L8: Yes mam. I will try harder next time.

Researcher: What were some of the challenges experienced?

L8: I confused the planes. My understanding of alternate angles was not totally correct.

Researcher: How did the manipulative help you?

L8: The mathematics model helped me see better and I was allowed to touch the model. I also enjoyed working with my group and with the mathematics model. It helped me see things better.

I like to see more teachers use manipulatives.

Researcher: Yes more teachers should incorporate mathematics models in their lessons Thank you.

Learner 8- L8 response to interview question three

It can be seen that Learner eight engaged (Jones and Tiller, 2017) with the manipulative as he/she has drawn different triangles in the various planes. This is where the metamorphosis from seeing and doing to reflecting can embed the learning into real time absorption of materials and mathematics. Learner eight enjoyed looking and touching the manipulative and this is consistent with Hunt *et al.* (2011) who state that that the use of manipulatives allow for information to be received visually and kinesthetically. Manipulative support education targets in more than one sense organ (Tertemiz, Celik and Dogan, 2014).

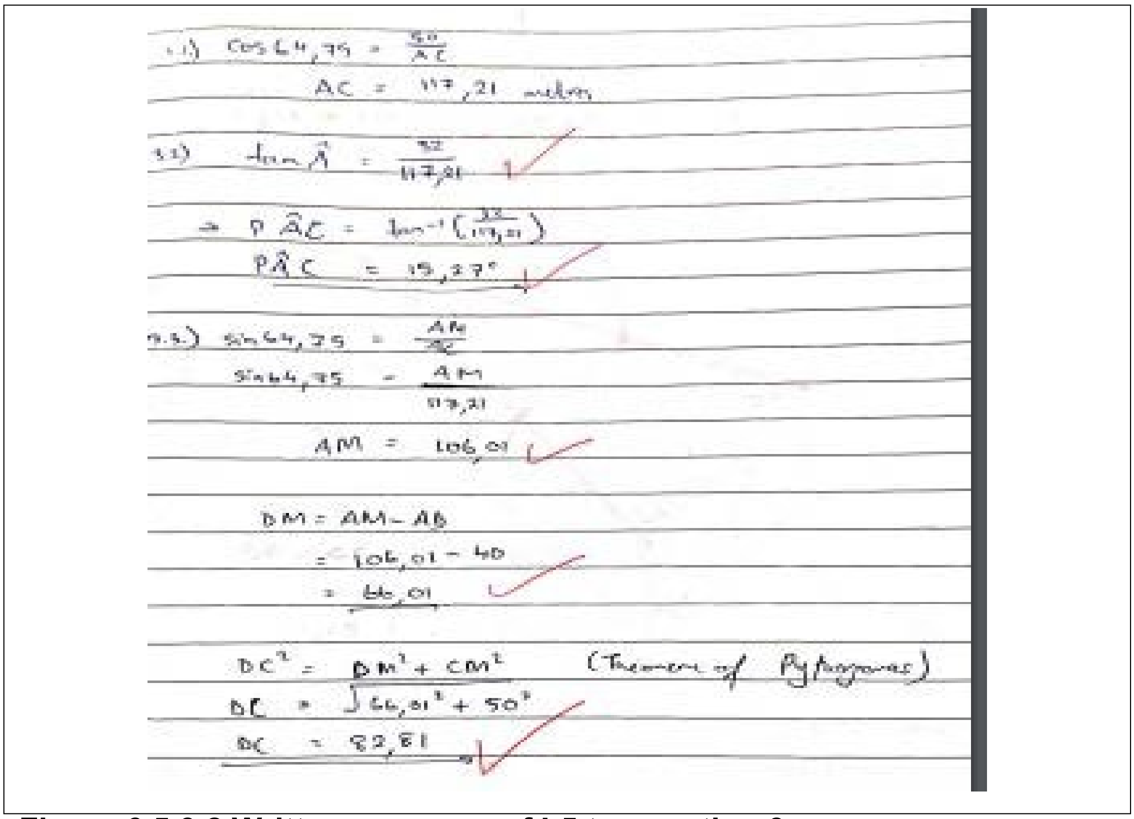


Figure 6.5.3.2 Written response of L5 to question 3

Learner five displayed a sound knowledge and application of trig ratios. Learner five extracted Δ CAM and calculated CM. Learner five was able to identify cos θ = $\frac{adj}{hyp}$ trigonometry ratio, perform familiar procedures and manipulate theof formula to calculate AC. Once again in question 3.2 Learner five was successful in extracting triangle PAC, and then applying the tan θ = $\frac{opp}{adj}$

ratio. In question 3.3 Learner five clearly displayed a through understanding and was able to use the manipulative to visualize and correctly perform a multiple step solution to calculate magnitude of AM and DM and finally apply the theorem of Pythagoras to calculate the magnitude od DC.

The data obtained from learner responses to question three revealed that no responses where partially correct. Refer to table 6.4 above.

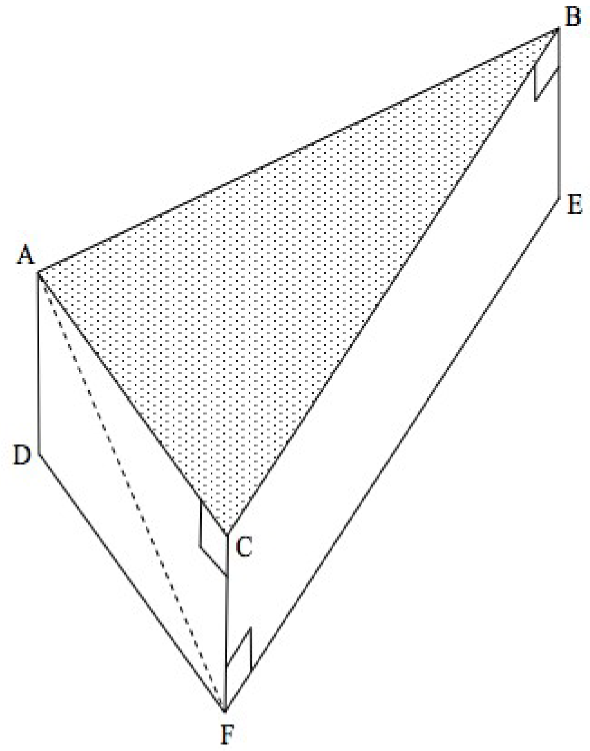
Figure 6.5.3.3 Learners engage with the manipulative and work with the activity sheet.

6.5.4 Analysis of question Four

QUESTION 4(NSC PAPER 2-Nov 2011)

The figure below represents a triangular right prism with

BA = BC = 5 units, < ABC = 50° and <FAC = 25°.

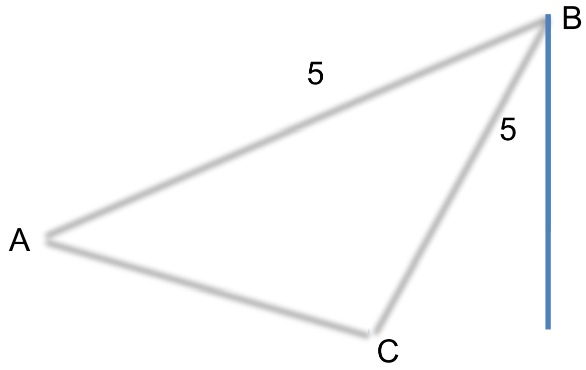


4.1 Determine the area of Δ ABC.

4.2 Calculate the length of AC.

4.3 Hence, determine the height FC of the prism.

Given two sides and an angle the area rule can be applied or an alternate method half base x height x 2 (two triangles)



Area = $\frac{1}{2}$ a.c. sin b

Area Δ ABC = $\frac{1}{2}$ AB X BC. sin c

= $\frac{1}{2}$ (5) . (5) sin 50^0

=9, 58 units

4.2 Level of this question can be classified as Knowledge- cognitive level 1. Straight recall, since two sides and an included angle are given. This would require the identification of the correct formulae, which is the cosine rule or sine rule, could be applied.

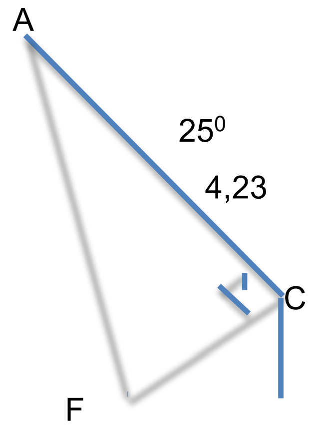
$AC^2 = 5^2 + 5^2 -2(5)(5) \cos 50^0$

AC^2 = 17,8606

AC = 4, 23 units

4.3 Level of this question can be classified as a routine procedure. Requires conceptual understanding. Learners need to identify and extract Δ ACF. Hence would imply use of previous answer so learners would be required to reason and carry forward the answer obtained for the length of AC in question 4.2. Now learners need to apply the correct trigonometry ratio to obtain height FC. The trigonometry ratio to be utilized would be:

tanθ = $\frac{OPP}{ADJ}$



tan 25^0 = $\frac{CF}{AC}$ ← Carried forward

CF = tan 25^0 x AC ………make CF subject of the formula.

= tan 25^0 x 4,23

= 1, 97 units

or alternate method

Some learners may opt to use the sine rule

< AFC = $180^0 - (25^0 + 90^0) = 65^0$

$$\frac{FC}{sssn25^0} = \frac{4,23}{sssn65^0}$$

FC = $\frac{4,23\ XX\ sssn25^0}{sssn65^0}$

= 1,97 units

Table 6.5 Learner performance in question four

Question One	Cognitive demand level(K/RP/CP/PS)	Correctly answered	Partially correct	Incorrect	Not answered
4.1	K	16 (100%)	0 (0%)	0 (0%)	0 (0%)
4.2	K	16 (100%)	0 (0%)	0 (0%)	0 (0%)
4.3	RP	15 (94%)	1 (6%)	0 (0%)	0 (0%)

CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedures

Question 4 was level one and level two-type question and the results in table 6.4 demonstrate that question 4 was answered well by most of the learners.

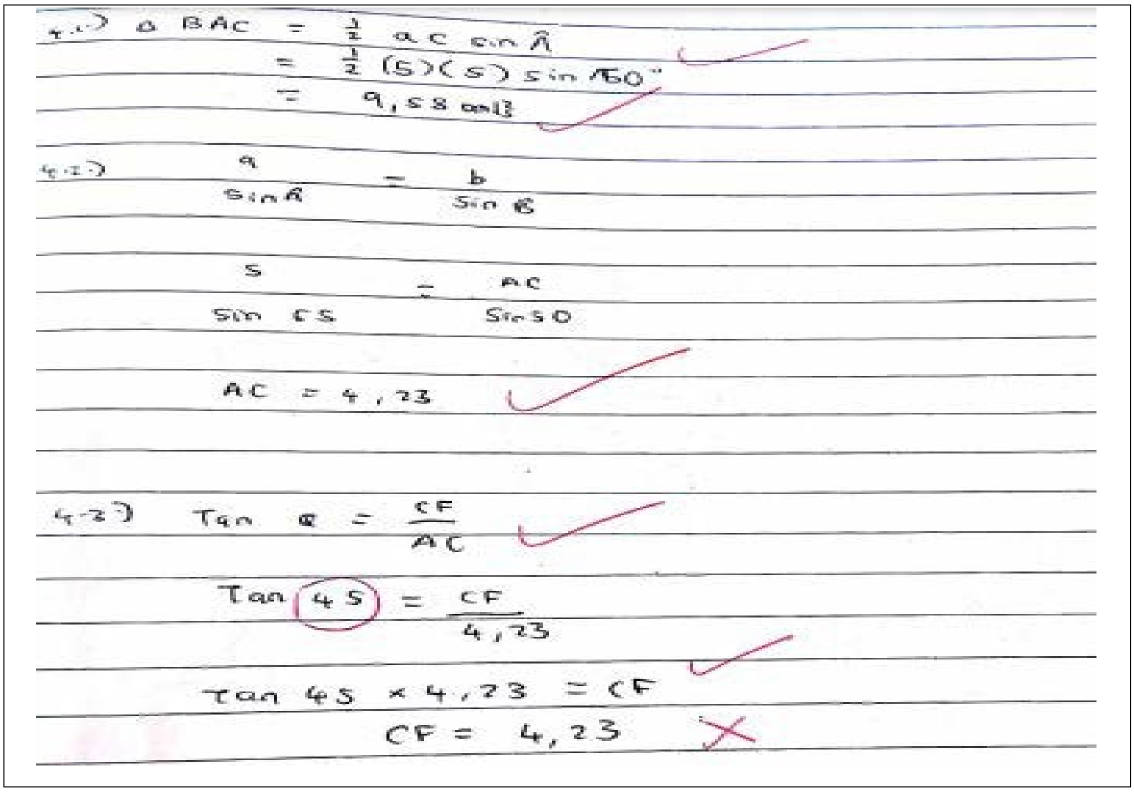


Figure 6.5.4.1 Written response of L11 to question 4

Learner eleven for question 4.1 above was able to select the correct formula, substitute correctly and arrive at the correct answer. In question 4.2, Learner eleven executed the problem correctly by applying the sine rule. However, in question 4.3, Learner eleven used an incorrect angle of 45^0 instead of 25^0, which resulted in the incorrect height of FC of the prism. Learner eleven on the diagram had indicated 25^0 but may have made a careless mistake. In this question as table 6.5 indicated that this was a knowing (level 1) and routine procedure (level 2) type of question.

Researcher: What trigonometric concepts or rules did you need to solve the question? L11: Err…. I needed to use the area rule, the sine rule and simple trig ratios. Researcher: What are simple trig ratios? L11: My "baby trig". In a right angled triangle sin,cos,tan. SOH CAH TOA mam ! Researcher: OH ok! Can you please explain how you answered question 4.1. L11: I had to use the area rule in order to work out area because two sides and an angle was given. This angle is formed between both arms. Researcher: How did you answer 4.2 L11: Well I had two angles and one side and needed to calculate another side, so I guess I had to use the sine rule mam. Researcher: Well done! I want to know why did you write tan 45^0 instead of 25^0. L11: I think I was thinking that the diagonal bisects the angles in at the corner. But that does not apply to rectangles. It works for squares. I think I was over thinking and made a careless mistake mam. You can see that I cross - multiplied correctly but with the wrong angle. Researcher: What were some of the challenges you experienced? L11: Looking at the area rule in a specific order and determining which side to use. Also mam… when using the sine rule trying to know what to substitute in it. Researcher: Did the use of manipulatives make it easier or helped you to solve the problem?

L11: Yes mam. Reason being it helps me identify each triangle or shape with its own angle individually. So when I see the model I am able to separate and identify and solve. I was able to touch and move the model around and that was nice mam.

Researcher: Thank you!

Learner L11 interview response to question 4

Learner eleven explained their experience with the manipulative and the enjoyment encountered and this is in keeping with claims by Carbonneau, Wong & Borysenko (2020) who explain that the use of concrete manipulatives in classrooms provides opportunities for learners to interact physically with abstract content, which they normally would be unable to visualize or touch. Learners obtain enjoyment when taught Mathematics through participatory and interactive ways that use manipulatives (Choden & Chalermnirundorn, 2021; Day & Hurell, 2019; Stiegelmeier and Moore, 2019; Strom, 2009; Reimer & Moyer, 2005). The use of manipulatives (Kontas, 2016) discloses the idea that they are beneficial for concretizing abstract topics. An increased usage of manipulatives in teaching mathematical concepts leads to a better understanding when manipulatives are used to help learners incorporate their knowledge and link them with their thoughts.

Written response of L6 to question 4

Learner six was able to visualize better with the 3D manipulative and then proceeded to highlight triangle ABC in green and triangle ACF in pink. L6 was better able to see the triangles. She executed question 4.1 and 4.2 well by applying the area rule and cosine respectively. In question 4.3 she selected the correct trigonometric ratio, substituted correctly and was able to successfully manipulate the equation to calculate the height FC of the prism.

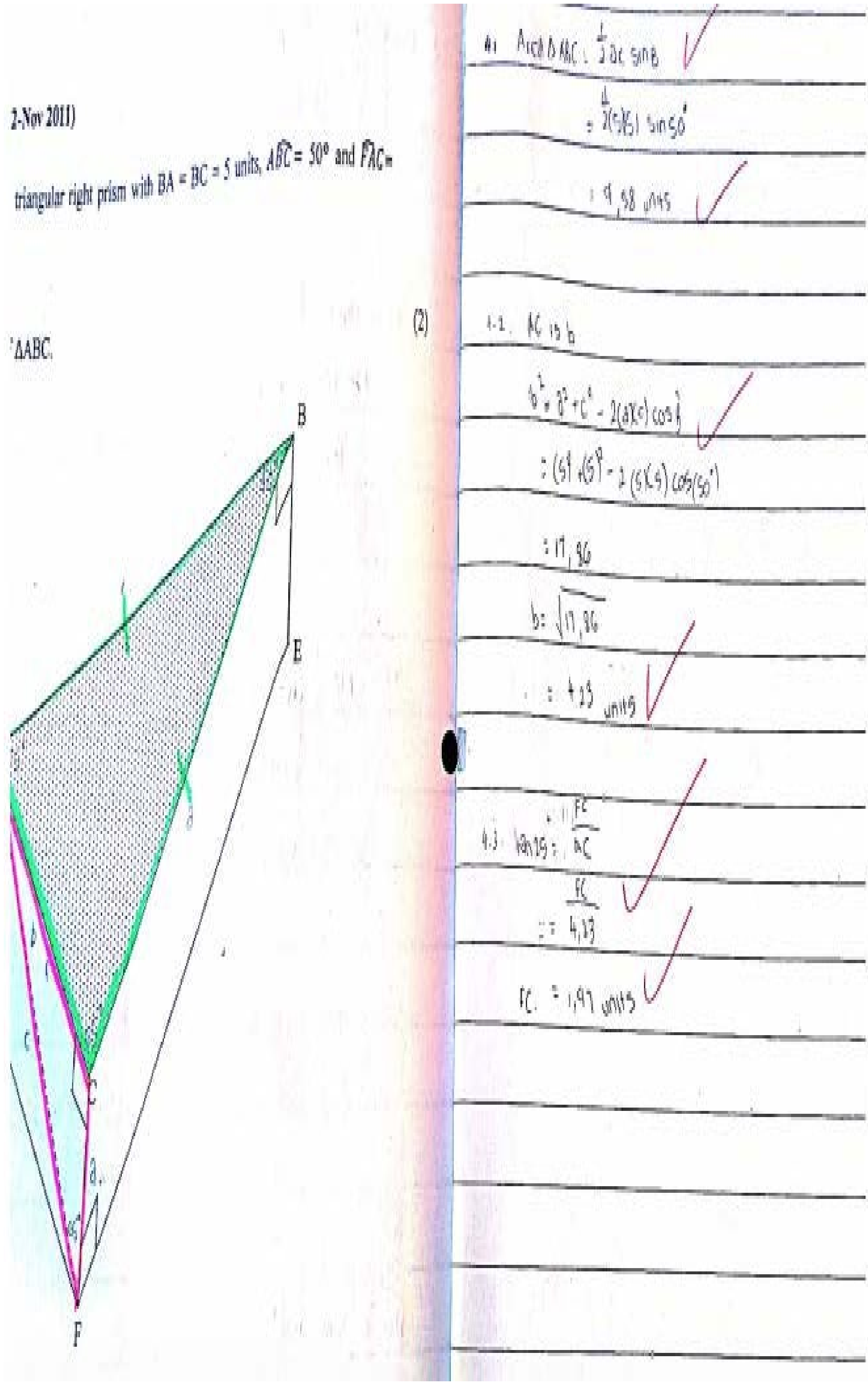


Figure 6.5.4.2 **Written response of L6 to question 4**

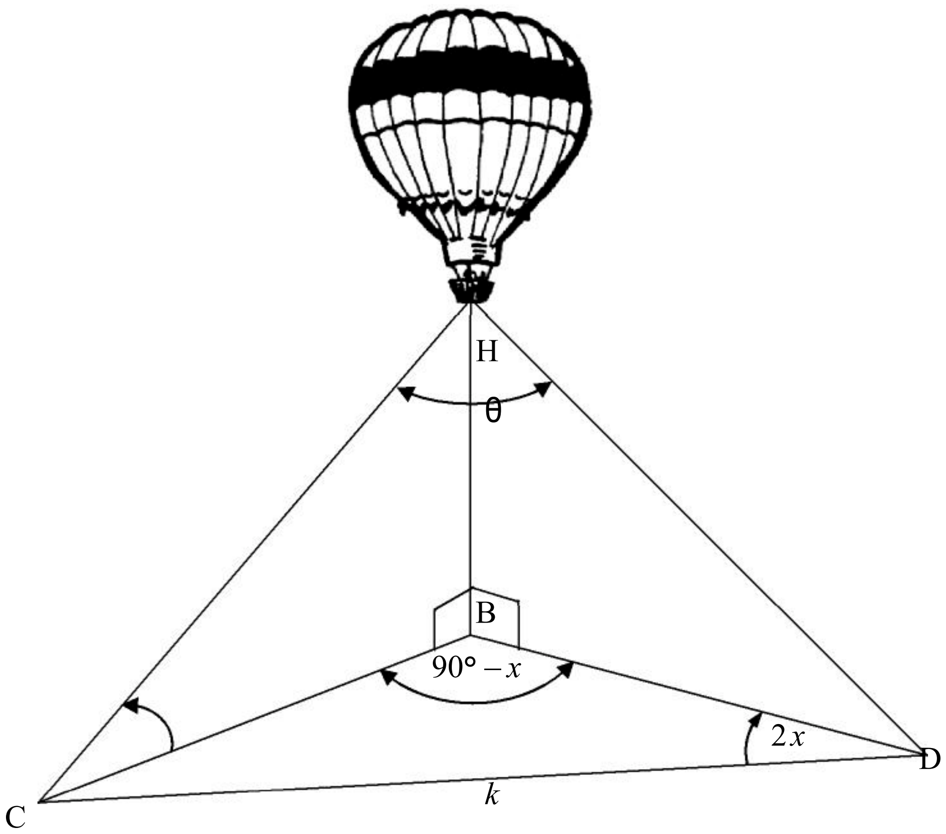
Figure 6.5.4.3 Learners engage with the manipulative and work with the activity sheet.

The data obtained from learner response to question four revealed that no responses where partially correct or incorrect. Refer to table 6.5 above.

6.5.5 Analysis of question Five

QUESTION 5 (NSC PAPER 2-NOV 2012)

CDB = $2x$ and CB D = $90° - x$. The distance between C and D is k metres.



A hot-air balloon H is directly above point B on the ground. Two ropes are used to keep the hot-air balloon in position. The ropes are held by two people on the ground at point C and point D. B, C and D are in the same horizontal plane. The angle of elevation from C to H is x.

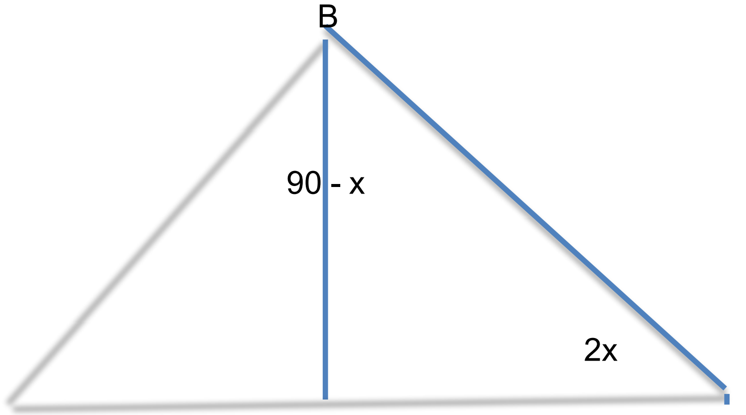
5.1 Show that CB = $2k \sin x$.
(5)

5.2 Hence, show that the length of rope HC is $2k \tan x$.
(3)

5.3 If $k = 40$ m, $x = 23°$ and HD = 31,8 m, calculate θ, the angle between the two ropes.
(4)

This problem involves complex calculations and higher order reasoning.

5.1 Use ΔCBD on the horizontal plane.



Given two angles and a side and required to find a side would imply the use of the sine rule.

$$\frac{CB}{SIN\ D} = \frac{CD}{SIN\ B}$$

$$\frac{CB}{SIN\ 2X} = \frac{K}{SIN(90^0 - X)}$$

use of the reduction formula and knowledge of double angles are required.

$$\frac{CB}{2SINX..COOSX} = \frac{K}{COOS}$$

CB = $\frac{k.2 \sin x.\cos x}{\cos x}$

CB = 2k. sin x

CB = $\frac{k.2 \sin x.\cos x}{\cos x}$

CB = 2k. sin x

CB = $\frac{k.2 \sin x.\cos x}{\cos x}$

CB = 2k.
sin x

CB = $\frac{k.2}{\sin x . \cos x}$

c
o
s
x

CB = 2k.
sin x

CB = $\frac{k.2}{\sin x . \cos x}$

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o
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CB = 2k.
sin x

CB = $\frac{k.2}{\sin x . \cos x}$

c
o
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x

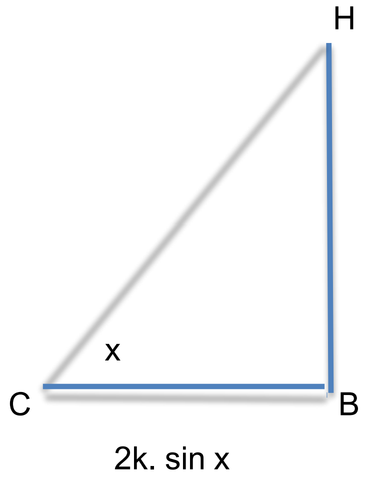
CB = 2k.
sin x

CB = $\frac{k.2}{\sin x . \cos x}$

c
o
s
x

CB = 2k.
sin x

5.2 Extract ΔCHB on the vertical plane.



$\cos x = \frac{CB}{CH}$

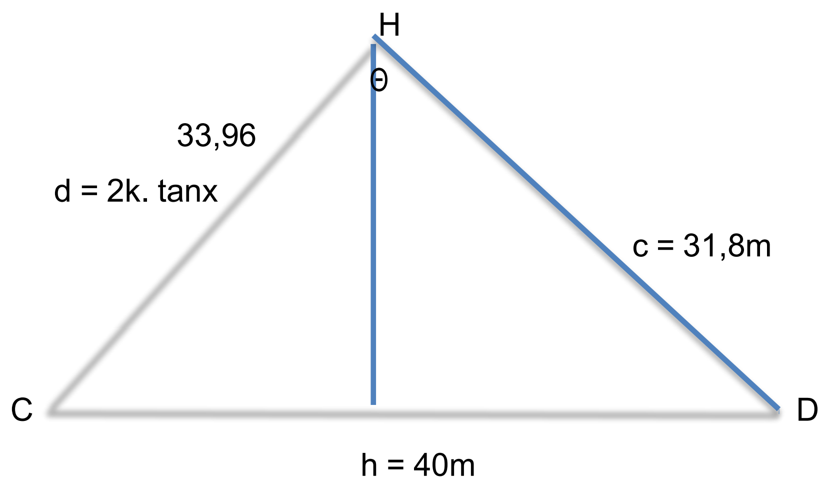
$\cos x = \frac{2k.\sin x}{CH}$

CH. cos x = 2k. sinx

$C = \frac{2k.\sin x}{\cos x}$ ← quotient identity applied

CH = 2k . tan x

5.3 Extract ΔCHD on the slant plane,



Calculate value of side HC

HC = 2k.tanx

= 2(40) .tan 23^0

=33.96

Given 3 sides the cosine rule can be used to calculate the angle

$$CD^2 = CH^2 + HD^2 - 2.CH.HD.\cos\theta$$

$$\cos\theta = \frac{CH^2+HD^2-CD^2}{2.CH.HD}$$

$$\cos\theta = \frac{33{,}96^2+31{,}8^2-40^2}{2.(33{,}96)(31{,}8)}$$

$\cos\theta = 0{,}2613$

$\theta = 74{,}85^0$

Table 6.6 Learner performance in question five

Question One	Cognitive demand level(K/RP/CP/P)	Correctly answered	Partially correct	Incorrect	Not answered
5.1	**K**	**12 (75%)**	**3 (19%)**	**1 (6%)**	**0 (0%)**
5.2	**RP**	**11 (69%)**	**2 (13%)**	**3 (19%)**	**0 (0%)**
5.3	**CP**	**11 (69%)**	**0 (0%)**	**5 (31%)**	**0 (0%)**

CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedures

Learners struggled with concepts that needed deeper conceptual understanding. Questions, which required interpretation of information or justification, posed the greatest challenges.

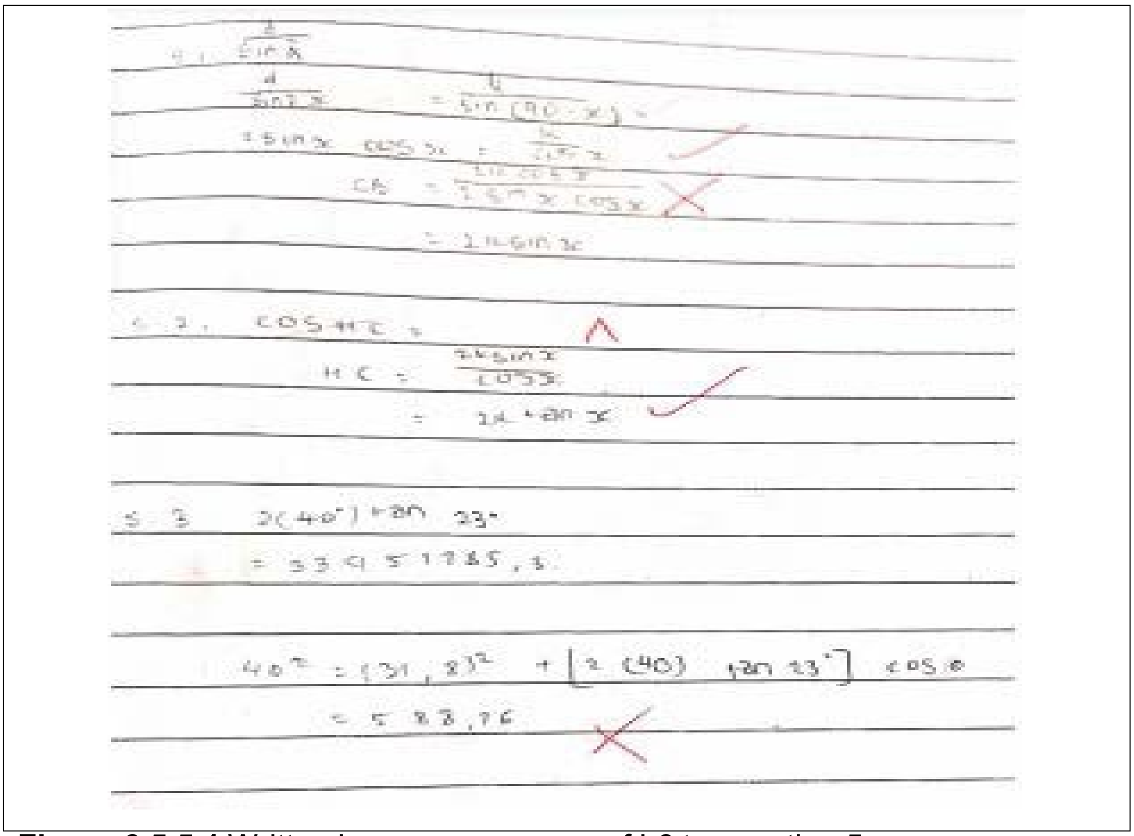


Figure 6.5.5.1 Written Learner response of L6 to question 5

Learner six choose the sine rule and substituted correctly. However, Learner six was not able to manipulate the equation in the third line. Learner six omitted replacing d with CB and incorrectly cross - multiplied the equation. Learner six was unable to logically calculate the distance between C and D in terms of *k* metres and merely wrote 2k sinx, which shows no logical link from the second last step to the last step.

Question 5.2 Learner six did not write the trig ratio applied in solving the question. The solution contained six steps but Learner six merely wrote the last two steps. A possible reason could be that Learner six copied the answer from the group without understanding. Question 5.3 Learner six was unable to use the calculator correctly to calculate HC, in addition the cosine rule was incorrect. It is evident that learner Learner six struggled with concepts that needed deeper conceptual understanding. Questions, which required interpretation of information or justification, posed, as the greatest challenges to Learner six.

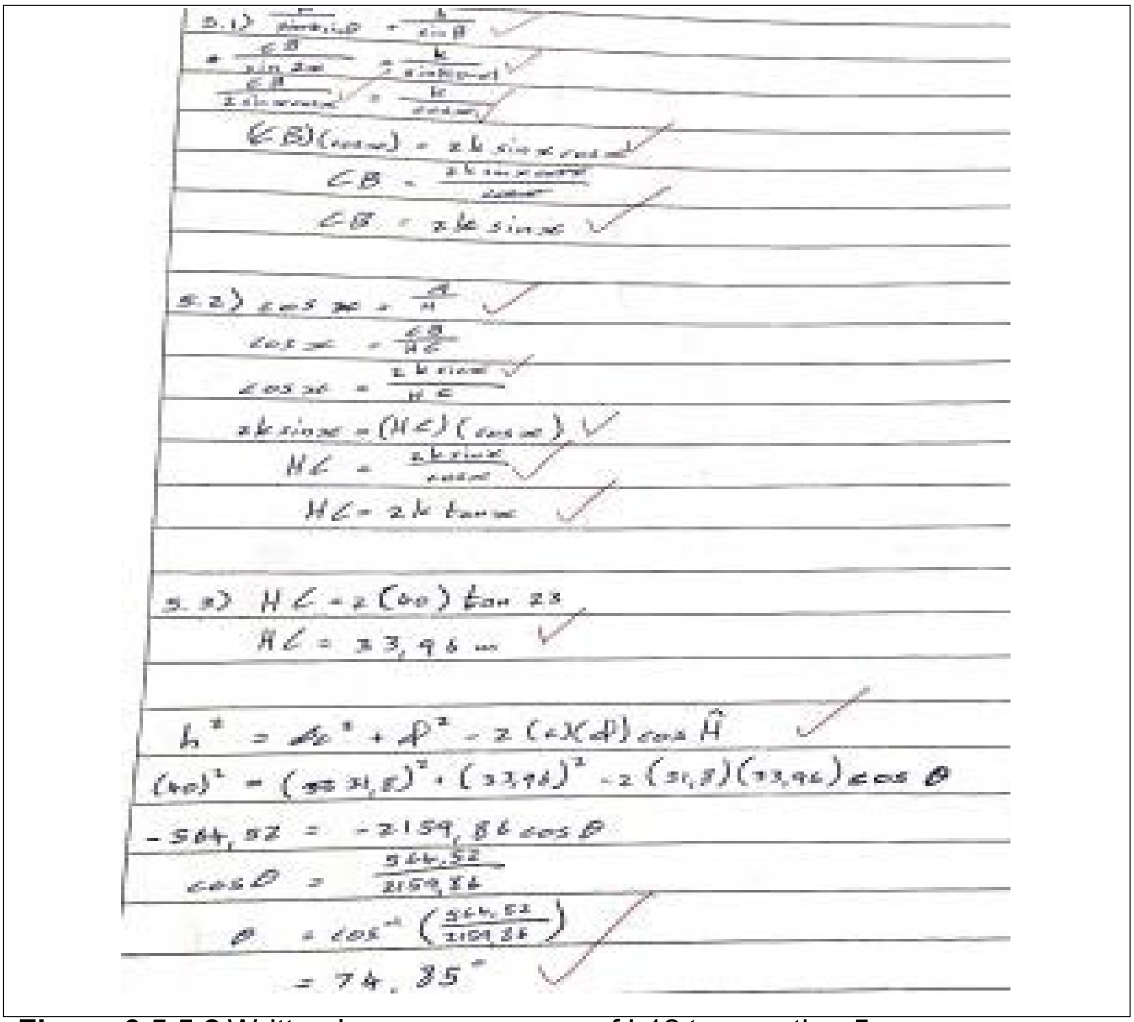


Figure 6.5.5.2 Written learner response of L12 to question 5

Learner twelve displayed skills and the ability to solve this problem that involved complex calculations and higher order reasoning. Learner twelve was able to use ΔCBD on the horizontal plane. Learner twelve applied the sine rule to arrive at the correct answer in question 5.1. In question 5.2, Learner twelve extracted ΔCHB on the vertical plane and executed the cos x = $\frac{CB}{CH}$ ratio. Learner twelve was able to apply algebraic skills and manipulate the equation to prove that the magnitude of HC = 2k tan x. In question 5.3 Learner twelve was once again able to extract ΔCHD on the slant plane, Learner twelve successfully calculated the value of side HC using HC =

2k.tanx. Given 3 sides Learner twelve then applied the cosine rule to calculate the angle.

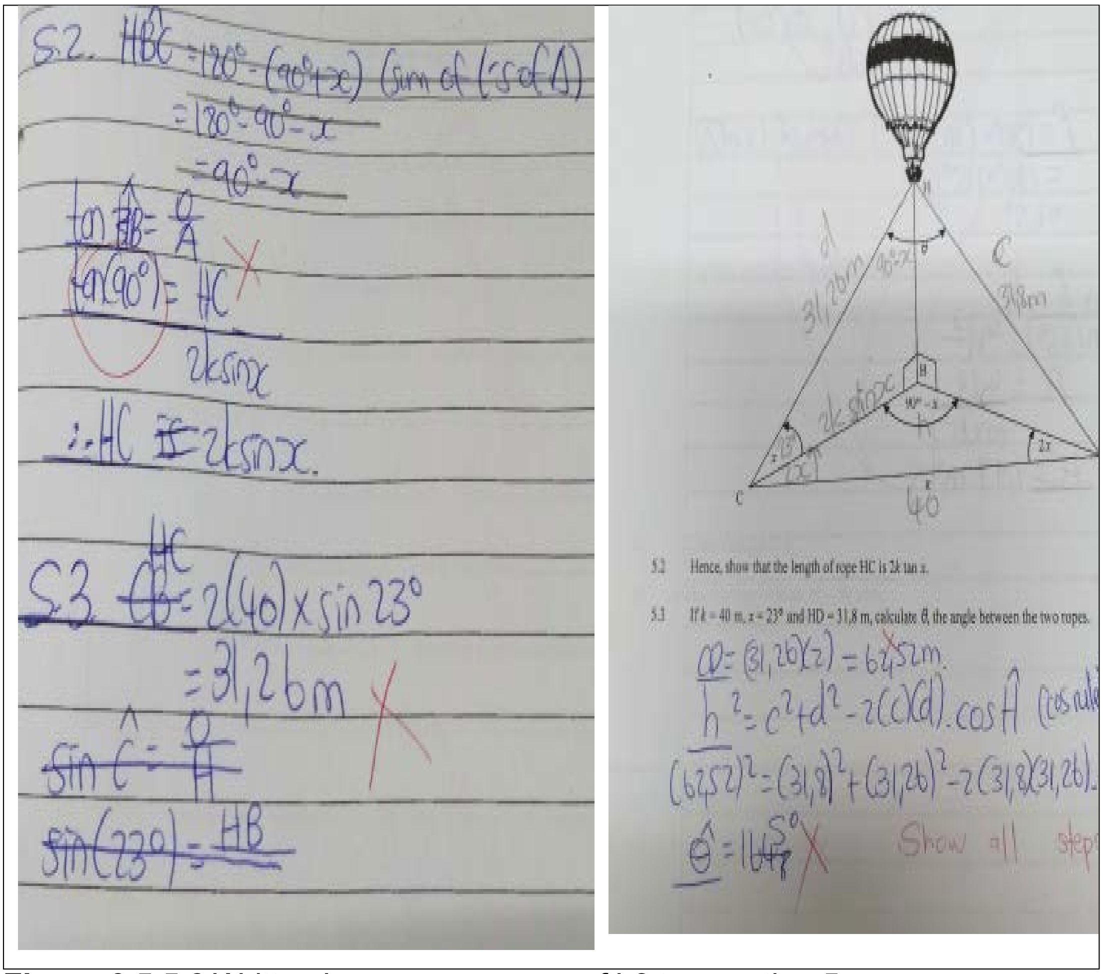


Figure 6.5.5.3 Written Learner response of L8 to question 5

Learner eight displayed poor skills and the inability to solve this problem that involved complex calculations and higher order reasoning. Learner eight was unable to use ΔCBD on the horizontal plane. Learner eight applied the sine rule to arrive at the correct answer in question 5.1.

From the diagram it was given that <HBC was 90^0 but Learner eight used the sum of angles of a triangle and discovered <HBC was 90 – x. Learner eight realized that this was incorrect and cancelled the solution. Learner eight did a second attempt. In question 5.2 Learner eight used incorrect angle and an incorrect trigonometric ratio. Learner eight used tanθ instead of cosθ. Learner

eight failed to realize that tan 90^0 is undefined and should have immediately seen the error. L8 continued to cross- multiply and obtained an incorrect answer. In the solution it would appear that tan 90^0 reduced to 1, which is incorrect, sin 90^0 reduces to 1. Learner eight merely cross -multiplied and wrote an incorrect answer which contained sinx instead of tanx as asked in the question 5.2 to prove HC = 2k tanx. It was observed that this learner opted to work alone and did not enjoy group work.

In question 5.3 Learner eight continued to carry incorrect answer from 5.2. HC = 2(40) x tan 23^0. Learner eight used sin 23^0 which resulted in an incorrect answer. CD=k was given as 40cm. Learner eight indicated this measurement on the diagram but proceeded to calculate CD doubling CH. Learner eight was able to identify the cosine rule needed to be applied, given three sides of a triangle and required to calculate and angle. The incorrect substitution of h resulted in the incorrect answer. Further more the learner did not show all steps of manipulation of the equation and rushed to obtain a final answer. It was also observed that the learner put <BCD = 2x.

Researcher: In question 5.2, why did you use tan 90^0 ?

L8: I saw that the question wanted me to prove HC = 2k tan x. there was a tan x there…

Researcher: Use you calculator and calculate tan 90^0. What do you get?

L8: (Learner uses calculator) Mam I think something is wrong with my calculator. It is giving me an error.

Researcher: Nothing is wrong with your calculator. Recall your trigonometry graphs, 90^0 is an asymptote. Therefore tan 90^0 is undefined.

L8: Oh yes mam… Now I remember…..(pause). I thought tan 90^0 is equal to one. I see now that I made a mistake. Sin 90^0 is equal to one. I see now that I have made a mistake.

Researcher: You still continued solving the problem but did not prove HC = 2k tanx. Even though you were unable to prove HC = 2k tan x, why did you not use it in question 5.3?

L8:…Err… I'm not sure. I got confused. There were no numbers, only variables and I had to prove.

Researcher: Did you not check with the rest of the group?

L8: No! I don't like working in groups. Never did. I like to work on my own…

Researcher: In the diagram I noticed that you put <BCD =2x. Why did you do that? This was not given in the question.

L8:In the diagram and the manipulative it looked like an isosceles triangle. Err…..I think I assumed that side BC equal side BD.

Researcher: You know in mathematics we cannot make assumptions,

L8: Yes mam… I remember you keep telling us that we cannot assume things….

Researcher: In question 5.3 you doubled CH to obtain CD but it was given that CD equals 40m. Why did you do that?

L8: I don't know why I did such a silly thing mam. I think I saw x and 2x and then decided to double 31,26.

Researcher: I see you were able to correctly chose the cosine rule…

L8: Yes mam… That was easy. I had three sides and needed to find an angle.

Researcher: I'm glad you were able to select the cosine rule correctly. Unfortunately you substituted incorrectly which resulted in an incorrect answer for the angle. Remember to show all steps.

L8: Yes mam….I will remember to do that.

Researcher: Had you worked with your group perhaps you would have had a different result.

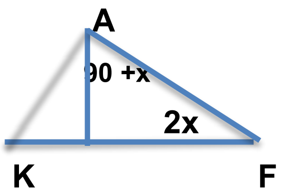
L8:….(Silence) I guess so. I thought I did not need their help.

Learner 8 - L8 interview response to question 5

Learner eight can be described as a solitary learner. This type of learner pursues their own interest and has a clear, deep understanding. Learner eight would certainly enjoy individualized projects and working on her own. This learner enjoys own space and should be encouraged to socialize (Mantle, 2001). Learner eight is a reminder that any group of learners always has a diversity of learning styles. This learner confirms the claim by Carbonneau, Zhang & Ardasheva (2018) that manipulatives are not a panacea to all students' learning.

Observation

Learners discussed and interacted with the artifact. They discovered that a two-mark question, implied that it was straightforward and was merely a knowing question and did not require them to allocate much time solving it. Learners were clearly able to establish that A was the apex. The learners repeatedly read the question to make sense of what was given and what they were required to solve. They closely looked at how many sides and angles were given. Learners began drawing triangles and placing information on it.



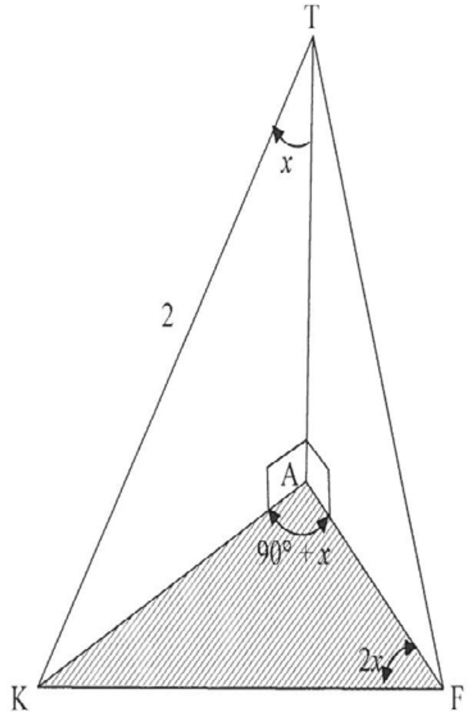
Learners were asked why they used the sign rule? Their response was that they were given 2 angles and 1 side. Several learners began to immediately substitute in the formula, which seemed to cause confusion and careless mistakes. The learners were encouraged to first write down the formula first and thereafter substitute.

Figure 6.5.5.4 Learners engage with the manipulative and work with the activity sheet.

6.5.6 Analysis of question Six

Question Six (**NSC PAPER 2-**Feb/March 2015)

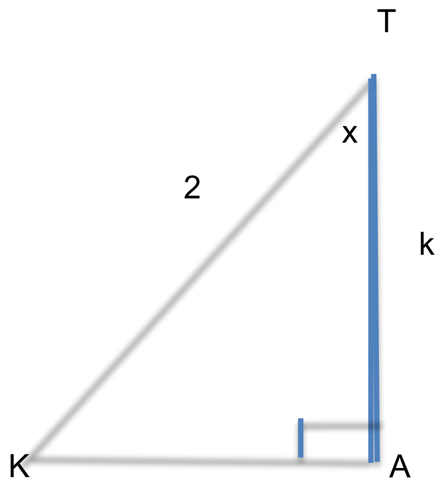
6.1 In the figure, points K, A and F lie in the same horizontal plane and TA represents a vertical tower. $A\hat{T}K = x$, $K\hat{A}F = 90° + x$ and $K\hat{F}A = 2x$ where $0° < x < 30°$. TK = 2 units.



6.1.1 Express AK in terms of $\sin x$. (2)

6.1.2 Calculate the numerical value of KF. (5)

6.1.1 Involved complex procedures. Learners had to visualize and extract ΔTAK on the vertical plane.



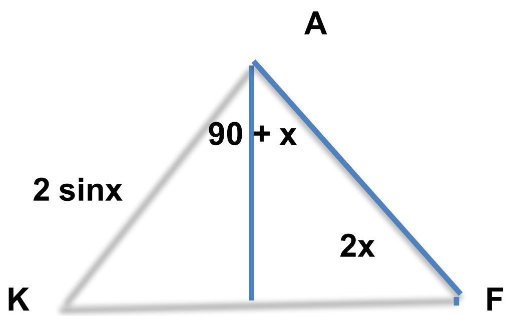
Since 90^0 Δ, use of basic trigonometry ratios $\frac{sssnx}{1} = \frac{KA}{2}$

AK . 1 = 2. sin x

AK =2. sinx

6.1. 2 This problem involved complex procedures. Higher order reasoning was required.

Learners had to extract Δ KAF on the horizontal plane.



Reduction formula knowledge was required for sin (90^0 + x) which reduces to the trig ratio cos x.
In addition, knowledge of double angle formula was needed for expansion of sin 2x was required. Knowledge of manipulation of equation, cross multiplication and solving of numerical value of KF was further required.

Given two angles and one side, the sine rule can be applied.

$$\frac{KF}{\sin\,(90^0+x)} = \frac{2\sin x}{\sin 2x}$$

$$\frac{KF}{\cos x} = \frac{2\ \sin x}{2\ \sin x.\cos x}$$

$$KF = \frac{2\sin x.\cos x}{2\sin x.\cos x}$$

$$KF = 1$$

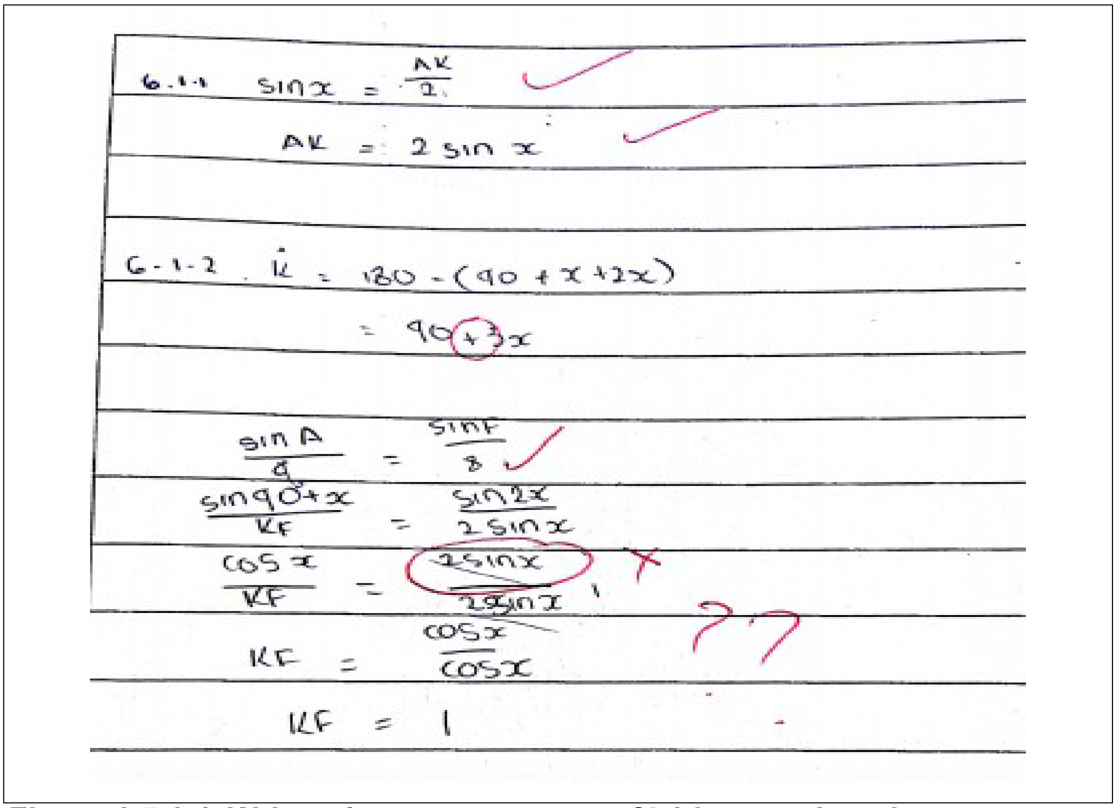


Figure 6.5.6.1 Written learner response of L6 in question six

Learner six was successful in applying the sine rule and expressing AK in terms of sin x however, Learner six displayed difficulty in simplifying the equation to calculate angle K. Learner six further displayed lack of knowledge of double angle formula and reduced sin 2x to 2 sinx. The later part of Learner six response lacks logic and cannot explain the derivation of cos x /cosx.

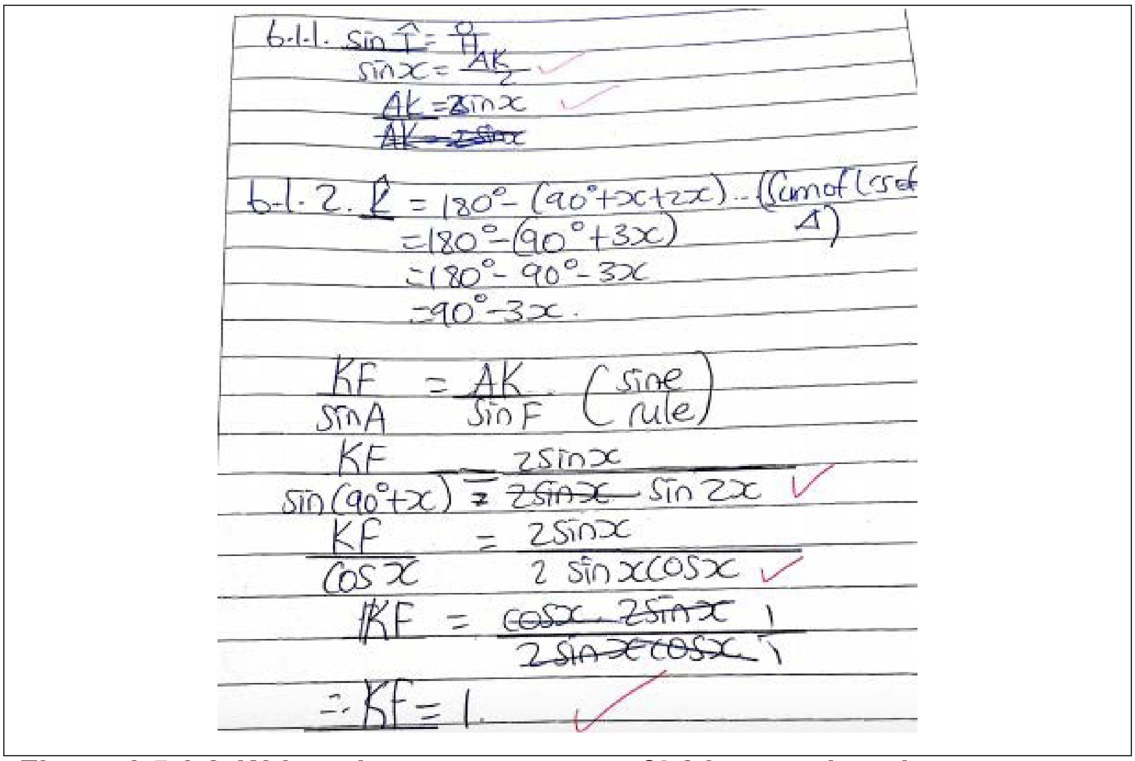


Figure 6.5.6.2 Written learner response of L8 in question six

Learner eight executed question 6 well. Question 6.1.1 involved complex procedures and Learner eight was clearly able to visualize and extract ΔTAK on the vertical plane. Thereafter Learner eight applied the sine rule and was able to express AK in terms of sinx.

Question 6.1.2, the problem involved complex procedures and Learner eight displayed higher order reasoning. Learner eight successfully extracted Δ KAF on the horizontal plane. Reduction formula for sin (90^0 + x) and double angle formula for expansion of sin 2x was applied by L8. Knowledge of manipulation of equation, cross multiplication and solving of numerical value of KF was further evident in Learner eight's response above. Given two angles and one side, Learner eight applied the sine rule and was successful in obtaining a numerical value for KF.

Researcher: What trigonometric concepts or rules did you need to solve question 6? L8: I ….err needed to know double angles, expansion, sine rule and trig ratios sinθ = O/H Researcher: Question 6.1.1, was it easy for you to answer and did the manipulative help you? L8: Yes. It was not too bad. The different colours on the model helped me see the ΔKAT on the vertical plane and so this was a 90^0 triangle. I….(pause) I used sinθ = O/H Researcher: I noticed you calculated <K but did not use it anywhere to solve for KF. L8: Yes mam. I thought that I would need it. Researcher: Explain your answer to question 6.1.2 L8: I could clearly see the different colour triangle KAF on the horizontal plane in the model. I used sine rule. The sin (90^0+x), I had to use the reduction formula. Then things were not cancelling out and I began to get worried. After some time I notice sin 2x and remember double angle formula must be used. When I did that I was able to cancel and arrive at a numerical value of 1. Researcher: Well done. You show good understanding and a sound knowledge of your trigonometry. Keep up the interest.

Learner 8-L8 interview response to question six

Learner eight shows evidence of internalizing mathematics processes and procedures. Procedural fluency (Kilpatrick *et al*., 2001) is clearly demonstrated in the learner's solution. Through the interaction with the 3D maths model, Learner eight accumulated physical experiences and was able to conceptualize (Cockett, 2015) the abstract mathematical concepts. Learner eight was therefore able to construct new knowledge.

The data obtained from learner response to question six revealed that no responses were completely incorrect. Refer to table 6.7 below.

Table 6.7 Learner performance in question six

Question One	**Cognitive demand level(K/RP/CP/PS)**	**Correctly answered**	**Partially correct**	**Incorrect**	**Not answered**
6.1	K	16 (100%)	0	0 (0%)	0 (0%)
6.2	CP	12 (75%)	4 (25%)	0 (0%)	0 (0%)

CP, Complex problems; K, Knowledge; PS, problem solving; RP; routine procedures

The analysis of question Six shows that all sixteen participants were successful in answering question 6.1.Learners were able to apply the trig ratio sin θ =$\frac{opp}{hyp}$.

Only twelve leaners were able to correctly answer question 6.2. the other four learners failed once again in the application of algebraic skills in solving equations.

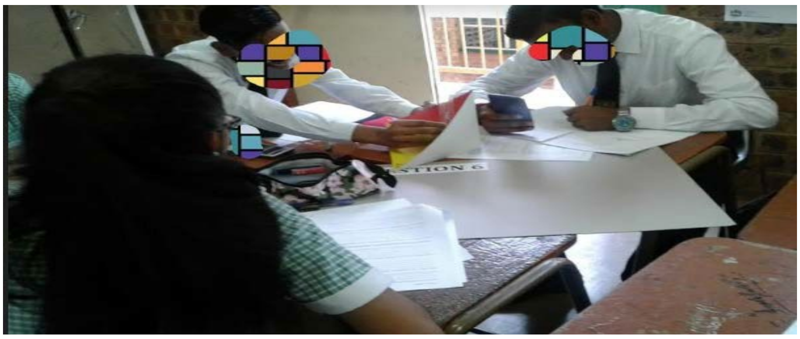


Figure 6.6 Learners engage with the manipulative and work with the activity sheet.

Learners were allocated time to interact with the manipulative. Figure 6.6 shows the interaction between the learners and the manipulative. The grade

12 learners pointed at, touched and engaged with the manipulative and with each other. In addition, they spoke aloud, debated, described and discussed what they saw and made input to the solving process.

6.6 Conclusion

The manipulatives used in this study allowed for learners to better understand abstract concepts in trigonometry particularly proofs that contained variables. The findings concur with Hollard (2020) who claims that research has revealed that manipulatives are useful tools in the mathematics classroom, enabling learners better understand abstract mathematical concepts. The findings of this study revealed that several learners were able to answer knowledge type questions (level1).

Tall(2008) states that construction of new knowledge is built on the construction of previous knowledge. The findings revealed gaps in the knowledge construction of previous concepts that resulted in learners not being able to make the necessary mental constructions for the solution of 3D trigonometry problems and triangles. Learners knew the trigonometry and the correct formula to apply. What failed some of them were their algebraic skills. Manipulation of formula to make a specific variable the subject of the formula posed as a common problem experienced by several learners.

A lack of computational fluency was present. The development of fluency is a goal of mathematics teaching. Learners who develop fluency, efficient, accurate and flexible ways of dealing with numbers are better able to cope with tasks that involve everyday life mathematics (Watson and Sullivan, 2008). A learner becomes proficient in mathematics when they are able to think mathematically, fluently make choices between strategies and engage in mathematical discussions with other learners (Kilpatrick, Swafford & Findell, 2001). Cartwright (2018: 208) claims that “ mathematical fluency results when learners’ strategies and ability to reason are concurred with their conceptual understanding". Computational fluency can thus be described as having the ability to apply more than one method to solve a problem, possessing the ability to use the most efficient method and the ability to execute the solutioncorrectly. It was observed that learners used different methods to arrive at theiranswers.

In this study it was expected that grade 12 learners should have a thorough knowledge of solving algebraic equations, solving for variables, manipulating equations and calculating algebraic terms. The results show that the grade 12 learners experienced many difficulties with basic algebra, cross multiplication, making a variable the subject of the formula, squaring a binomial and a monomial that contained surds as shown in the learners' responses.

In addition, it was discovered that the learners had experienced difficulty with processing the language and the way the questions were structured particularly when several triangles were connected to form a 3D shape. Even though some learners expressed an understanding of the trigonometry rules, they lacked the development of relational thinking and this resulted in hindering their ability to apply the trigonometry rules to deconstruct the 3D into 2D as to make the problem easier to solve.

This chapter 6 discussed learners' responses and reveals that the use of manipulatives aids the understanding of mathematical concepts when taught using activities containing 3D trigonometric problems, which includes the transition from concrete to abstract. The chapter offered graphs, tables, transcripts and written responses of grade 12 mathematics learners solutions and responses to solving 3D trigonometric problems using area rule, sine rule, cosine rule and basic trigonometric ratios along with the application of theorem of Pythagoras. This information served to explore the conceptual understanding of trigonometric concepts and the influence the mathematical manipulatives have on the grade 12 learners understanding. The next chapter discusses the themes that emerged from this study.

CHAPTER SEVEN: FINDINGS AND DISCUSSIONS

7.1 Introduction

Chapter 6 discussed the analysis of the main study and thequalitative methods employed to obtain data. The data was analysed to ascertain how the purposely-designed mathematics manipulatives enhanced the learning of trigonometry among grade 12 learners. In addition the data wasfurther analysed to determine the impact manipulatives had on the grade 12 learners' understanding and learning. This chapter discussed learners' responses and revealed that the use of manipulatives aids the understandingof mathematical concepts when taught using activities containing 3D trigonometric problems, which includes the transition from concrete to abstract. The chapter offered graphs, tables, transcripts and written responses of grade 12 mathematics learners' solutions and responses to solving 3D trigonometricproblems using area rule, sine rule, cosine rule and basic trigonometric ratios along with application of theorem of Pythagoras. This information served to explore the conceptual understanding of trigonometric concepts and the influence the mathematical manipulatives have on the grade 12 learners' understanding.

In this chapter the results will be analysed via thematic content analysis within the context of the literature review contained in Chapter Two. Themes that emerged from the data obtained are discussed in detail and related to similar research carried out. An overview of the study is given on the advantages of manipulatives and how they improve performance and increase understanding of abstract concepts. The research questions are addressed. Thematic content analysis and steps in the analysis process are discussed. From the study eight themes emerged and they are discussed in detail. Learning modes, utility of manipulatives in relation to mathematics itself and the analysis flow chart are explained.

7.2 Overview

Education and mathematics competencies and skills have become the most central elements that impact on the development of any nation, particularly in science and technology. International and national studies have shown that South African learners have poor mathematics skills (Bansilal, James & Naidoo, 2010). Jojo (2019) claims that in South Africa the teaching of mathematics has been shown to be among the worst in the world. The unacknowledged poor quality of mathematics teaching methods that exist in public schools has disadvantaged learners and resulted in a deprivation of several learner entries to both higher education and modern knowledge intensive work skills. Teachers therefore need to use better methods toimprove their teaching skills and help learners understand trigonometry better. This study, which involved the use of manipulatives in equipping grade 12 learners with the necessary cognitive skills, is aimed at providing teachers with the knowledge, use and the benefits that manipulatives have to offer.

The findings of this study confirms the benefits of engaging grade 12 mathematics learners through the constructivist learning approach, through the use of concrete mathematic manipulatives in solving three-dimensional trigonometric problems. Although there are several recommendations and reports by credible sources on the use of manipulatives (Munday, 2019; Vang, 2017; Willingham (2017); Cockett, 2015; Mustafa al-Absi & Nofal, 2010) and validated studies (Carbonneau, Choden & Chalermnirundorn, 2021; Madonsela, Ndlovu, Brijlall, 2020; Silva, Costa & Martins, 2020; Wong & Borysenko, 2020; Brijlall & Niranjan, 2015;) there still exists the continued perpetuation of the traditional methods such as memorization, use of algorithms and focus on procedural skills which can be described as being less effective and overshadows the opportunities for learners to be actively involved in constructive and active learning (Moyer, 2001).

7.3 Research Question

This study made use of artefacts (manipualtives) in exploring the conceptual understanding shown by grade 12 mathematics learners in the application of 3 - dimensional trigonometric problems

The research question addressed by this study are:

> How did Mathematical models help learners to adopt more active approaches towards learning of three-dimensional trigonometric problems amongst Grade 12 learners?

To answer the research question: Through the interviews and learner responses and observation, the researcher discovered that the grade 12 learners in the study were primarily concerned about their mathematics fluency and their mathematical modeling abilities and ability to use Mathematics in real life situations. Learners emphasized that mathematics manipulatives were essential in teaching mathematics. It was observed that before the use of manipulatives, learners tend to be passive in learning and then after introducing manipulatives, learners became more active and enthusiasticabout learning. Based on learners' responses and explanations it can be concluded descriptively that the use of manipulatives effectively increases learner understanding of mathematical concepts and impact student learning outcomes. The results of this study align with previous research (Buasen *et al.,* 2020; Peltier *et al.*, 2020; Ulyani & Qohar, 2020; Marasigan *et al.*, 2019; Hidayah, Dwijanto & Istiandaru, 2018; Liggett, 2107; Larbi & Mavis, 2016) that manipulatives can improve learners' mathematical and learning outcomes. The learners' actions depicted that of a flowchart to assist in selecting correct formula and method. See figure 7.1 below.

Findings show learners are better able to visualize and internalise the problem. This is evident when learners extracted triangles in the different planes, namely, the horizontal plane, vertical plane and slant plane. They socialized, exchanged ideas and worked in a collaborate mannerwithin their group to achieve a common goal. Their learning was based on Piaget's theory based on constructivism. The learners constructed new knowledge from previous knowledge, coming into grade 10 with knowledge and application of basic trigonometric ratios, area rule, sine rule and cosine rule in two dimension and now applying their knowledge to three dimensional problems. Learners gained confidence and felt that they would answer this section better in the National Senior Certificate examinations (NSC).

Learner responses:

> *Learner 11: Err…. I needed to use the area rule, the sine rule and simple trig ratios.*
>
> *Learner 11: My "baby trig". In a right angled triangle sin,cos,tan. SOH CAH TOA mam !*
>
> *Learner 15: Yes the manipulatives helped me to visualize better and see things better in perspective. I was better able to see the triangles in the vertical plane, horizontal plane and the slope plane. Working in a group with my classmates was fun and enjoyable.*
>
> *Learner 4: It was easier to find as visual made problem become alive and easier to see. My model helped me fill information on the diagram and it made it less confusing.*
>
> *Learner 4: Mmmm ... (pause) I enjoyed working with the maths model and with my friends in the group. I was comfortable and we exchanged ideas. I was not scared to give the wrong answer. The model helped me see things better.*
>
> ***Learner 12**:"Yes. This exercise provided good practice in the relevant questions."*
>
> ***Learner 13**:"Yes, this refreshed my memory for the NSC exams."*

Learner 14: *"Yes. With the understanding of the work done I am able to do the working better because I would be confident."*
Learner 15*: "Yes, if I continue using examples like this, I will be more prepared and confident to write the NSC November exam."*

7.4 Thematic content analysis

Clarke and Braun (2017) states that thematic analysis can be described as a method for identifying, analyzing and interpreting patterns of meanings known as themes within qualitative data. Thematic analysis can be used to analyse large and small data sets, from a case study research consisting of 1-2 participants to large interview studies with sixty or more participants (Cederval & Aberg, 2010).

Thematic content analysis is the method most suited for my research study, which involved eliciting and analyzing the responses of the grade 12 mathematics learners in the Pinetown District. The thematic approach analysis consists of six phases. Braun & Clarke(2006) define them as: Phase 1: Familiarizing yourself with the data; Phase 2: Generating initial codes; Phase 3: Searching for themes; Phase 4: Reviewing potential themes; Phase 5: Defining and naming themes and Phase 6: Producing the report.

7.5 Themes identified

In this section the researcher presents the results of the thematic analysis and claims that the following themes are representative of the most common and important beliefs of manipulatives among grade12 mathematics learners in this study. Eight themes were identified from the four groups and the data obtained in this study.

The following themes were identified in this study and are described in detail and analysed:

- **Theme one:** Interaction/Engagement/ collaborative learning.

- **Theme Two:** Fun ,enjoyable and interesting.
- **Theme Three:** Link from concrete to Abstract.
- **Theme Four:** Lack of use of manipulatives by educators.
- **Theme Five:** Reduction of Task difficulty.
- **Theme Six:** Increased confidence in Problem Solving.
- **Theme Seven**: Increases spatial skills -Helps Visualise.
- **Theme Eight:** Promotes mathematical fluency.

In the ensuing discussions the above themes are further elucidated.

7.5.1 Theme one: Interaction/Engagement/ collaborative learning

The findings of this research confirm that the use of manipulatives gives opportunities to the learners to observe and to pay attention to the teacher's instructions, statements and questions. The presence of these 3 dimensional mathematical models assisted learners to think and recall previous knowledge on which to build and construct newly acquired knowledge. The class buzzed with constructive noise. From the observation notes and video recordings, it was observed that the learners were actively engaged with each other and with the manipulatives. They discussed, argued, defended their answer and respected each other at the same time. They touched and pointed at the manipulative and explained to each other what they saw, which angle or side on the horizontal plane, vertical plane or slant plane was required to be calculated.

Based on the semi-structured interviews, observation, audio data and learner responses to the activity sheet from the grade 12 learners, themetwo revealed that the grade 12 learners were actively engaged and fully occupied during the solving of three-dimensional trigonometry problems when using the mathematical manipulatives. Carbonneau, Wong & Borysenko (2020) explain that the use of concrete manipulatives in classrooms provides opportunities for learners to interact physically with abstract content, which they normally would be unable to visualize or touch. Burns (2007) and Cocket & Kilgour (2015) add that through this process learners become more engaged with the learning material and are more successful in interpreting abstract mathematical concepts based on concrete manipulatives (Willingham 2017; Sarama & Clements 2009).

Collaborative learning can be described as any form of instructional activity that occurs when learners work together and interact closely in order to exchange ideas and thoughts to achieve a common goal (Laal & Ghodsi, 2012; Prince, 2004). Extensive readings on collaborative learning has been researched and has been proven to be effective in increasing learner engagement and learning performance (Van Leeuwen & Janssen, 2019 Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt & Wenderoth, 2014). Loes, An, Sachaie & Pascarella (2017) add the point that collaborative learning is particularly useful as itguides learners to various perspectives, which potentially helps their understanding of the material they are being taught or to which they are exposed. Carbonneau, Wong & Borysenko (2020: 2) explain that learning with concrete manipulatives highly depends on students' interpretation of the manipuatives. Collaborative learning is one strategy that can be easily integrated within the classroom. Collaborative learning may be particularly beneficial in the context of learning with perceptually rich manipulatives. Learners who become overly fixated on surface properties of objects may benefit from engaging in discussions with peers who are more adept at recognizing the abstract representation of the manipulatives.

Based on the semi-structured interviews, observation, video/audio recordings and learner responses to the activity sheet from the grade 12 learners, theme two revealed that the grade 12 learners were actively engaged and fully occupied during the solving of three-dimensional trigonometry problems when using the mathematical manipulatives.

7.5.2 Theme Two: Fun ,enjoyable and interesting

Learners explained that the use of manipulatives aided them in understanding trigonometry in a fun way. Very little or no learning takes place when the environment is threatening and therefore educators are requested to create a productive and conducive learning environment (Larbi & Mavis, 2016)

The learners experienced enjoyment and pleasure working withmanipulatives. Learners indicated that when they saw the manipulatives and touched them, they became interested and eager to learn. Visual aids arouseinterest in learners and helps teachers to teach concepts in an easy way (Shabiralyani *et al.*, 2015). Learners obtain enjoyment when taught mathematics through participatory and interactive ways that use manipulatives (Choden & Chalermnirundorn, 2021; Day & Hurell, 2019; Stiegelmeier & Moore, 2019; Strom, 2009; Reimer & Moyer, 2005).

Some learner responses:

> ***Learner 4:*** *"Very interesting."*
>
> ***Learner 7:***"*They are fun."*
>
> ***Learner 10:*** *"It is interesting as it forces you to see things differently."*
>
> ***Learner 11: "****Its interesting. It allows you to see things differently."*

7.5.3 Theme Three: Link from concrete to Abstract CRA

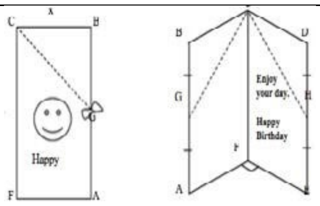
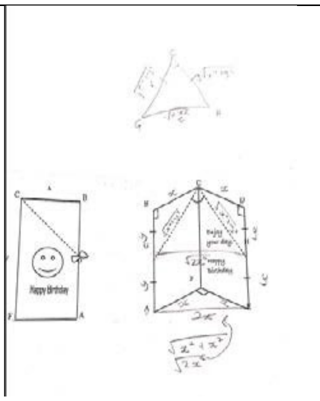
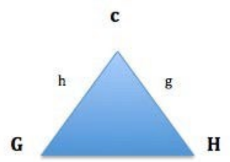
Witzel (2005) describes the concrete representational, abstract (CRA) instruction as a process for teaching and learning mathematical concepts. Beginning with the manipulation of concrete materials, the process then transfers learners to the representational level and then climaxes at the abstract level. CRA instruction makes provision for students to make associations from one stage of the process to the next.When learners are first permitted to develop a concrete understanding of the mathematical concept, they then become more likely to successfully carry out a mathematics skill and genuinely understand mathematical concepts at an abstract level.

> ***L9:*** *“Yes! Reason being it helps me identify each triangle or shape with their own angles individually. So when I see the model I am able to separate and identify and solve.”*

Manipulatives are considered to be effective in fostering the development and enhancement of conceptual understanding in Mathematics as they assist learners to link and relate concrete ideas to abstract ideas (Uribe-Florez & Wilkins, 2010; Allsopp, 2007). Jones and Tiller (2017) claim that by making use of hands–on , concrete manipulatives during mathematics teaching time could result in learners having a higher retention rate and develop a more positive attitude towards their education. Smith (2009) further adds that well designed manipulative bridges open or close the gap between formal Mathematics and informal Mathematics.

Specific information for the stages of the CRA is shown in table 7.1

Stages	**Key elements**	**Sample problem**	**Explanation**
Concrete	A 3D mathematical model of a birthday card	 	Here a colourful birthday card was placed on a flat surface.

Transition to representational	Use of concrete and representational materials together		Once learners engaged with the concrete manipulative, they began to draw the triangles in the various planes
RepresentationaL	Triangles in the horizontal plane, vertical plane and Slant plane		At the representational stage of CRA the learners are comfortable using the given information and working with the triangles in the various planes
Transition to abstract	Use of representational and abstract materials together	Extract Triangle CGH on the slant plane 	• Visualise triangle CGH on the slant plane • Link side GH on horizontal plane equal side GH on the slant plane. • Given two sides and required to

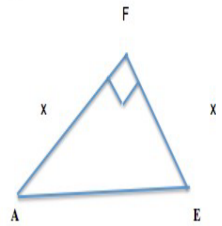
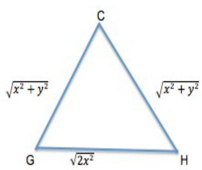
			find an angle meant that the cosine rule had to be applied.
Abstract	Apply theorem of Pythagoras Now applying the cosine rule in ΔGCH	GC = h = $\sqrt{x^2 + y^2}$. ...Pythagoras Link had to be made GH = h = AE  AE = $\sqrt{x^2 + x^2} = \sqrt{2x^2}$ ....Pythagoras Link has to be made, side GH = c = AE Now applying the cosine rule in ΔGCH Link has to be made, side GH = c = AE Now applying the cosine rule in ΔGCH  $c^2 = h^2 + g^2$ -2hg.cos c	Students at the abstract level of CRA no longer require the assistance of the manipulative to solve the problem

Table 7.1 Specific information for the stages of CRA.

7.5.4 Theme Four: Lack of use of manipulatives by educators

The data extracted from the interview responses of the learners revealed that educators did not use manipulatives in their classroom instruction. It was found that majority of the learners were not exposed to manipulatives and mathematics problem solving. Fifteen of sixteen participants indicated that their educators did not use manipulatives and examples of their responses included:

> *Learner 16" No, teachers didn't use as trig was left for last section thus time did not permit";*
>
> *Learner 11: " No, we only used diagrams";*
>
> *Learner 3 " No, was never exposed to this before. Until now."*

The reoccurring focus by participants reporting that their teachers have not used manipulatives demonstrates that possibly teachers do not know how to integrate the use of manipulatives in their classes and how to manage time.

Research on the use of manipulatives has yielded several benefits. However, evidence has shown that teachers' conception of using manipulatives is limited in their classroom instruction (Furner & Worrell 2017; Marzola 2006). Ndlovu & Chiromo (2019) claim that the reasons for the lack of effective use of manipulatives has not been thoroughly researched and that literature shows that the lack of teacher knowledge or expertise in a specific area is the main reason why topics are not taught effectively. Puchner *et al.* (2013) assets that teachers experienced problems in using manipulatives when teaching. Ndlovu & Chiromo (2019) further suggests that since the use of manipulatives has been identified as potential resources that can assist learners understand abstract mathematics, it is therefore vital that teachers posses the required knowledge and expertise necessary to use this by themselves, to solve mathematical concepts which they could apply in their teaching.

7.5.5 Theme Five: Reduction of Task difficulty

The majority of the participants indicated that using manipulative to solve 3D trigonometry problems reduced the difficulty of the problem and made it easier to understand. Examples of their responses are

> L*earner* 9:"Yes was easy to solve problems when using manipulatives. Reason being it helps me identify each triangle or shape with their own angles individually. So when I see the model I am able to separate, identify and solve";
> *Learner 5:"It was completely easy, however it has made understanding three dimensional problems much better"; L1:"Yes. They helped me to conceptualise the problem and it made it easier to answer these questions."*

The reoccurring focus by the participants that the use of manipulatives demonstrates a reduction of task difficulty indicates that there exists genuine benefits to the use of these manipulatives. It aided learners in breaking down the question and being successful in attaining the correct answer to the problem. All educators should use a method that assists a learner.

7.5.6 Theme Six: Perseverance and increased confidence in Problem Solving

Perseverance is a key process through which Mathematics can be learned with understanding. However, withstanding such uncertainty can be difficult for learners to endure and therefore necessitates support. Teachers are required to offer support and provide some form of scaffolding so as the ease the experience. The findings of this study showed that learners had persevered significantly. Learners' perseverance increased over time as they interacted together in a collaborative manner. Whilst persevering, their self-confidence also increased and they became more confident in tackling the three dimensional trigonometric problems. Some of the learners' responses from the interview include:

> ***Learner 1****: "I am more confident than I was previously";*
> ***L****earner* ***2****:"I am more confident in answering these questions.";*
> ***L****earner* ***4****: "Yes! This activity was enjoyable, boosts my confidence in attacking 3D problems. I can visualize and conceptualize better.";* ***L****earner* ***14:*** *"Yes. With the understanding of the work done I am able to do the working better because I would be confident.";*
> ***L****earner* ***15****: "Yes, if I continue using examples like this, I will be more prepared and confident to write the NSC November exam."*

The findings suggest that manipulatives support perseverance in solving three-dimensional trigonometric problems as learners encouraged each other to revisit their conceptual thinking and to re-initiate and re-sustain their productive struggle by exploring a different set of mathematical ideas and approaches. In addition, the data revealed malleability of perseverance, which implied that learners could increase and improve their perseverance in solving three-dimensional trigonometry problems over a period of time through the use of carefully designed mathematical manipulatives.

The idea of struggle has long been recognized as key to learning Mathematics with understanding (Festinger, 1957; Dewey, 1910). Researchers (Carbonneau, Marley & Selig, 2013; Saitta, Gittings & Geiger, 2011; McNeil, Uttal, Jarvin & Sternberg, 2009) state that learners' activities that integrate the use of manipulatives may improve learning as they provide more opportunities and exposure for learners to interact and engage with each other. There has been a profound impact on learners' learning achievement when there is an increase in learner engagement in STEM (Science, Technology, Engineering and Mathematics) learning (Freeman *et al.*, 2014). Lee (2014) claims that behavioural engagement can be considered to be one component of student engagement that examines and explores the amount of effort and perseverance a student applies in learning. Farrington, Roderick, Allensworth, Nagaoka, Keyes, Johnson & Beechum, (2012) define academic perseverance as being the likelihood that students will finish a task to the best of their abilities in spite of obstacles and challenges they encounter on their path. Carbonneau et al, (2020,3) claims that since manipulatives are concrete objects designed to firstly represent abstract concepts that may otherwise be challenging to grasp and secondly increase student engagement while learning, argues that learning using manipulatives can potentially influence students' perseverance. Belenky & Schalk (2014) agree with Carbonneau, Wong and Borysenko (2020) by stating that learning with manipulatives may aid in promoting academic perseverance by increasing learners' interest and engagement in their learning. Research has revealed that the use of well-designed external knowledge representations such as manipulatives may capture the student's interest that may have previously not been interest in the topic or hold the continued interest in some learners. The use of manipulatives to trigger interest among learners in a topic is important as students' level of interest are predictive of their commitment, their engagement and their perseverance in learning (Hay, Callingham & Carmichael, 2015).

Sengupta- Irving & Agarwal (2017) claim that it is expected that students are to learn Mathematics in such a manner that when they are exposed to challenging problems, they will persist and not surrender. The creation of opportunities for students to persist in problem solving is therefore argued as being important to effective teaching and to learners acquiring positive dispositions in mathematics learning.

7.5.7 **Theme Seven:** Spatial skills are improved - Helps Visualise

The spatial skills had improved. The use of manipulatives assisted the participants in visualising the problem and understanding the problem better.

They were better able to extract triangles in the various planes, namely, the horizontal plane, vertical plane and the slant plane.

Lohman (1996) defines spatial skills as a person's skills to generate, retain, retrieve and transform well-structured visual images. Spatial skills encompass cognitive skills associated with spatial visualization, mental rotation and spatial orientation (Uttal & Cohen, 2012). Spatial visualization is the process of constructing, maintain and manipulating 2D and 3D objects in one's mind (Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013; Cracow & Sorby, 2008). Mental rotation is described as the rotation of mental representations of 2D or 3D objects to determine their images from various viewing angles (Ha & Fang, 2016). According to Lin, Chen, & Lou (2014) spatial orientation involves the change of location in space in relation to two-dimensional (2D) or three dimensional (3D) objects that a person can see.

Research carried out has revealed that spatial skills play a crucial role in developing expertise and success in science, engineering, and technology and fields of mathematics study. Examples of some of the learners' responses in this study are:

> *Learner 1: " They gave a sort of scenario, which made it easier to view each triangle and the information that was given."*
> L*earner* 2," *It gave better visuals to interpret the questions. Allowed us to manipulate the objects in order to understand the question.*";
> L*earner* 4: " *Helps to visualize.";*
> L*earner* 6, " *It helped me to see the different angles and shapes.*"; L*earner* 9: " *It helped to separate each shape in order to work out answers on their own. Then put everything together and see the bigger picture.";*
> *Learner 11: " It gave me a visual representation of the diagram to understand better.";*

The findings of this research confirm that the use of manipulatives improves spatial skills that are fundamental to higher level thinking, reasoning and creative processes.

7.5.8 **Theme Eight:** Promotes mathematical fluency

From the learners' responses in Chapter Six it was evident that several learners had attained mathematical fluency. Computational fluency can be described as having the ability to apply more than one method to solve a problem, possessing the ability to use the most efficient method and the ability to execute the solution correctly. It was observed that learners used different methods to arrive at their answers.

A learner becomes proficient in Mathematics when they are able to think mathematically, fluently make choices between strategies and engage in mathematical discussions with other learners (Kilpatrick, Swafford & Findell, 2001). Watson and Sullivan (2008) concur with Kilpatrick *et al.* by agreeing that learners who develop fluency, efficient, accurate and flexible ways of dealing with numbers are better able to cope with tasks that involve everydaylife mathematics Cartwright (2018: 208) claims that " mathematical fluency is the result when learners' strategies and ability to reason are concurred with their conceptual understanding".

7.6 Traditional Instructional practice versus Constructivist approach with use of manipulatives

The application of manipulatives supports the constructivist approach as it supports Piaget's and Bruner's theories. Table 7.2 below provides traditional Instructional practice versus Constructivist approach with use of manipulatives

Traditional Instructional Practice	**Constructivist approach using manipulatives**
Competitive classroom	Co-operative classroom
Individual work	Collaborative work
Lecture and worksheet	Discussion and hands on direct experiences
Little to no use of manipulatives	frequent use of manipulatives
Abstract mathematical concepts	Link from concrete to abstract mathematical concepts
Less effective traditional practices like memorisation, focus on procedural skills and use of algorithms	Active learning and construction of new knowledge from prior knowledge

Table 7.2 Traditional Instructional practice versus Constructivist approach with use of manipulatives

7.7 Learning modes

Maz-Machado, Madid, Mantero & Fanjul (2019) refer to learning modes as practical lessons which make the learning of mathematics possible. Figure 7.1 depicts learning modes making learning easier or more dynamic, manipulatives facilitating understanding of abstract concepts or manipulatives facilitating meaningful learning. It can be seen that co-occurrence occurs between *Manipulatives facilitate understanding of abstract concepts* and *Learning is easier*. The use of manipulatives in this study allowed for various learning modes to occur and addressed the research question 2 of this study: How did the use of models improve the learning of Mathematics?.

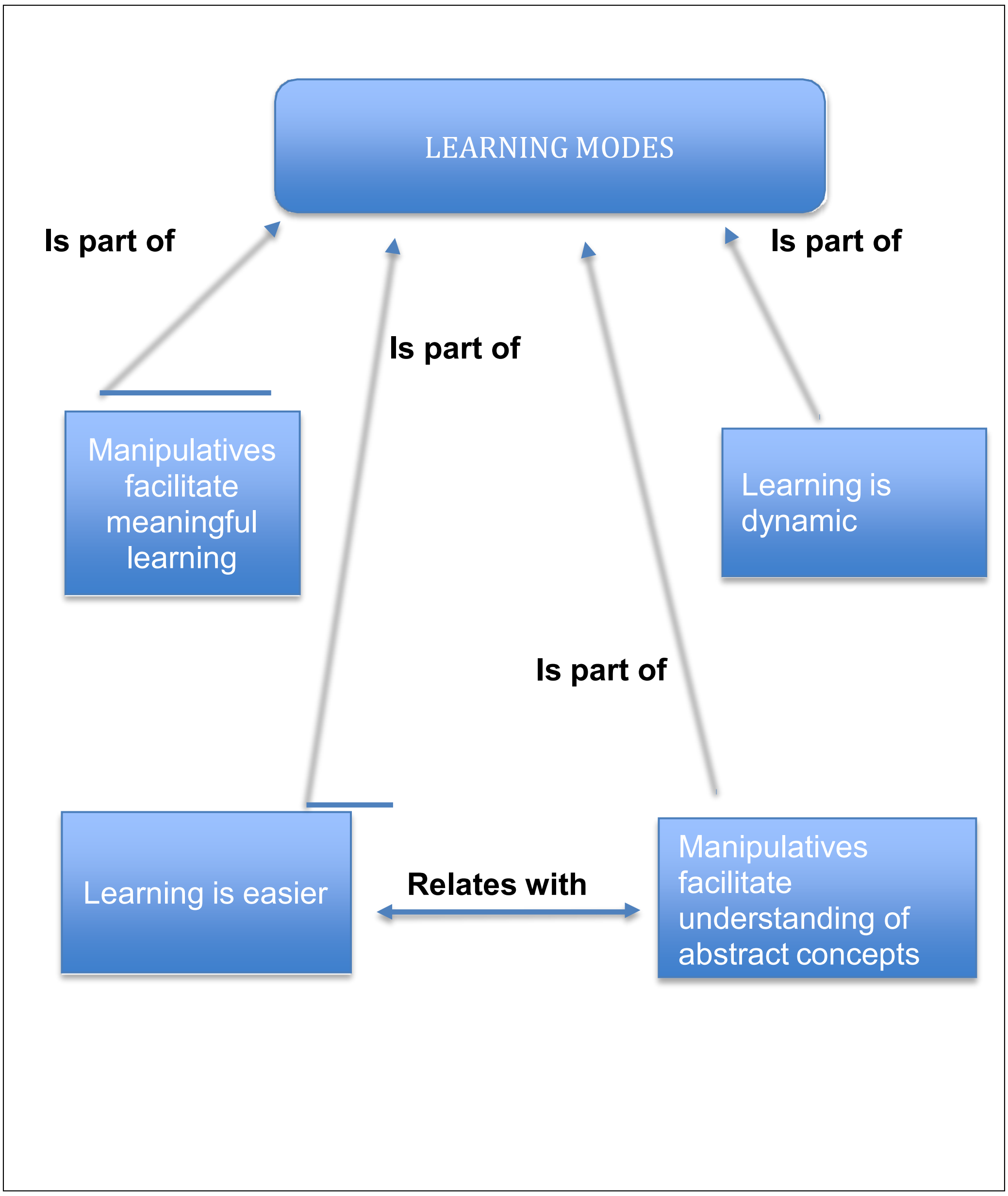


Figure 7.1 Utility of practical sessions regarding modes of learning (Adapted from Maz-Machado, Madid, Mantero & Fanjul, 2019)

Some of the learner responses, which relates to Figure 7.1, (Utility of practical sessions regarding modes of learning):

Learner 1: *"They gave a sort of scenario, which made it easier to view each triangle and the information that was present."*

Learner 2:"*It gave better visuals to interpret the questions. Allowed us to manipulate the objects in order to understand the question."*

Learner 3:"*It shows the way the shape is and helps applying those shapes properties in an easier way."*

Learner 4: *"Helps to visualise."*

Learner 5: *"It was completely easy, however it has made understanding three dimensional problems much better."*

Learner 6: *"It helped me to see the different angles and shapes."*

Learner 9: "*It helped to separate each shape in order to work out answers on their own. Then put everything together and see the better picture."*

Learner 10: "*I was able to see what was happening in the problems."*

Learner 11: *"It gave me a visual representation of the diagramme to understand better."*

Learner 12:"*We identified given angle more clearly."*

Learner 13:"*I saw what happened in the problem."*

Learner 16: " *It shows a bigger picture and it is understandable."*

7.8 Utility of Manipulatives in relation to Mathematics itself

This category clusters ideas linked with labels in which students reflect about Mathematics itself. The face-to-face lessons which the learners attended show the utility of Mathematics, foster learner engagement with mathematics, reveal that mathematics is more familiar with learners and allows hands-on manipulation to visualise Mathematics. Figure 7.2 below explains the flow chart of manipulatives in relation to Mathematics itself.

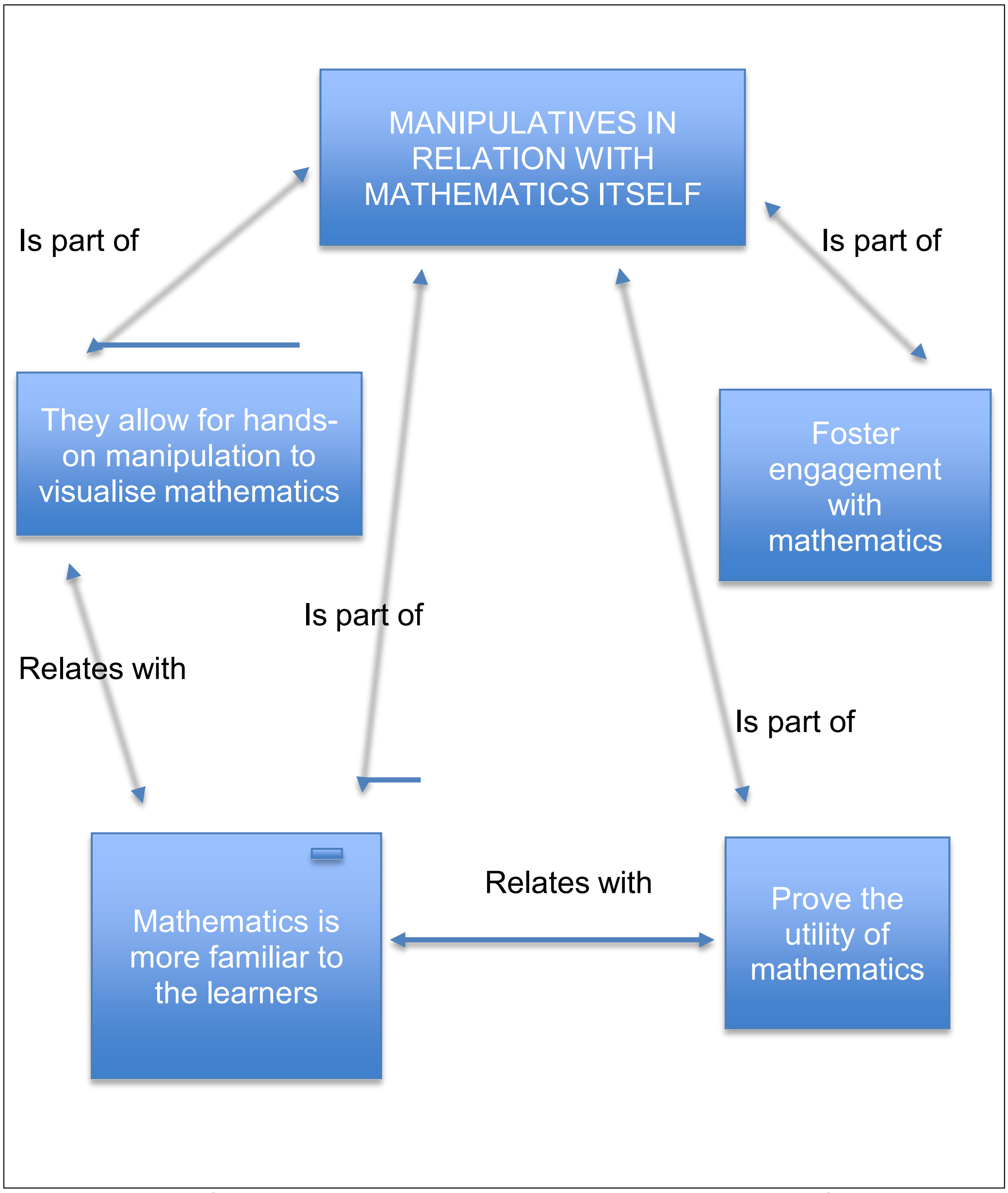


Figure 7.2 Utility of Manipulative in relation to Mathematics itself

(Adapted from Maz-Machado, Madid, Mantero & Fanjul, 2019)

7.9 Analysis Flow Chart

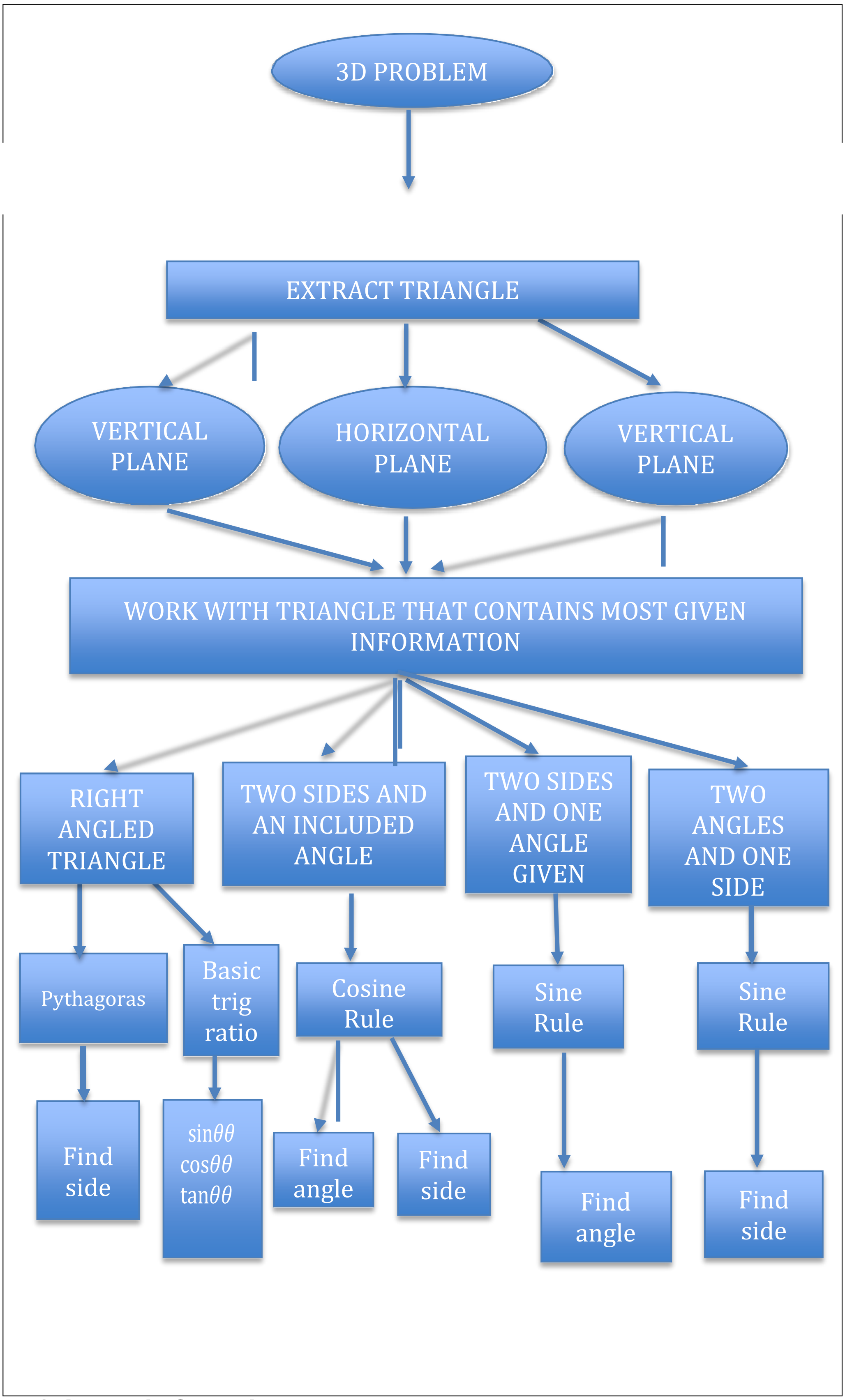


Figure 7.3 Analysis flow chart

To unpack the necessary mental constructions required to solve the 3D trigonometric problems, the analysis flow chart below depicts the learners ' thought processes. During observation of the learners, it was noted that the learners' communication with others had increased and they began to draw a flow chart to determine the path they would follow.

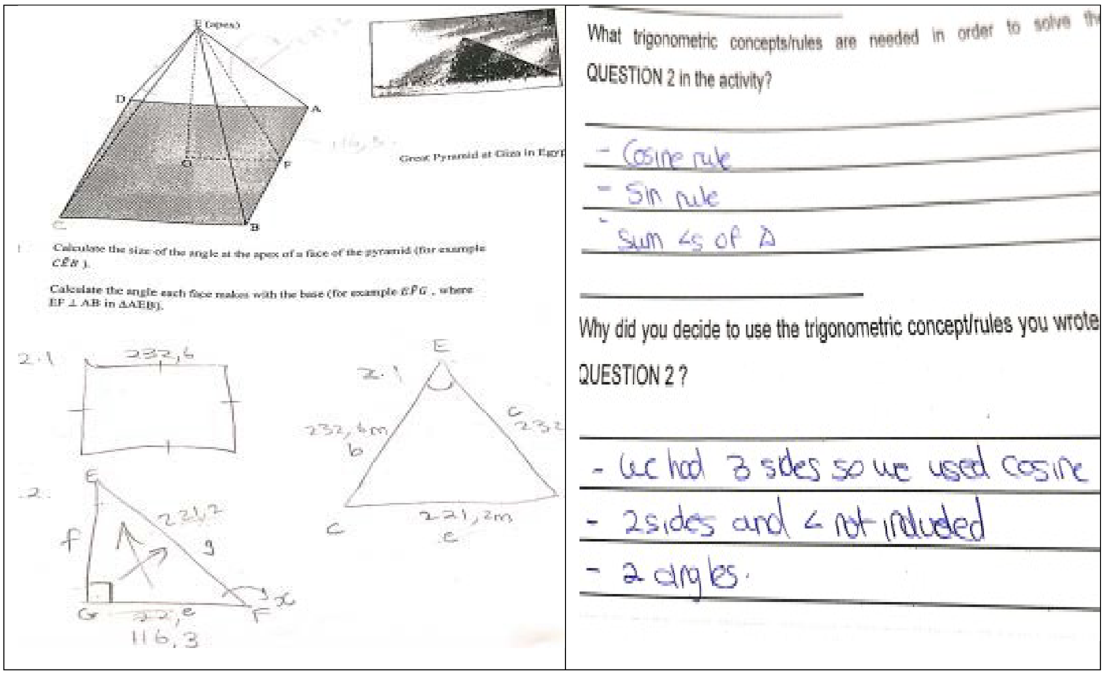


Figure 7.4 Extract Showing learners use of the analysis flow chart

From the extract above it can be seen that the learner internalised the problem and was able to extract the triangles and square base in the various planes. The learner concluded that the cosine rule had to be applied as 3 sides were given and the calculation of an angle was required.

Teachers are always in search of reaching mathematical fluency in their learners and are in search of methods and techniques to assist their learners in transitioning from concrete to abstract. Teachers, therefore, need to develop strategies to be used when solving right-angled triangles and triangles that are not right angled. Learners are to be taught the conditions that determine which rule should be selected in order to solve the problem. Learners struggled with concepts in the curriculum that required deep conceptual understanding and higher order thinking skill (HOTS). Questions where learners had to interpret information or provide proofs presented the greatest challenges. The analysis flow chart was used to analyse the mental construction displayed in learners' responses and also in the interview process, to probe learners' responses. The use of the flow chart offers direction and a path to follow to achieve the desired outcome. It is recommended that high school educators encourage the use of an analysis flow chart as this provides an entry point from the given information and an end point. This analysis flow chart would scaffold the actions of the learner.

7.10 Conclusion

The findings of this study confirms the value and benefits of engaging grade 12 mathematics learners through active learning strategies, through the use of concrete mathematics manipulatives in solving 3D trigonometric problems. Despite reports and recommendations by credible sources (DoBE Angie Motshekga) and validated studies (Munday, 2019; Willingham & Vang, 2017; Cockett, 2015; Mustafa al-Absi & Nofal, 2010), the continued use of the less effective traditional practices such as memorisation, focus on procedural skills and use of algorithms, often dominates the opportunities for learners to be engaged in active learning (Moyer, 2001). The next chapter concludes the study by discussing the findings in this qualitative research, the researchers thoughts, recommendations and limitations of the study.

CHAPTER EIGHT: CONCLUSIONS, RECOMMENDATIONS AND LIMITATIONS

8.1 Introduction

The previous chapter analysed the results via thematic content analysis with the literature reviewed and the theoretical framework discussed. The eight themes that were identified from the data obtained were discussed in detail. Anoverview of the study was presented on the advantages of manipulatives andhow they improve performance and increase understanding of abstract concepts. The research questions were addressed. Learning modes, utility of manipulatives in relation to mathematics itself and the analysis flow chart are explained.

The use of sound teaching and learning principles fosters an environment where learners are motivated to reach their full potential. The findings in this study are important for the South African education system because changing school climates and improving learning strategies with the use of mathematics manipulatives in Mathematics is very important in improving learner performance and the Mathematics results in our country. Underperformance in Mathematics has persistently been a challenge in South African schools. This study was conducted to add to current research concerning the use of manipulatives in mathematics classrooms in secondary schools. In addition this study was carried out to:

Determine how Mathematical models/manipulatives helped grade 12 learners to adopt more active approaches towards learning of three-dimensional trigonometric problems and how did the use of models improve the learning of mathematics?

This chapter presents (1) an overview of the study, (2) conclusions to the thesis, (3) the contribution to research and recommendations, (4) the limitations of the study and (5) a summary with concluding remarks of the thesis.

8.2 Summary to the study

Chapter One the researcher set out to present the research process as it unfolded. That chapter provided an overview of the study. The background and purpose of this study was explained. The motivation for conducting this study and the nature of Mathematics with respect to the study were discussed. Research questions and key terminology in the study were presented and explained. Thereafter, the National Performance in the Eleven most Popular subjects in 2015-2019 and performance trends were analysed. This was followed by summaries of successive chapters.

Chapter Two began with the history of Mathematics in South Africa and 4IR, highlighting international and national studies conducted. Thereafter the state of current mathematics was described. Major systemic problem areas impacting the achievement in Mathematics were discussed. The state of current mathematics teaching and learning practices in South African schools was discussed. The challenges experienced amongst learners of Mathematics: content knowledge gaps that exist in mathematical knowledge, challenges teachers experience and the impact of the Gross Domestic Product (GDP) of the country. Pedagogical content knowledge was addressed. Research on the factors associated with mathematics performance was explained. Perspectives of the use of manipulatives was defined.

Thereafter, the context of the study was defined. Performance trends of Grade12 mathematics Paper 2 from 2016 to 2020 are interrogated, an overview of learner performance in Paper 2 was presented and suggestions for improvement recommended. The descriptions of the different learning styles that exist and which apply to this study were discussed. In this chapter the literature and relevant studies on the use of manipulatives in Mathematics were cited and discussed. The 2018,2017 and 2016 National Senior Certificate (NSC) Diagnostic report for paper 2 were analysed, offering an overview of learner performance in paper 2. Suggestions for improvement are discussed and the importance of introducing manipulatives into the classroom. A comprehensive literature review and findings were summarized. Recent South African and international studies were deliberated. In the study advantages, drawbacks, benefits and mistakes of using manipulatives were discussed. The chapter concluded with a summary of Lesh's Translation Model and its application to this study.

In Chapter Three the theories of Piaget and Vygotsky and Kolb's Experiential Learning Theory (ELT) underpinning this research were reviewed. In this study, information processing constructivism was used. Constructivism as a paradigm suggests that learning is an active process. Knowledge is constructed and not directly perceived through the senses. In this study the researcher acknowledges that by using manipulatives in the research allowed for learners to construct new knowledge and for social interaction and exchange of ideas to take place through group discussions.

Chapter Four continued with a discussion of the research methodology employed. Methodological issues relevant to this study were considered. The critical research questions and research instruments were explained to be in keeping with the dictates of some experts in the field of education research. The interpretive paradigm was explained in detail and proven to coinside with the theoretical framework adopted for this study. The data capture strategies were aligned to a qualitative approach used in this study. In addition reliability and validity were discussed.

Chapter Five provided details of a pilot study undertaken by the researcher in a school in the Pinetown District, in search of validation of the research instruments used in this research and thus placing it in context. The pilot study made use of mathematical models/manipulatives to teach grade 12 learners to solve three-dimensional trigonometric problems. In this chapter, discussions and results on the pilot study are reported.

In the pilot study qualitative methods were employed and data collected through the use of mathematical models, activity worksheets, semi-structured interview schedules, observation and video/audio recordings that were administered to grade 12 mathematics learners (n=9). The semi-structured interview schedule was designed to give an insight into the learners' experiences and knowledge of working with mathematical models and their cognitive processes when solving the three-dimensional trigonometric problems. Modifications to the interview schedule were discussed and findings and recommendations of the pilot study were presented.

The emphasis in Chapter Six was on the analysis of the data. Qualitative methods were applied in this study and the data was retrieved through activity sheets, semi-structured interviews and observations. The learner responses in the activity sheets were analysed to ascertain how the purposely designed manipulative, enhanced the learning of trigonometry among grade 12 learners and how the use of manipulatives impacted the learners' understanding and learning. The semi -structured interview was designed to acquire an in-depth insight into the learners' knowledge of and skills in solving three-dimensional trigonometric problems. While learners were interacting and engaging with the manipulative, the researcher observed the learners.

Chapter Seven began with the discussion of learners' responses and revealed that the use of manipulatives aids the understanding of mathematical concepts when taught using activities containing 3D trigonometric problems, which includes the transition from concrete to abstract. The chapter offered graphs, tables, transcripts and written responses of grade 12 mathematics learners solutions and responses to solving 3D trigonometric problems using area rule, sine rule, cosine rule and basic trigonometric ratios along with application of theorem of Pythagoras. This information served to explore the conceptual understanding of trigonometric concepts and the influence the mathematical manipulatives have on the grade 12 learners understanding. In this chapter the results were analysed via thematic content analysis within the context of the literature review contained in Chapter Two. Themes that emerged from thedata obtained were discussed in detail and related to similar research conducted. An over view of the study was given on the advantages of manipulatives and how they improve performance and increases understanding of abstract concepts. The research questions were addressed.Thematic content analysis and steps in the analysis process were discussed. The eight themes that emerged form this study were discussed in detail. Learning modes, utility of manipulatives in relation to mathematics itself and the analysis flow chart were explained.

The final chapter, Chapter Eight captured the researcher's views and thoughts of the study. Initially the chapter summarized the research providing an overview of the study and thereafter, subsequent sections concluded the research.

8.3 Research questions and focus of enquiry

This study made use of concrete mathematical manipulatives in exploring conceptual understanding of grade 12 mathematics learners in learning to solve 3-dimensional trigonometric problems. The research questions addressed in this study were:

8.3.1 Research Question One: How will Mathematical models help learners to adopt more active approaches towards learning of three-dimensional trigonometric problems amongst Grade 12 learners?

The use of manipulatives encouraged learner interaction, engagement and collaborative learning among the learners. The findings of this research confirmed that the use of manipulatives had provided opportunities to the learners to observe and to pay attention towards the teacher's instructions, statements and questions. The presence of these 3 dimensional mathematical models aided learners to think and recall previous knowledge on which to build from and construct newly acquired knowledge. From the observation notes and video recordings, it was observed that the learners were actively engaged with each other and with the manipulatives. Learners discussed, argued, defended their answers and maintained respect for each other at the same time. They touched and pointed at the manipulative and explained to each other what they had seen, which angle or side on the horizontal plane, vertical plane or slant plane was required to be calculated.

In addition, the semi-structured interviews, observation, video/audio recordings and learner responses to the activity sheet from the grade 12 learners, revealed that the grade 12 learners were actively engaged and fully occupied during the solving of three-dimensional trigonometry problems whenusing the mathematical manipulatives. Carbonneau, Wong & Borysenko(2020) explain that the use of concrete manipulatives in classrooms provides opportunities for learners to interact physically with abstract content, which they normally would be unable to visualize or touch. Burns(2007) and Cocket& Kilgour (2015) add that through this process learners become more engaged with the learning material and are more successful in interpreting abstract mathematical concepts based on concrete manipulatives(Willingham 2017; Sarama & Clements 2009).

Collaborative learning can be described to any form of instructional activity that occurs when learners work together and interact closely in order to exchange ideas and thoughts to achieve a common goal (Laal & Ghodsi, 2012; Prince, 2004). Extensive research on collaborative learning has been researched and has been proven to be effective in increasing learner engagement and learning performance (Van Leeuwen & Janssen, 2019; Freeman, Eddy, McDonough, Smith, Okoroafor, Jordt &Wenderoth, 2014).

Loes, An, Sachaie & Pascarella (2017) add that collaborative learning is particularly useful to learners, which potentially helps their understanding of the material they are being taught or to which they are exposed. Carbonneau, Wong & Borysenko (2020, 2) explain that learning with concrete manipulatives highly depends on students' interpretation of the manipuatives. Collaborative learning is one strategy that can be easily integrated within the classroom. Collaborative learning may be particularly beneficial in the context of learning with perceptually rich manipulatives. Learners who become overly fixated on surface properties of objects may benefit from engaging in discussions with peers who are more adept at recognizing the abstract representation of the manipulatives.

Learners explained that the use of manipulatives aided them in understanding trigonometry in a fun way and that the lessons were enjoyable and interesting. Very little or no learning takes place when the environment is threatening and therefore educators are requested to create a productive and conducive learning environment (Larbi & Mavis, 2016). The learners experienced enjoyment and pleasure working with manipulatives. Learners indicated that when they saw the manipulatives and touched them, they became interested and eager to learn. Visual aids arouse interest in learners and helps teachers to teach concepts in an easy way (Shabiralyani *et al*., 2015). Learners obtain enjoyment when taught mathematics through participatory and interactive ways that use manipulatives (Choden & Chalermnirundorn, 2021; Day & Hurell, 2017; Stiegelmeier & Moore, 2019; Strom, 2009; Reimer & Moyer, 2005).

The use of mathematical models/ manipulatives helped learners develop perseverance and increased their confidence in problem solving of three-dimensional trigonometric problems.

Perseverance is a key process through which Mathematics can be learned with understanding. However, withstanding such uncertainty can be difficult for learners to endure and therefore necessitates support. Teachers are required to offer support and provide some form of scaffolding so as the ease the experience. The findings of this study showed that learners had persevered significantly. Learners' perseverance increased over time as they interacted together in a collaborative manner. Whilst persevering their self-confidence also increased and they became more confident in tackling the three-dimensional trigonometric problems. Some of the learner's responses from the interview include:

> ***Learner 1****: "I am more confident than I was previously";*
> ***L****earner* ***2****:"I am more confident in answering these questions.";*
> ***L****earner* ***4****: "Yes! This activity was enjoyable, boosts my confidence in attacking 3D problems. I can visualize and conceptualize better.";* ***L****earner* ***14:*** *"Yes. With the understanding of the work done I am able to do the working better because I would be confident.";*
> ***L****earner* ***15****: "Yes, if I continue using examples like this, I will be more prepared and confident to write the NSC November exam."*

The findings in this study have suggested that manipulatives supported perseverance in solving three-dimensional trigonometric problems as learners encouraged each other to revisit their conceptual thinking and to re-initiate and re-sustain their productive struggle by exploring a different set of mathematical ideas and approaches. In addition the data revealed malleability of perseverance, which implied that learners could increase and improve their perseverance in solving three-dimensional trigonometry problems over a period of time through the use of carefully designed mathematical manipulatives. Sengupta- Irving & Agarwal (2017) claim that it is expected that students are to learn Mathematics in such a manner that when they are exposed to challenging problems, they will persist and not surrender. The creation of opportunities for students to persist in problem solving is therefore argued as being important to effective teaching and to learners acquiring positive dispositions in mathematics learning.

The idea of struggle has long been recognized as key to learning mathematics with understanding (Festinger, 1957; Dewey, 1910). Researchers (Carbonneau, Marley & Selig, 2013 ; Saitta, Gittings & Geiger, 2011; McNeil, Uttal, Jarvin & Sternberg, 2009) state that learners' activities that integrate the use of manipulatives may improve learning as they provide more opportunities and exposure for learners to interact and engage with each other. There has been a profound impact on learners' learning achievement when there is an increase in learner engagement in STEM (Science, Technology, Engineering and Mathematics) learning (Freeman *et al.,* 2014). Lee (2014) claims that behavioural engagement can be considered to be one component of student engagement that examines and explores the amount of effort and perseverance a student applies in learning. Farrington, Roderick, Allensworth, Nagaoka, Keyes, Johnson & Beechum, 2012) define academic perseverance as being the likelihood that students will finish a task to the best of their abilities in spite of obstacles and challenges they encounter on their path. Carbonneau *et al.*, (2020:3) claims that since manipulatives are concrete objects designed to firstly represent abstract concepts that may otherwise be challenging to grasp and secondly increase student engagement while learning, argues that learning using manipulatives can potentially influence students' perseverance. Belenky & Schalk (2014) agrees with Carbonneau *et al.* by stating that learning with manipulatives may aid in promoting academic perseverance by increasing learners' interest and engagement in their learning. Research has revealed that the use of well-designed external knowledge representations, such as manipulatives, may capture the student's interest in the topic or hold the continued interest in some learners. The use of manipulatives to trigger interest among learners in a topic is important as students level of interest are predictive of their commitment, their engagement and their perseverance in learning (Hay, Callingham & Carmichael, 2015).

The study revealed that there was a reduction of task difficulty. Several of the participants indicated that using manipulative to solve 3D trigonometry problems reduced the difficulty of the problem and made it easier to understand. Examples of their responses are :

> L*earner* 9:"Yes was easy to solve problems when using manipulatives. Reason being it helps me identify each triangle or shape with their own angles individually. So when I see the model I am able to separate, identify and solve";
> *Learner 5:"It was completely easy, however it has made understanding three dimensional problems much better"; L1:"Yes. They helped me to conceptualise the problem and it made it easier to answer these questions."*

The reoccurring focus by the participants that the use of manipulatives demonstrates a reduction of task difficulty indicated that there does exist genuine benefits to the use of these manipulatives. It aided learners in breaking down the question and being successful in arriving at the correct answer to the problem. All educators could use this method in their teaching so as to assist their learners.

8.3.2 Research question two: How will the use of models improve the learning of mathematics?

The use of the models/ manipulatives promoted mathematical fluency.
From the learners' responses in Chapter Six, it was evident that several learners had attained mathematical fluency. Computational fluency can be described as having the ability to apply more than one method to solve a problem, possessing the ability to use the most efficient method and the ability

to execute the solution correctly. It was observed that learners used different methods to arrive at their answers.A learner becomes proficient inMathematics when they are able to think mathematically, fluently make choices between strategies and engage in mathematical discussions with other learners (Kilpatrick, Swafford & Findell, 2001). Watson and Sullivan (2008) concur with Kilpatrick *et al*. by agreeing that learners who develop fluency, efficient, accurate and flexible ways of dealing with numbers are better able tocope with tasks that involve everyday life Mathematics. Cartwright (2018: 208)claims that " mathematical fluency is the result when learners' strategies and ability to reason are concurred with their conceptual understanding".

The findings of this research confirmed that the use of manipulatives improved spatial skills that are fundamental to higher level thinking, reasoning and creative processes. The spatial skills of the learners had improved. The use of manipulatives helped the participants in visualising the problem and understanding the problem better. The learners were better able to extract triangles in the various planes namely the horizontal plane, vertical plane and the slant plane. Lohman (1996) defines spatial skills as a person's skills to generate, retain, retrieve and transform well-structured visual images. Spatial skills encompass cognitive skills associated with spatial visualization, mental rotation and spatial orientation (Uttal & Cohen, 2012). Spatial visualization is the process of constructing, maintain and manipulating 2D and 3D objects in one's mind (Uttal, Meadow, Tipton, Hand, Alden, Warren & Newcombe, 2013; Cracow & Sorby, 2008). Mental rotation is described as the rotation of mental representations of 2D or 3D objects to determine their images from various viewing angles (Ha & Fang, 2016). According to Lin, Chen, & Lou (2014) spatial orientation involves the change of location in space in relation to two-dimensional (2D) or three dimensional (3D) objects that a person can see.

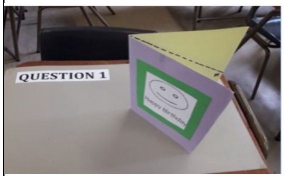
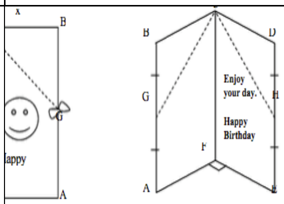
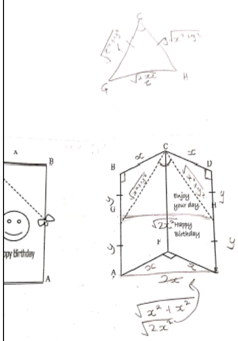
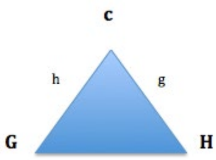
Research carried out has revealed that spatial skills play a crucial role in developing expertise and success in science, engineering, and technology and mathematical fields of study. Examples of some of the learners' responses in this study are contained in Chapter Seven.

In this study the results showed that the use of mathematics manipulatives aided learners link concrete to abstract concepts (CRA). Witzel (2005) described the concrete representational, abstract (CRA) instruction as a process for teaching and learning mathematical concepts. Beginning with the manipulation of concrete materials, the process then transfers learners to the representational level and then climaxes at the abstract level. CRA instruction makes provision for students to make associations from one stage of the process to the next. When learners are first permitted to develop a concrete understanding of the mathematical concept, they then become more likely to successfully carry out a mathematics skill and genuinely understand mathematical concepts at an abstract level.

> ***L9:*** *"Yes! Reason being it helps me identify each triangle or shape with their own angles individually. So when I see the model I am able to separate and identify and solve."*

Manipulatives are considered to be effective in fostering the development and enhancement of conceptual understanding in Mathematics as they assist learners to link and relate concrete ideas to abstract ideas (Uribe-Florez & Wilkins, 2010; Witzel and Allsopp, 2007;). Jones and Tiller (2017) claim that by making use of hands–on, concrete manipulatives during mathematics teaching time could result in learners having a higher retention rate and develop a more positive attitude towards their education. Smith (1999) further adds that a well designed manipulative bridges or closes the gap between formal Mathematics and informal Mathematics.

Specific information for the stages of the CRA is shown in table 8.1

ages	**y elements**	**mple problem**	**planation**
ncrete	3D mathematical model of a birthday card	 	re a colourful birthday card was placed on a flat surface.
ANSITION TO REPRESENTATIONAL	e of concrete and representational materials together		ce learners engaged with the concrete manipulative, they began to draw the triangles in the various planes
PRESENTATIONAL	angles in the horizontal plane, vertical plane and Slant plane		the representational stage of CRA the learners are comfortable using the given information and working with the triangles in the various planes
ANSITION TO ABSTRACT	e of representational and abstract materials together	tract Triangle CGH on the slant plane 	• Visualise triangle CGH on the slant plane • Link side GH on horizontal plane equal side GH on the slant plane. • Given two sides and required to find an angle meant that the cosine rule had to be applied.

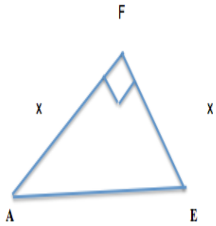
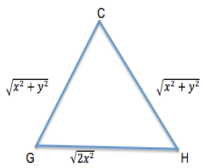
STRACT	Apply theorem of Pythagoras Now applying the cosine rule in ΔGCH	$= h = \sqrt{x^2+y^2}$. ...Pythagoras k had to be made GH = h = AE  $= \sqrt{x^2+x^2} = \sqrt{2x^2}$ ....Pythagoras k has to be made, side GH = c = AE w applying the cosine rule in ΔGCH as to be made, side GH = c = AE pplying the cosine rule in ΔGCH  $+ g^2$ -2hg.cos c	udents at the abstract level of CRA no longer require the assistance of the manipulative to solve the problem

Table 8.1 Specific information for the stages of CRA.

It is evident from table 8.1 that learners where able to link the concrete mathematical concept to the abstract mathematical concept.

Figure 8.1 below explains the flow chart of manipulatives in relation with mathematics

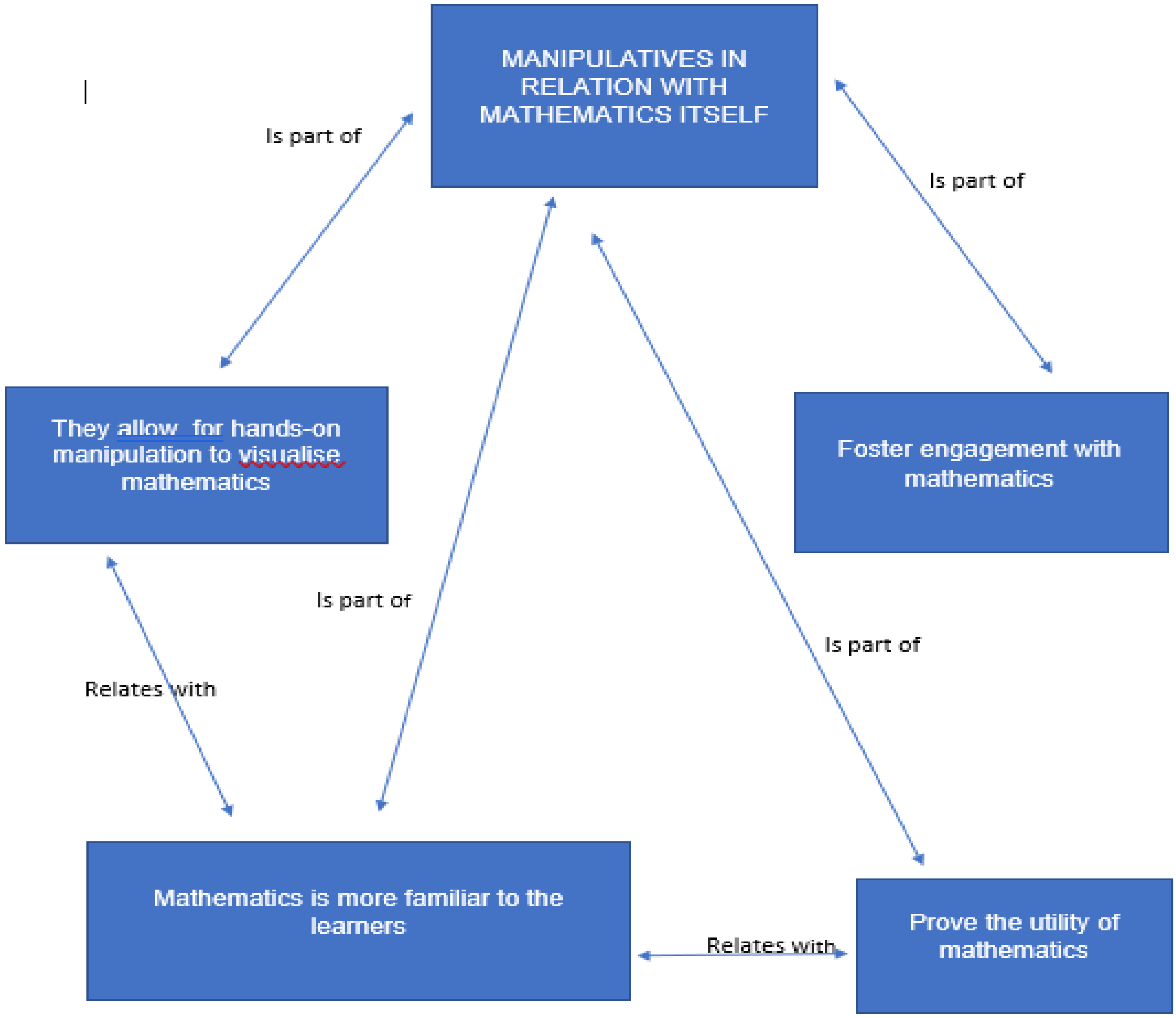


Figure 8.1 Utility of Manipulative in relation to mathematics itself (adapted from Maz-Machado, Madrid, Mantero & Fanjul, 2019)

This category groups ideas linked with labels in which students reflect about mathematics itself. The face-to-face lessons which the learners attended show the utility of Mathematics, foster learner engagement with Mathematics, revealthat Mathematics is more familiar with learners and allows hands-on manipulation to visualize mathematics. This allowed for the use of models to improve learning of Mathematics.

Finally, to unpack the necessary mental constructions required to solve the 3D trigonometric problems, the analysis flow chart below depicted the learners‘ thought processes. During observation of the learners, it was noted that the learners’ communication had increased and they began to draw a flow charts to determine the path they would follow.

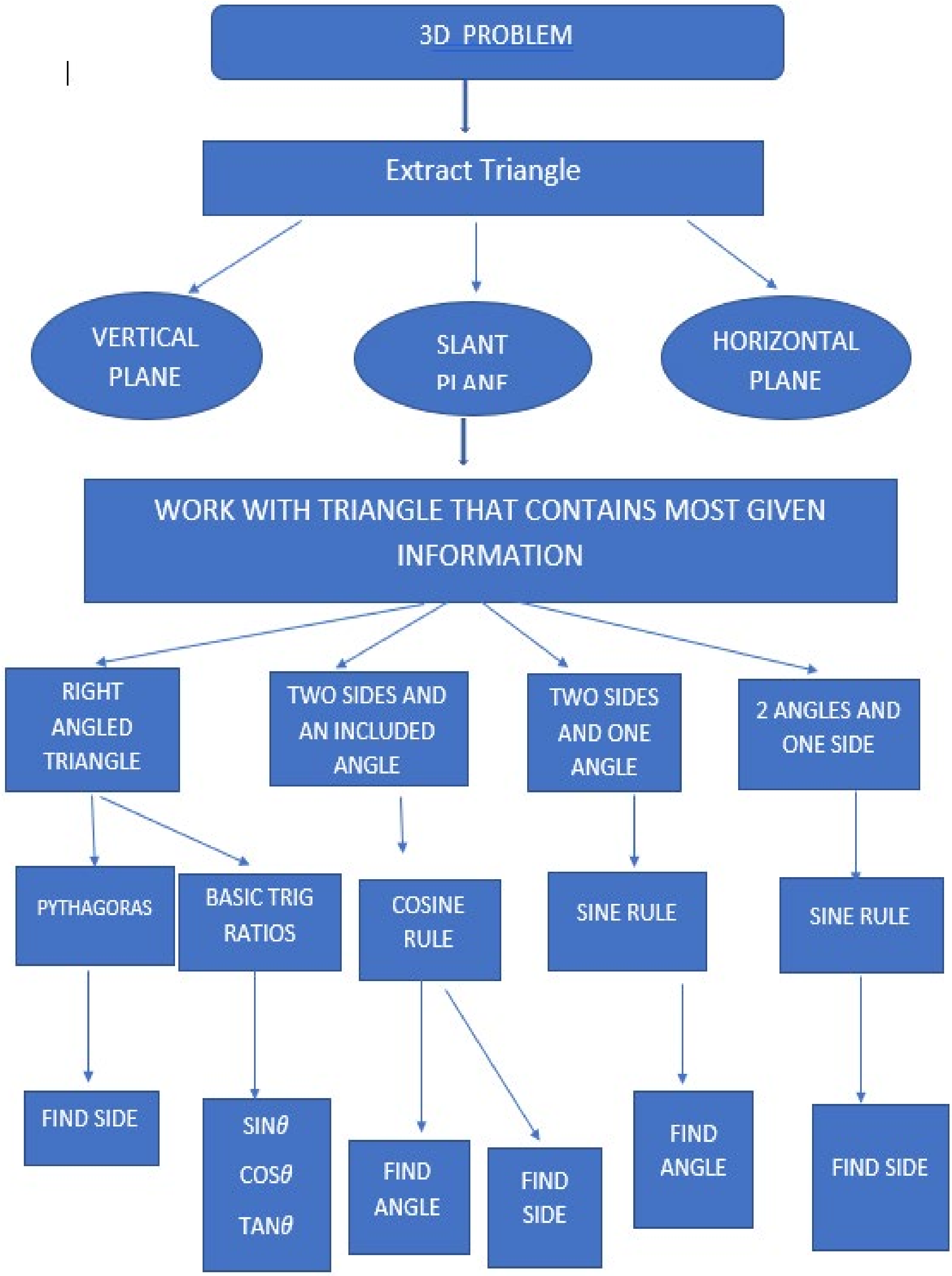


Figure 8.2 Analysis flow chart

The use of this flow chart allowed for learners to clearly and strategically decompose the three dimensional problem and in addition it provided a lens to aid learners.

8.4 Contributions Made By This Study

The teaching and learning setting in South African has been transformed by policy-makers revising curricula in accordance with the Curriculum and Assessment Policy Statement (DoBE, 2012). According to Van Laren (2012: ;203) there are revision plans to improve teaching and learning in Mathematics. This study has provided an alternate approach to teaching trigonometry by developing and using mathematical three-dimensional models to extend teachers' pedagogical content knowledge for Grade 12 three-dimensional (3D) problem solving in trigonometry. The study addressed the challenge experienced by the Minister of Education, this challenge being that teachers' knowledge and the supply of quality learning support material was inadequate in the area of trigonometry amongst other teaching areas (Motshekga, 2012: 2). The resultant effect was poor performance by learners in Mathematics. According to the National Senior Certificate (NSC) School Subject Report (2019, 4) the report presents performance in eleven gateway subjects. The report was to be used by school managers, subject advisors, district planners and curriculum specialists to analyse a specific subjects overall performance within a school, cluster, circuit or district level. The report assisted education stakeholders to identify the specific subject that was displaying poor performance and provided and ensured appropriate interventions were immediately introduced. Mathematics clearly revealed a downward trend and all stakeholders needed to identify factors that had contributed to the poor performance and to arrive at solutions to best address the challenges that existed. If poor performance was detected early then, it would allow for stakeholders to put in place effective interventions early enough to see a positive impact thus retaining high performance levels and also increasing performance. It was evident from the above NSC reports that intervention strategies have to be put in place to ensure learners become more confident in solving 3D trigonometric problems. This study made use of manipulatives to aid students to visualise the 3-D figure and extract the triangles in the horizontal, vertical and slant planes. The use of manipulatives aid learners in deeper conceptual uderstanding . The manipulatives made learning more meaningful and provided a link from concrete to abstract. Motshekga in DoBE (2017: 174) provided recommendations for improvement which included the need for teachers to develop strategies to be used when solving right-angled triangles and triangles that are not right-angled. Teachers should use models to show learners the different planes in a 3-D shape. This would assist learners to identify the different triangles and place angles and side in perspective.

The low Mathematics results exposed the challenges in Mathematics education throughout the basic education sector in South Africa (Collet & Steyn, 2017). Olivier (2018) added that challenges amongst Mathematics learners included the learners' lack of interest and self-directed learning (SDL), inability to learn at the levels they are being taught, low confidence and a presence of significant content knowledge gaps in their mathematical knowledge. The challenges amongst teachers are the lack of skills and confidence to teach the Mathematics curriculum effectively and the widespread reliance of several on traditional teacher-centered pedagogies (Collett & Steyn, 2017; Olivier, 2016).

The use of manipulatives in this study has proven to aid learners in solving 3D trigonometry problems. This study has produced findings similar to that of other national and international researchers (Johnson, O' Meara & Leavy, 2021; Hidayah and Prayoga, 2021; Carbonneau, Wong & Borysenko, 2020)

The literature review has argued for methods to address the study's research opportunity, which was to evaluate the effectiveness of manipulatives in Mathematics. The emergence of the themes all point to the advantages and benefits of using mathematics manipulatives in teaching.

The following themes were identified in this study: (1) **Theme one**: Interaction/Engagement/ collaborative learning. (2) **Theme Two:** Fun, enjoyable and interesting. (3) **Theme Three:** Link from concrete to Abstract.

(4) **Theme Four:** Lack of use of manipulatives by educators. (5) **Theme Five:** Reduction of Task difficulty. (6) **Theme Six:** Increased confidence in Problem Solving. (7) **Theme Seven**: Increases spatial skills -Helps Visualise . (8) **Theme Eight:** Promotes mathematical fluency In the ensuing discussions the above themes are further elucidated in 7.5.1 to 7.5.8.

Effective mathematics pedagogy has been described by several different terms. Newmann and Wehlange (1998) describe effective mathematics pedagogy as standard authentic instruction, active learning for Mathematics (Smith, 1999) and professional standards for teaching Mathematics (Martin & Speer, 2009). It can be clearly seen that these descriptions of effective pedagogy have a common concept, which is the creation of a learning environment that allows both the learner and the educator to participate in constructive mathematical concepts. The use of mathematical manipulatives in this study allowed for the creation of a learning environment that encouraged learner participation and the construction of mathematical knowledge. The emergence of the themes mentioned in the paragraph above confirms that manipulatives do encourage participation and construction of knowledge.

Benning (2020) provides six categories of effective mathematics pedagogy namely: Creating a Mathematical Setting, Worthwhile Mathematics Tasks, Mathematical Connections, Mathematical Discussions, Assessment of Students' Learning and Pedagogical Content Knowledge.

Creating a Mathematical Setting

Choosing mathematical tasks that are rich and creating strong learning environment promotes successful mathematics learning (Gervasoni, Hunter, Bicknell & Sexton, 2012; Kilpatrik, Swafford & Findel, 2001). Benning (2020) claims that effective mathematics pedagogy relates to the creation of an environment to induct learners into Mathematics learning. It deals with the actions an educator applies before, during and after a lesson. The construction of a mathematical setting is viewed to incorporate both knowledge and skills of the educator to create a learning environment that sustains the learners' interest and attention. The use of manipulatives in this study strongly supports the creation of a mathematical setting. Learners sustained interest and attention when engaging with the manipulatives.

Worthwhile Tasks

Anthony and Walshaw (2009) claim that selecting a worthwhile mathematical task or problemorientated task in the classroom enhances learners productive dispositions where learners can cultivate an inclination towards the worthwhileness of mathematics. Benning (2021) reiterated that a task can be considered worthwhile if the objectives and activities related with the task involve learners in discovering procedures, mathematical concepts and relationships.

Ingram (2013) adds that worthwhile mathematical tasks assists learners to develop skills such as perseverance, mathematical intimacy and integrity, concentration, independence, reflection and cooperation. Hiebert & Grouws (2007) claim that an important aspect of pedagogy is that a task should provide opportunities to learners to apply their mathematical knowledge and thinking skills to solve challenging problems.

Mathematical Connections

Benning (2021) explains mathematical connectivity as to the process a teacher engages the learners to apply to approaches that are not rehearsed in order to produce several solutions to a task. Jacobs & Spangler (2017) add that the inclusion of open-ended tasks assists learners to make sense of new mathematical concepts.

Mathematical Discussions

Having class discussions to assess learners' mathematics understandingcould enhance their adaptive reasoning where learners can increase their mathematical knowledge to unfamiliar situations. Stein, Engle, Smith & Hughes (2008) describe mathematical discussion as providing opportunities for learners to communicate their conjectures, concepts and mathematical dispositions. There are five important key practices for composing mathematical discussion in the classroom. These comprise of being able to predict learners' responses, monitoring both group work and individual work, choosing work for whole class discussion, sequencing the work submitted by groups or individuals and finally facilitating the connection-making between learners' responses.

Assessment of Students' Learning

According to Benning (2021) assessment is a broad topic. Assessment is used to monitor a learner's understanding and mathematical understanding as well as to provide feedback to the teachers future instruction and teaching methods, updates parents on their childs performance, provides policy direction in education and determines the mathematical achievement of the learners in a specific country.

Pedagogical Content Knowledge

Teachers knowledge of the content of Mathematics and what is important for students to learn and the decisions taken by the teacher about classroom instruction are all important in the construction of knowledge. This study has offered an alternate approach to teaching trigonometry by developing and using mathematical three-dimensional models to extend teachers' pedagogical content knowledge.

The Mathematics models in this study were based on three dimensional problems that were selected from past NSC papers and these problems showed learners the relevance of trigonometry and its use and need in everyday life situations. Learners were better able to appreciate the relevance of trigonometry.

From the discussion above, the creation of a conducive learning environment, selection of rich and worthwhile tasks, mathematical discussions, mathematical connections, assessment of learners' work and educators' pedagogical content knowledge are all deemed important to mathematics pedagogy.

Table 8.2 Elements of effective pedagogy using mathematical manipulatives in trigonometry

Elements of effective pedagogy	Mathematical manipulative use in trigonometry
• Creating a classroom atmosphere that promotes the needs of individual students	-Selecting good problems that all - for the exploration of mathematical concepts -sustain interest and attention
• Building on learners' thinking	- Allows students a chance to absorb and extend their knowledge
• Worthwhile mathematical tasks questions that appeared in past year NSC questions	- the models were designed from
• Making connections	- the use of the manipulatives encouraged visualisation - manipulatives encouraged learners to explore various solutions when solving 3D trigonometry problems - Manipulatives fostered deep thinking and linking with other ideas within mathematics
• Assessment for learning	-The use of the mathematics manipulatives/models

	allowed for observation and questioning to facilitate learners' reasoning and learning
• Mathematical communication	- The use of manipulatives/ Mathematics models allowed for group work and class discussions for the grade 12 learners to communicate their mathematical ideas
• Mathematical Language	- The use of manipulatives models its own mathematical language and accommodates learners with different home language - unique mathematical language provides strategies for solving challenging 3D trigonometry problems
• Teacher knowledge and learning	- Manipulatives brings to the classroom a variety of representations to promote an assortment of mathematical perspectives.

Adapted from Benning (2021)

8.5 Recommendations

Further research into the use of manipulatives in secondary schools is required. This research should focus on learners' perceptions of manipulatives and how they contribute to improved understanding and attitudes. The findings of this type of research should be communicated to teachers form both primary and secondary schools to highlight the identified benefits of the use of manipulatives and to motivate and encourage teachers to use manipulatives as learners transition from the GET phase to the FET phase. Research on secondary educators beliefs in the use of mathematics manipulatives in the classroom is another area that needs to be researched. Mathematics manipulatives proved to be beneficial in Mathematics. Future research could investigate if the use of manipulatives in other learning areas would improve learning and build deeper understanding.

Considering the results of this study, the recommendations below have been developed for policy makers, researchers and practitioners:

(1) Educators should be more willing to use manipulatives in their Mathematics lessons. In addition, they should be encouraged to train and develop themselves in the use of manipulatives in Mathematics lessons.

(2) In-service and distance education training should be provided so as to reduce the lack of knowledge in using manipulatives and it should be mandatory that almost all secondary school Mathematics teachers participate in the training.

(3) It is evident that exhibitions, symposiums and training about manipulatives should be organized in collaboration with universities. NGOs (Non-Governmental Organizations) can also be included in these organisations so as to reach a larger population.

(4) There is a need to offer professional opportunities that improve teachers understanding of good pedagogical practices around the use of manipulatives. Such opportunities should concentrate on being able to identify suitable manipulatives that would aid mathematical reasoning across all mathematical

domains and also engage teachers in discovering their use, as learners themselves so as to develop a better mathematical understanding. Teachers then need to be allocated time use and have a trial with manipulatives in their classrooms to experience positive outcomes of improved mathematical understanding.

(5) There is a need for the production and design of teacher support materials and guidelines that provide samples or exemplars of good practice in the manipulative usage in mathematics teaching and learning.

(6) For manipulatives to be effective and to meet the curriculum demand, it is recommended that the educator prepares the lesson plan and implements it carefully when the educator plans to use manipulatives integrated with a series of written and oral questions.

(7) Reaching mathematical fluency must be a priority for all educators to attain in their students.

(8) Algebraic skills and equations and manipulation skills must be thoroughly covered and mastered by learners in the GET and FET phases as this becomes prerequisites in solving 3D trigonometric problems.

If these recommendations are heeded, then the researcher strongly believes that manipulatives can become a prominent feature of Mathematics education in secondary schools. Research shows that, if or when this becomes a reality, the pedagogical gaps that currently exist, will reduce and learners will be better assisted as they transition from a concrete to a more abstract understanding of mathematics.

8.6 Limitations of the study

The following limitations existed in this study:

8.6.1 Covid 19 pandemic

Due to the arrival of the Covid 19 pandemic and strict restrictions being imposed, the sample therefore had to be drawn from a single school.

8.6.2 Sample

The sample was drawn from a single school. This study focuses on grade 12 Mathematics learners. It is argued that from a constructivist perspective, these perceptions on manipulatives may not accurately describe what mathematical beliefs learners really have about mathematics manipulatives.

8.6.3 Socio economic status

This school is situated in a poor socio economic area with limited resources and facilities. Learners' perceptions and beliefs of mathematics manipulatives may be influenced by their socio-economic status.

8.7 Summary and concluding remarks of the thesis

The researcher believes that these findings suggest productive pathways forward. It is intended that this research to assist policy makers, departmental heads and educators to modify and encourage more use of mathematics manipulatives in teaching strategies in Mathematics in secondary schools so as to help learners link mathematical concepts from concrete to abstract. Such changes could improve outcomes for learners from a purely mathematical perspective by improving learners' epistemic beliefs about Mathematics. This research has proven that using mathematics manipulatives is beneficial to all learners. The findings reflected in this study resonate with a number of studies both internationally and nationally. The findings of this study clearly show that manipulatives offer learners an additional lens through which to see mathematics concepts as well as providing an additional resource.

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LIST OF APPENDICES

Appendix A

Gatekeeper permission to department of education

OFFICE OF THE

HEAD OF DEPARTMENT

KWAZULU-NATAL PROVINCE
EDUCATION
REPUBLIC OF SOUTH AFRICA

Enquiries: Phindile Duma
Ref.:2/4/8/1753

Mrs C Niranjan

34 Tensing Way

Everest Heights

VERULAM

4339

Dear Mrs Niranjan

1 PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"THE USE OF MODELS DEVELOP SKILLS RO SOLVE 3DTRIGONOMETRY PROBLEMS: A CASE STUDY OF GRADE 12 LEARNERS IN SELECTED SCHOOLS IN THE PINETOWN DISTRICT"**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 31 May 2021 to 31 August 2023.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers above.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

____

2 Dr. EV NzamaHead of Department: EducationDate: 01 June 2021

Appendix B
Gatekeeper permission to principal of the school

Application to do Research in the Department of Education – Palmview Secondary school in the Pinetown District – Kwa Zulu Natal

To: The Principal

Year: 2021

Research Project: The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners at a school in the Pinetown District.,

Mrs C. Niranjan (PhD student) is doing a study through the **School of Education, Mathematics Education at the Durban University of Technology** with **Prof Deonarain Brijlall.** His contact numbers are **031-3732126** (work) and 083 555 2390. We want to research the use of artefacts/models in the learning of 3D trigonometry problems among grade 12 learners at a selected secondary school in the Pinetown District, KwaZulu-Natal: South Africa.

Learners and educators are asked to help by taking part in this research project, as it would be of benefit to interested educationists and/or mathematics teachers. The aim of this research is to determine the impact artefacts/models would have on the teaching of 3D trigonometry problems in grade 12.

The participants will be randomly selected from the grade 12 mathematics classes.

However, participation is completely voluntary and has no impact or bearing on evaluation or assessment of the learner in any studies or course while at school. Participants are required to interact with the manipulatives provided by the researcher and then complete the activity sheet. The activity worksheet would be of a duration of approximately three lesson. Participants will then be engaged in semi-structured interviews after the activity sheets have been completed. These interviews will be recorded by the researcher in the form of transcripts. The interviews would be of duration of approximately one hour. All participants will be noted on transcripts and data collections by a *pseudonym* (i.e. fictitious name). The identities of the interviewees will be kept strictly confidential. All data will be stored in a secured password and not been used for any other purpose except for the research. At the end of the research, data collected will be correctly disposed of. The responses to the activity sheets will be shredded and electronic files will be overwritten.

Participants may leave the study at any time by notifying the researcher. Participants have the right to withdraw from the research project without any consequences. Participants may review and comment on any parts of the researchers' written reports. A copy of the thesis will be placed in the schools library for the participants to view.

The following Covid 19 guidelines will be strictly followed:

- All surfaces will be disinfected
- Social distancing to be adhered to, 1,5m apart.
- Windows and doors to be open to allow cross and natural ventilation. No fans to be switched on.
- Sanitizing of hands and the maths models before and after use.
- Use of facemasks to cover nose and mouth.

____________________________ ____________________

(*Researcher's Signature*) (*Date*)

DECLARATION

I,___________________________ _____________________(NAME and SIGNATURE).

_____________________on this day of__________month_________2021, hereby grant permission to the researcher to go ahead with the research in the above mentioned school following the terms of reference noted in this request letter.

Appendix C
LETTER OF INFORMATION
Pre Study

LETTER OF INFORMATION Pre Study

Title of the Research Study : The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners at a selected school in the Pinetown District.

Principal Investigator/s/researcher: Caresse Niranjan, Masters in Mathematics Education

Co-Investigator/s/supervisor/s: Professor Deonarain Brijlall (PhD) and Dr Veronica Masuku (PhD)

Good day. How are you?

I am a student at the School of Education at the Durban University of Technology doing research for my Doctorate in Education degree.

I would like to invite you to participate in this pre study research. Research is the systematic investigation into and study of materials and sources in order to establish facts and reach new conclusions.

The purposeof this research is do determine the impact the use of models/manipulatives have on learners ability to solve 3D trigonometry problems. The pre study will be conducted to test the validity of the research instruments before the main study can be carried out.

Participants are randomly selected from the grade 12 mathematics classes. You are required to interact with the models/manipulatives provided by me and then complete the activity sheet. The activity worksheet will be of a duration of approximately three lessons. You will be observed. On completion of the activity sheet, you will then be engaged in a semi-structured interview. These interviews will be recorded by the researcher in the form of transcripts. The interviews will be of an approximate duration of one hour.

This research process will not pose any risk or discomfort to you. Covid 18 Protocol will be followed as follows:

- All surfaces will be disinfected
- Social distancing to be adhered to, 1,5m apart.
- Windows and doors to be open to allow cross and natural ventilation. No fans to be switched on.
- Sanitizing of hands and the maths models before and after use.
- Use of facemasks to cover nose and mouth.

Participation is completely voluntary and has no impact or bearing on evaluation or assessment of the learner in any studies or course while at school. You may leave the study at anytime by notifying me. You have the right to withdraw from the research project without any consequences.

Learners and educators are asked to participate in this research project, as it would be of benefit to interested educationists and/or mathematics teachers. Learners would benefit by being taught through the use of model

There is no remuneration in participating in this study.
You will not be expected to cover any costs towards the study.

You will be noted on transcripts and data collections by a pseudonym (i.e fictitious name). Your identity will be kept strictly confidential. All data will be stored in a secure password and not be used for any other purpose except for research.

You may review and comment on any parts of my written reports. A copy of the thesis will be placed in the school library for you to view.

This is not a high-risk research project and research related injuries is zero.

On completion of the research, all data collected will be correctly disposed of. The responses to the activity sheets will be shredded and electronic files will be overwritten.

Persons to contact in the Event of Any Problems or Queries: (Supervisor and details) Please contact the researcher (tel no.), my supervisor (tel no.) or the Institutional Research Ethics Administrator on 031 373 2375. Complaints can be reported to the Director: Research and Postgraduate Support Dr L Linganiso on 031 373 2577 or researchdirector@dut.ac.za.

Appendix D
Letter of Consent

CONSENT

Full Title of the Study: The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners in a selected school in the Pinetown District.

Names of Researcher/s: Caresse Niranjan

Statement of Agreement to Participate in the Research Study:

- I hereby confirm that I have been informed by the researcher, Caresse Niranjan, about the nature, conduct, benefits and risks of this study - Research Ethics Clearance Number: 124/21,
- I have also received, read and understood the above written information (Participant Letter of Information) regarding the study.
- I am aware that the results of the study, including personal details regarding my sex, age, date of birth, initials and diagnosis will be anonymously processed into a study report.
- In view of the requirements of research, I agree that the data collected during this study can be processed in a computerised system by the researcher.
- I may, at any stage, without prejudice, withdraw my consent and participation in the study.
- I have had sufficient opportunity to ask questions and (of my own free will) declare myself prepared to participate in the study.
- I understand that significant new findings developed during the course of this research which may relate to my participation will be made available to me.

______________________ ___________ ______ ________________

Full Name of Participant Date Time Signature / Right Thumbprint

I,_______________(name of researcher) herewith confirm that the above participant has been full informed about the nature, conduct and risks of the above study.

____________________ ___________ _____________________

Full Name of Researcher Date Signature

____________________ ___________ _____________________

Full Name of Witness (If applicable) Date Signature

__________________ __________ ___________________

Full Name of Legal Guardian (If applicable) Date Signature

Appendix E
Information letter to parents

LETTER OF INFORMATION Pilot Study

Dear Parent/Guardian

Warm greetings!

My name is Caresse Niranjan. I am studying towards a Doctor in Education degree at the University of Technology. Below you will find more information about the study.
Title of the Research Study : The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners at a selected school in the Pinetown District.
Principal Investigator/s/researcher: Caresse Niranjan, Masters in Mathematics Education
Co-Investigator/s/supervisor/s: Professor Deonarain Brijlall (PhD) and Dr Veronica Masuku (PhD)
Brief Introduction and Purpose of the Study: The purpose of this research is do determine the impact the use of models/manipulatives have on learners ability to solve 3D trigonometry problems. The pilot study will be conducted to test or validate the research tools before the main study can be carried out. I would like to invite your child/ward to participate in this pre study research. Research is the systematic investigation into and study of materials and sources in order to establish facts and reach new conclusions.

Participants are randomly selected from the grade 12 mathematics classes. Your child/ward is required to interact with the models/manipulatives provided by me and then complete the activity sheet. The activity worksheet will be of a duration of approximately three lessons. Your child/ward will be observed. On completion of the activity sheet, your child/ward will then be engaged in a semi-structured interview. These interviews will be recorded by the researcher in the form of transcripts. The interviews will be of an approximate duration of one hour.

This research process will not pose any risk or discomfort to your child/ward. The following Covid 19 guidelines will be followed:

- All surfaces will be disinfected
- Social distancing to be adhered to, 1,5m apart.
- Windows and doors to be open to allow cross and natural ventilation. No fans to be switched on.
- Sanitizing of hands and the maths models before and after use.
- Use of facemasks to cover nose and mouth.

Participation is completely voluntary and has no impact or bearing on evaluation or assessment in any studies or course while at school. Your child/ward is free to withdraw out of the study at any time. There will be no problem for your child/ward if he/she decides to leave the study.

Learners and educators are asked to participate in this research project, as it would be of benefit to interested educationists and/or mathematics teachers. Your child/ward would benefit by being taught through the use of models.

Your child/ward will not get any money and/or gift to take part in the study. Participation is free, therefore your child/ward will not have to pay anything to be part of this study.

For confidentiality purposes, your child's/ward's name will not be written on any of the forms. Your child's/ward's personal details (surname and name) will also not be individually linked to the results of the study. Your child/ward will be noted on transcripts and data collections by a pseudonym (i.e fictitious name). Your child's/ward's identity will be kept strictly confidential. All data will be stored in a secure password and not be used for any other purpose except for research.

Your child/ward may review and comment on any parts of my written reports. A copy of the thesis will be placed in the school library for you to view.

This is not a high-risk research project and research related injuries are zero.

On completion of the research, all data collected will be correctly disposed of. The responses to the activity sheets will be shredded and electronic files will be overwritten.

Persons to contact in the Event of Any Problems or Queries: (Supervisor and details) Please contact the researcher (Caresse Niranjan 032 5333013), my supervisor (Prof Deonarain Brijlall 0835552390) or the Institutional Research Ethics Administrator on 031 373 2375. Complaints can be reported to the Director: Research and Postgraduate Support Dr L Linganiso on 031 373 2577 or researchdirector@dut.ac.za.

Appendix F
Informed consent parents

Appendix D
Informed consent for pilot study

CONSENT FOR PILOT STUDY

Full Title of the Study: The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners in a selected school in the Pinetown District.

Names of Researcher/s: Caresse Niranjan

Statement of Agreement to Participate in the Pilot Research Study:

- I hereby confirm that I have been informed by the researcher, Caresse Niranjan, about the nature, conduct, benefits and risks of this study - Research Ethics Clearance Number: 124/21____,
- I have also received, read and understood the above written information (Participant Letter of Information) regarding the study.
- I am aware that the results of the study, including personal details regarding my sex, age, date of birth, initials and diagnosis will be anonymously processed into a study report.
- In view of the requirements of research, I agree that the data collected during this study can be processed in a computerised system by the researcher.
- I may, at any stage, without prejudice, withdraw my consent and participation in the study.

- I have had sufficient opportunity to ask questions and (of my own free will) declare myself prepared to participate in the study.
- I understand that significant new findings developed during the course of this research which may relate to my participation will be made available to me.

_______________________ ___________ _______ _________________

Full Name of Participant **Date** **Time** **Signature / Right Thumbprint**

I,________________(name of researcher) herewith confirm that the above participant has b fully informed about the nature, conduct and risks of the above study.

____________________ ___________ ______________________

Full Name of Researcher **Date** **Signature**

____________________ ___________ ______________________

Full Name of Witness (If applicable) **Date** **Signature**

__________________ __________ ____________________

Full Name of Legal Guardian (If applicable) Date **Signature**

Appendix G
Information letter participants

LETTER OF INFORMATION Study

Warm greetings!
My name is Caresse Niranjan. I am studying towards a Doctor in Education degree at the University of Technology. Below you will find more information about the study.
Title of the Research Study : The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners at a selected school in the Pinetown District.
Principal Investigator/s/researcher: Caresse Niranjan, Masters in Mathematics Education
Co-Investigator/s/supervisor/s: Professor Deonarain Brijlall (PhD) and Dr Veronica Masuku (PhD)
Brief Introduction and Purpose of the Study: The purpose of this research is do determine the impact the use of models/manipulatives have on learners ability to solve 3D trigonometry problems. I would like to invite you to participate in this study research. Research is the systematic investigation into and study of materials and sources in order to establish facts and reach new conclusions.

Participants are randomly selected from the grade 12 mathematics classes. You are required to interact with the models/manipulatives provided by me and then complete the activity sheet. The activity worksheet will be of a duration of approximately three lessons. You will be observed. On completion of the activity sheet, you will then be engaged in a semi-structured interview. These interviews will be recorded by the researcher in the form of transcripts. The interviews will be of an approximate duration of one hour.

This research process will not pose any risk or discomfort to you. The following Covid 19 guidelines will be followed:

- All surfaces will be disinfected
- Social distancing to be adhered to, 1,5m apart.
- Windows and doors to be open to allow cross and natural ventilation. No fans to be switched on.
- Sanitizing of hands and the maths models before and after use.
- Use of facemasks to cover nose and mouth.

Participation is completely voluntary and has no impact or bearing on evaluation or assessment in any studies or course while at school. You are free to withdraw out of the study at any time. There will be no problem for you if you decide to leave the study.

Learners and educators are asked to participate in this research project, as it would be of benefit to interested educationists and/or mathematics teachers. You would benefit by being taught through the use of models.

You will not get any money and/or gift to take part in the study. Participation is free, therefore you will not have to pay anything to be part of this study.

For confidentiality purposes, you name will not be written on any of the forms. Your personal details (surname and name) will also not be individually linked to the results of the study. You will be noted on transcripts and data collections by a pseudonym (i.e fictitious name). Your identity will be kept strictly confidential. All data will be stored in a secure password and not be used for any other purpose except for research.

You may review and comment on any parts of my written reports. A copy of the thesis will be placed in the school library for you to view.

This is not a high-risk research project and research related injuries are zero.

On completion of the research, all data collected will be correctly disposed of. The responses to the activity sheets will be shredded and electronic files will be overwritten.

Persons to contact in the Event of Any Problems or Queries: (Supervisor and details) Please contact the researcher (Caresse Niranjan 032 5333013), my supervisor (Prof Deonarain Brijlall 0835552390) or the Institutional Research Ethics Administrator on 031 373 2375. Complaints can be reported to the Director: Rese

Appendix H
Informed consent of participants

CONSENT OF PARTICIPANT

Full Title of the Study: The use of models to develop skills to solve 3D trigonometry problems: A case study of Grade 12 learners in a selected school in the Pinetown District.

Names of Researcher/s: Caresse Niranjan

Statement of Agreement to Participate in the Research Study:

- I hereby confirm that I have been informed by the researcher, Caresse Niranjan, about the nature, conduct, benefits and risks of this study - Research Ethics Clearance Number: 124/21 _,
- I have also received, read and understood the above written information (Participant Letter of Information) regarding the study.
- I am aware that the results of the study, including personal details regarding my sex, age, date of birth, initials and diagnosis will be anonymously processed into a study report.
- In view of the requirements of research, I agree that the data collected during this study can be processed in a computerised system by the researcher.
- I may, at any stage, without prejudice, withdraw my consent and participation in the study.
- I have had sufficient opportunity to ask questions and (of my own free will) declare myself prepared to participate in the study.
- I understand that significant new findings developed during the course of this research which may relate to my participation will be made available to me.

______________________ ___________ ______ ________________

Full Name of Participant Date Signature Time / Right Thumbprint

I,________________(name of researcher) herewith confirm that the above participant has b fully informed about the nature, conduct and risks of the above study.

___________________ ___________ _____________________

Full Name of Researcher Date Signature

___________________ ___________ _____________________

Full Name of Witness (If applicable) Date Signature

_________________ __________ ___________________

Full Name of Legal Guardian (If applicable) Date Signature

Appendix I
IREC approval letter

Institutional Research
Ethics Committee
Research and Postgraduate Support Directorate2nd Floor, Berwyn Court Gate 1, Steve Biko Campus Durban University of Technology

P O Box 1334, Durban,

South Africa, 4001Tel:

031 373 2375
Email: lavishad@dut.ac.za
http://www.dut.ac.za/research/institutional_research_ethics

www.dut.ac.za

29 October 2021

MsC Niranjan

34 Tensing Way

Everest Heights

Verulam

4340

Dear Mrs Niranjan

The use of models to develop skills to solve 3D trigonometry problems: A case study ofGrade 12 learners in a selected school in the Pinetown District
Ethics Clearance Number: 124/21

The Institutional Research Ethics Committee acknowledges receipt of your final data collection toolfor review.

We are pleased to inform you that the data collection tool has been approved. Kindly ensure thatparticipants used for the pilot study are not part of the main study.

In addition, the IREC acknowledges receipt of your gatekeeper permission letter.

Please note that FULL APPROVAL is granted to your research proposal. You may proceed withdata collection.

Any adverse events [serious or minor] which occur in connection with this study and/or which may alter its ethical consideration must be reported to the IREC according to the IREC Standard Operating Procedures (SOP's).

Please note that any deviations from the approved proposal require the approval of the IREC asoutlined in the IREC SOP's.

Yours Sincerely,

Prof J K Adam
Chairperson: IREC

ENVISION2030 transparency • honesty • integrity • respect • accountability
fairness • professionalism • commitment • compassion • excellence

THE WORLD UNIVERSITY RANKINGS 2021 TOP 500

Appendix J
Introduction to Research Ethics Certificate


Zertifikat
Certificado
Certificat
Certificate
Promouvoir les plus hauts standards éthiques dans la protection des participants à la recherche biomédicale
Promoting the highest ethical standards in the protection of biomedical research participants

Clinical Trials Centre
The University of Hong Kong

Certificat de formation - Training Certificate

Ce document atteste que - this document certifies that

Caresse Niranjan

a complété avec succès - has successfully completed

Introduction to Research Ethics

du programme de formation TRREE en évaluation éthique de la recherche
of the TRREE training programme in research ethics evaluation

Release Date: 2021/04/28

Professeur Dominique Sprumont
Coordinateur TRREE Coordinator

Ce programme est soutenu par - This program is supported by :

Appendix K
Turnitin Report

Mathematics teaching

ORIGINALITY REPORT

13%	7%	2%	13%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Submitted to University of KwaZulu-Natal Student Paper	8%
2	Submitted to Durban University of Technology Student Paper	3%
3	researchspace.ukzn.ac.za Internet Source	2%

Exclude quotes Off
Exclude bibliography Off
Exclude matches < 2%

Appendix L
Questions, exam papers

INSTRUMENTS

Grade 12 Activity Trigonometry worksheet Solving 3 – Dimensional Trigonometric problems using models/artefacts

Group Number:___________

Instructions :

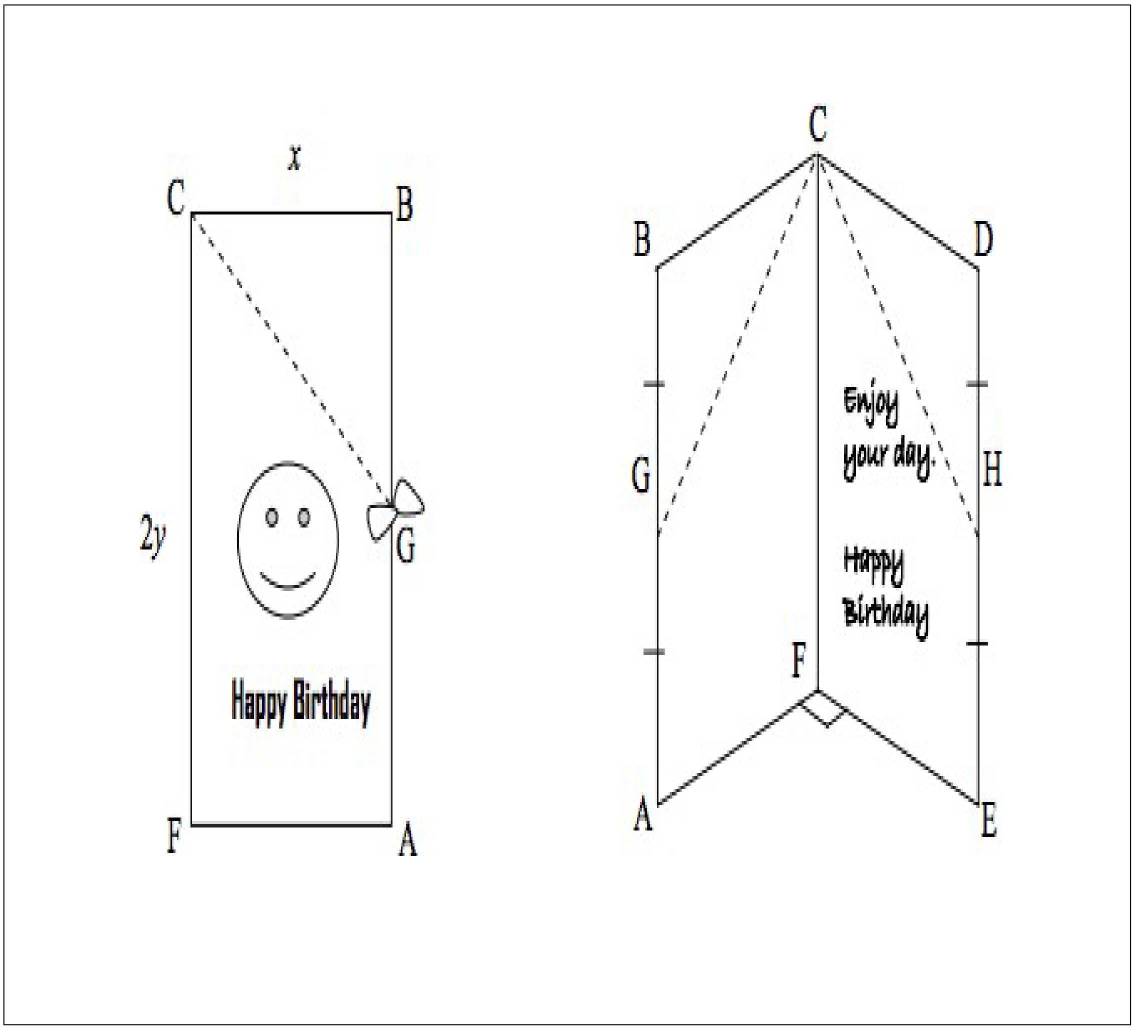
- **The following questions are designed to explore your range of understanding on solving 3D problems using trigonometric ratios, sine rule, cosine rule and area rule. Please answer all questions to the best of your ability.**
- **For each activity show in detail how you arrived at your answer.**
- **Please do not write your names on any of these pages.**
- **One student from each group may be selected to demonstrate their answer on the chalkboard.**

QUESTION 1 (NSC PAPER 2- FEB/MARCH 2012)

A rectangular birthday card is tied with a ribbon at the midpoints, G and H, of the longer sides. The card is opened to read the message inside and then placed on a table in such a way that the angle $A\hat{F}E$ between the front cover and the back cover of the card is 90°. The points G and H are joined by straight lines to the point C inside the card, as shown in the sketch.

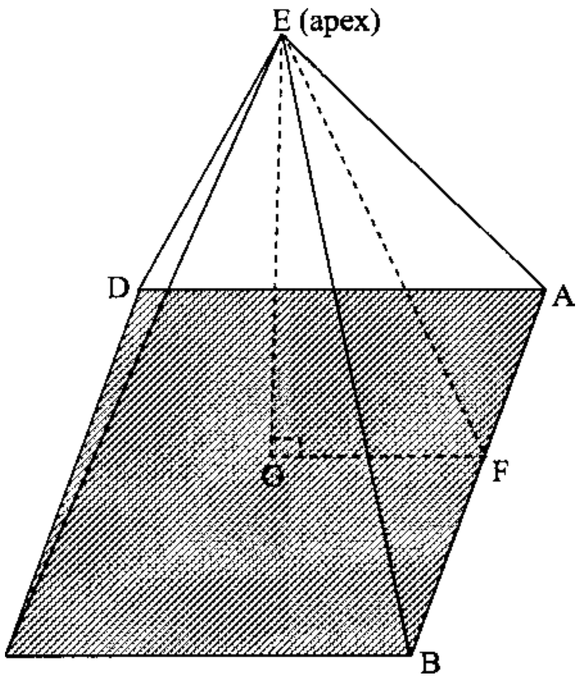
Let the shorter side of the card, BC = x, and the longer side, CF = 2

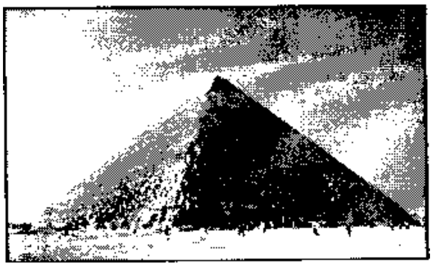
Prove that cos $GCH = \frac{y^2}{x^2+y^2}$



QUESTION 2 (NSC PAPER 2-NOV 2013)

The Great Pyramid at Giza in Egypt was built around 2 500 BC. The pyramid has a square base (ABCD) with sides 232,6 metres long. The distance from each corner of the base to the apex (E) was originally 221,2 metres.





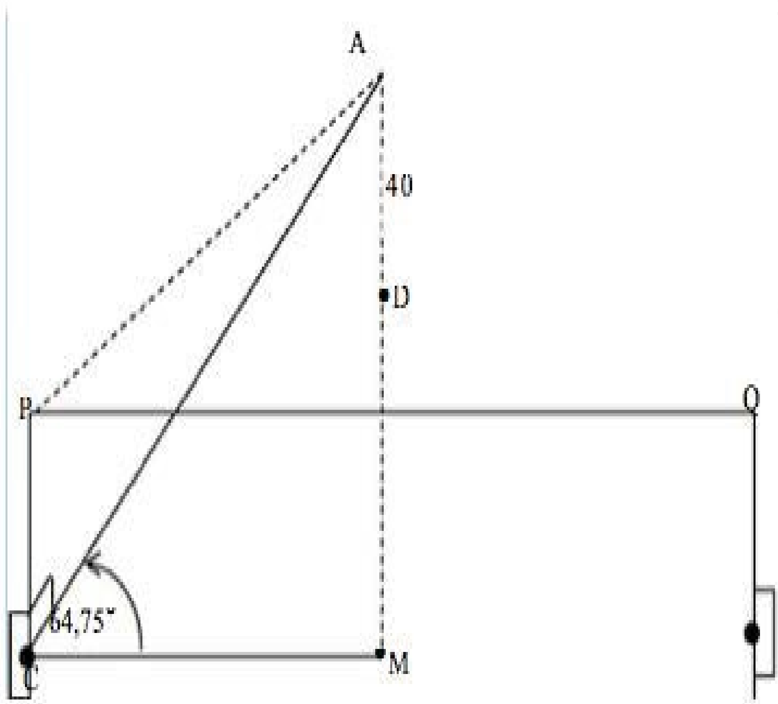
Great Pyramid at Giza in Egypt

2.1 Calculate the size of the angle at the apex of a face of the pyramid (for example $C\hat{E}B$). (3)

2.2 Calculate the angle each face makes with the base (for example $E\hat{F}G$, where EF ⊥ AB in ΔAEB). (6)

QUESTION 3 (NSC PAPER 2-Nov 2010)

The angle of elevation from a point C on the ground, at the centre of the goalpost, to the highest point A of the arc, directly above the centre of the Moses Mabhida soccer stadium, is 64,75°. The soccer pitch is 100 metres long and 64 metres wide as prescribed by FIFA for world cup stadiums. Also AC I PC. In the figure below PQ = 100 metres and PC = 32 metres.



3.1 Determine AC. (3)

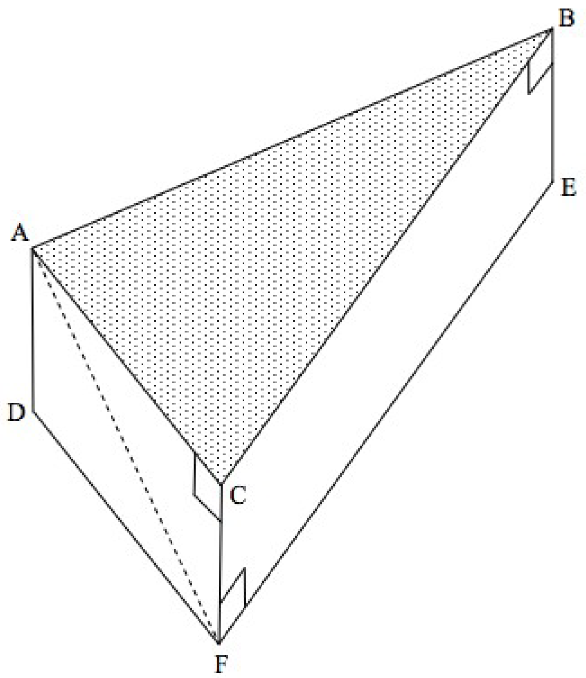
3.2 Calculate $P\hat{A}C$. (3)

3.3 A camera is positioned at point D, 40 metres directly below A. Calculate the distance from D to C. (4)

[10]

QUESTION 4(NSC PAPER 2-Nov 2011)

The figure below represents a triangular right prism with BA = BC = % units.
<ABC = 50^0 and < FAC = 25^0.



4.4 Determine the area of Δ ABC.

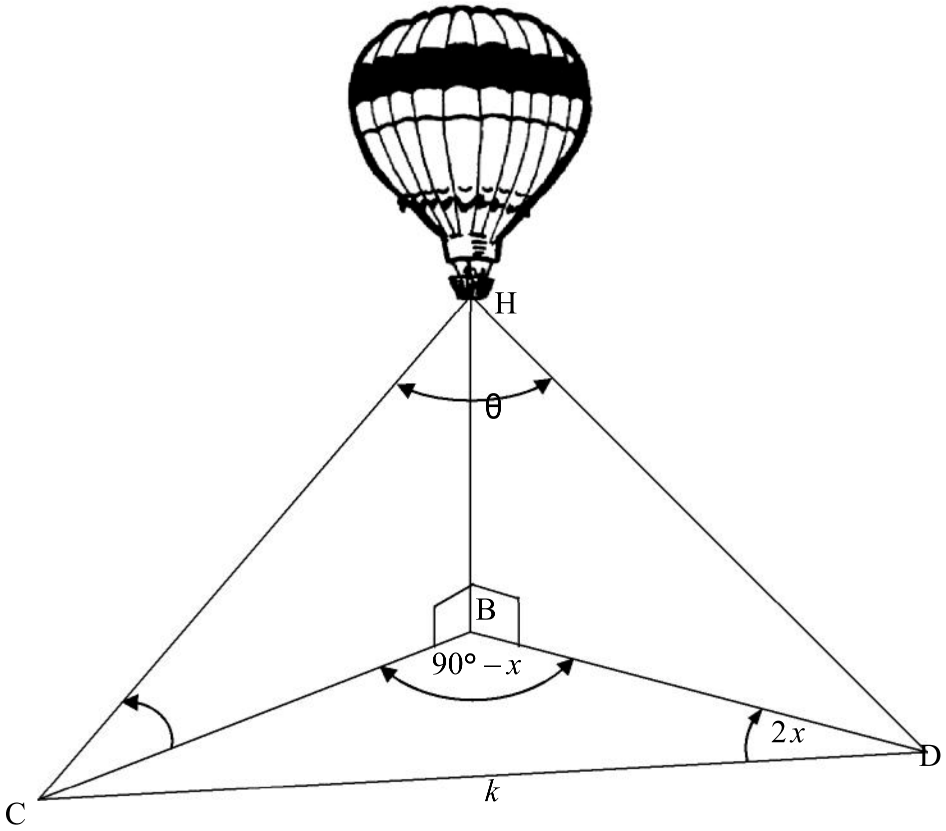
4.5 Calculate the length of AC.

4.6 Hence, determine the height FC of the prism.

QUESTION 5 (NSC PAPER 2-NOV 2012)

A hot-air balloon H is directly above point B on the ground. Two ropes

CDB = $2x$ and CB D = $90° - x$. The distance between C and D is k metres.



are used to keep the hot-air balloon in position. The ropes are held by two people on the ground at point C and point D. B, C and D are in the same horizontal plane. The angle of elevation from C to H is x.

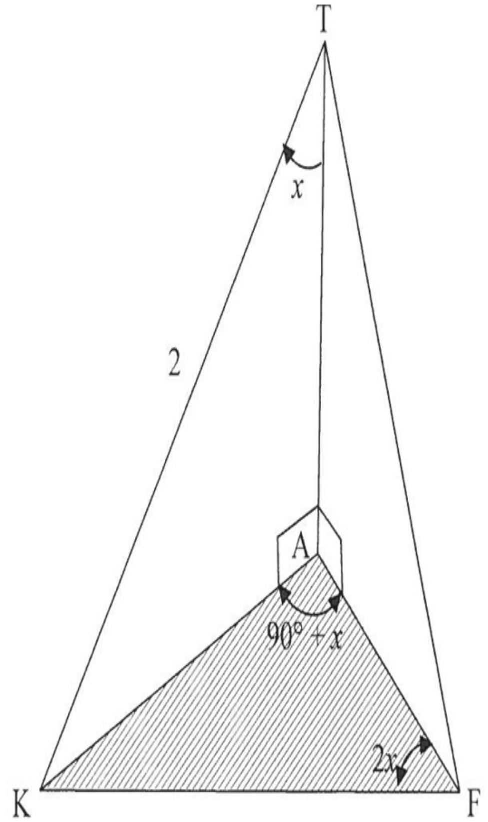
5.1 Show that CB = $2k \sin x$.
(5)

5.2 Hence, show that the length of rope HC is $2k \tan x$.
(3)

5.3 If $k = 40$ m, $x = 23°$ and HD = 31,8 m, calculate θ, the angle between the two ropes.
(4)

Question Six(**NSC PAPER 2-**Feb/March 2015)

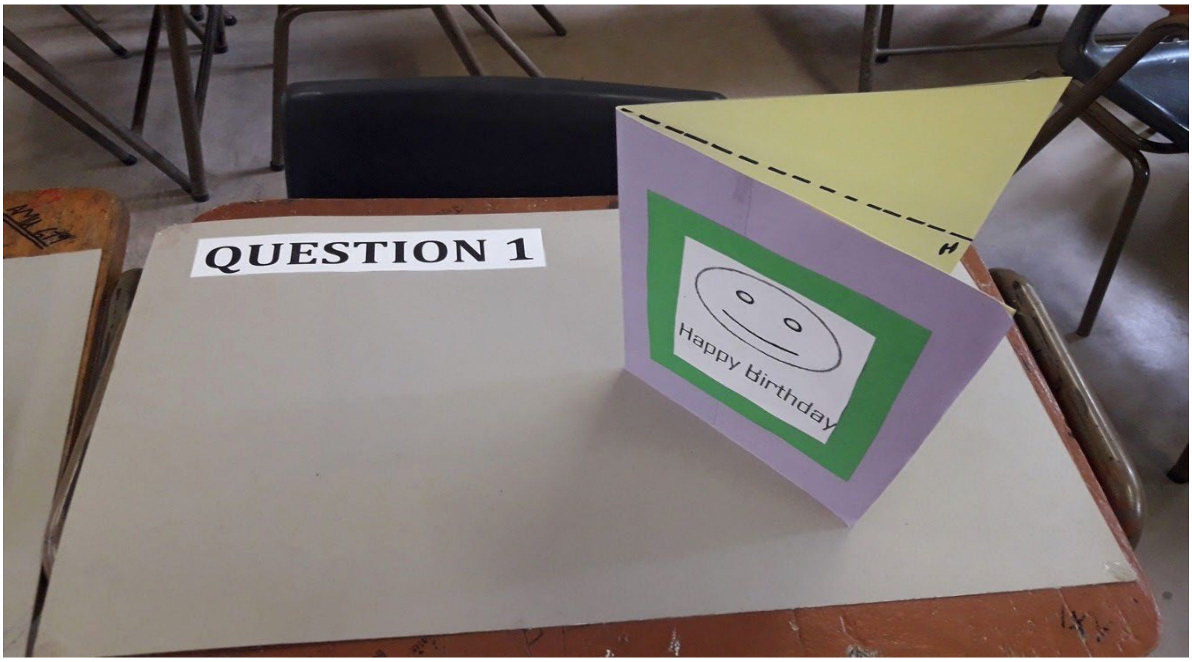
6.1 In the figure, points K, A and F lie in the same horizontal plane and TA represents a vertical tower. $A\hat{T}K = x$, $K\hat{A}F = 90° + x$ and $K\hat{F}A = 2x$ where $0° < x < 30°$. TK = 2 units.

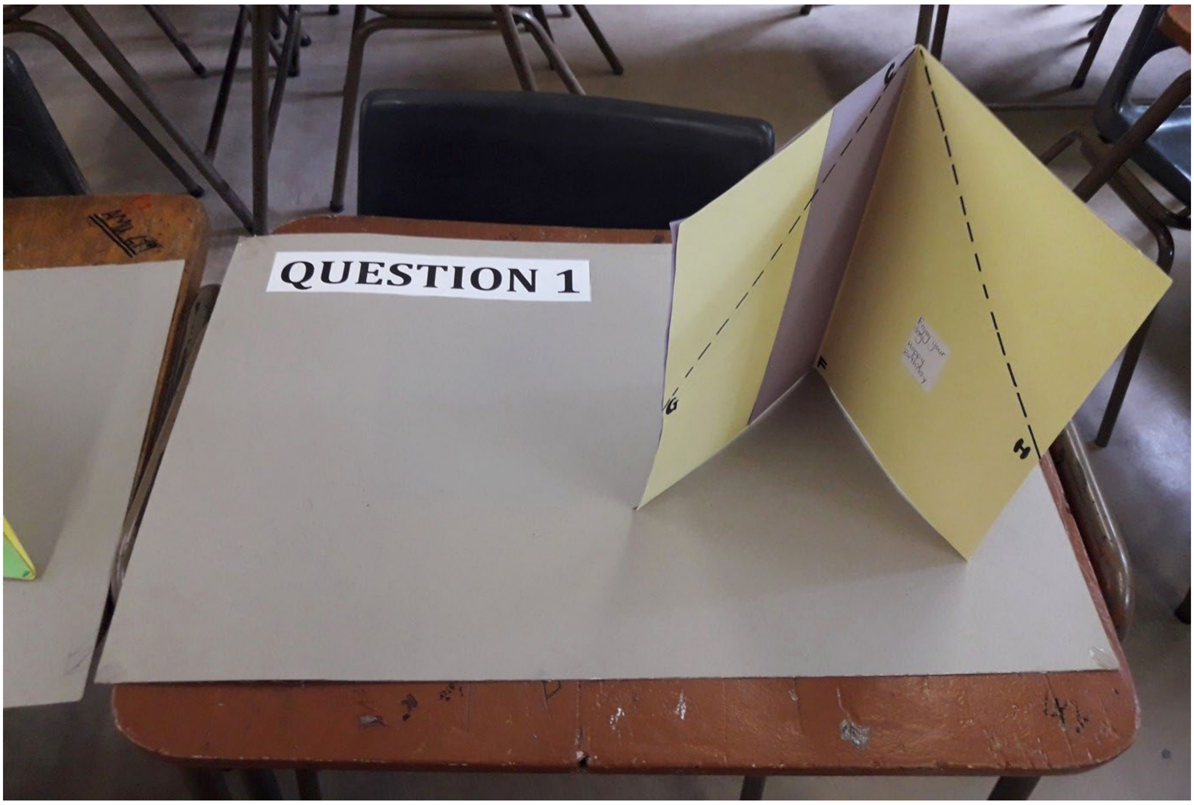


6.1.1 Express AK in terms of $\sin x$. (2)

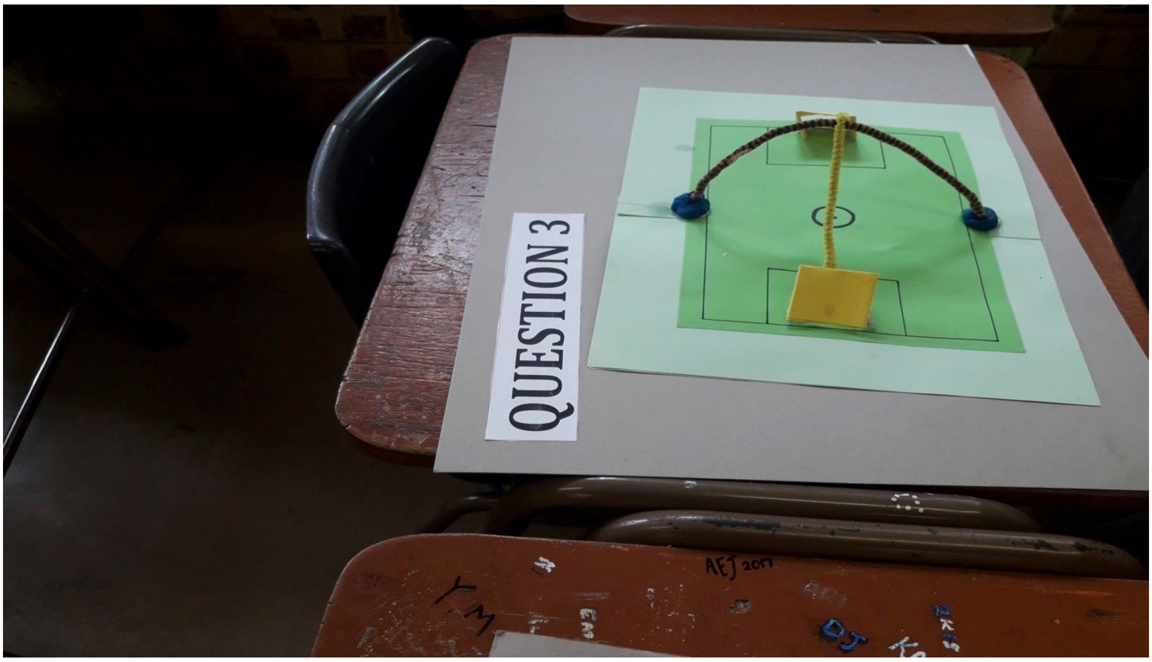
6.1.2 Calculate the numerical value of KF. (5)

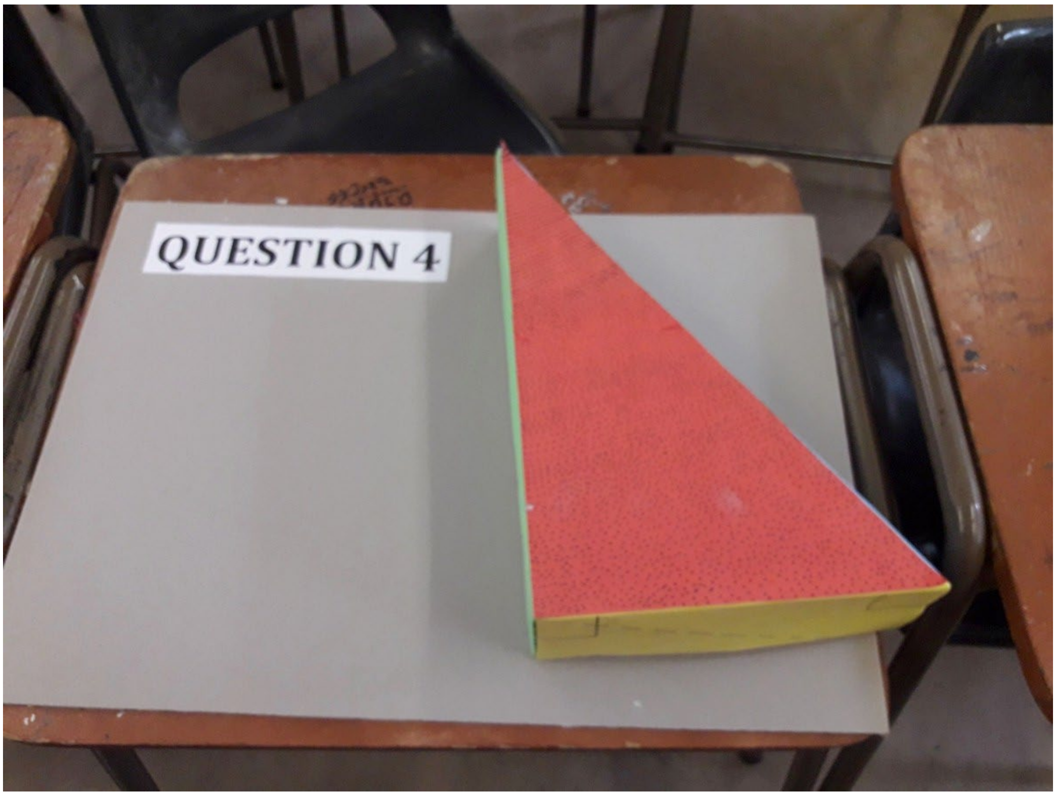
Appendix M
Models designed by researcher

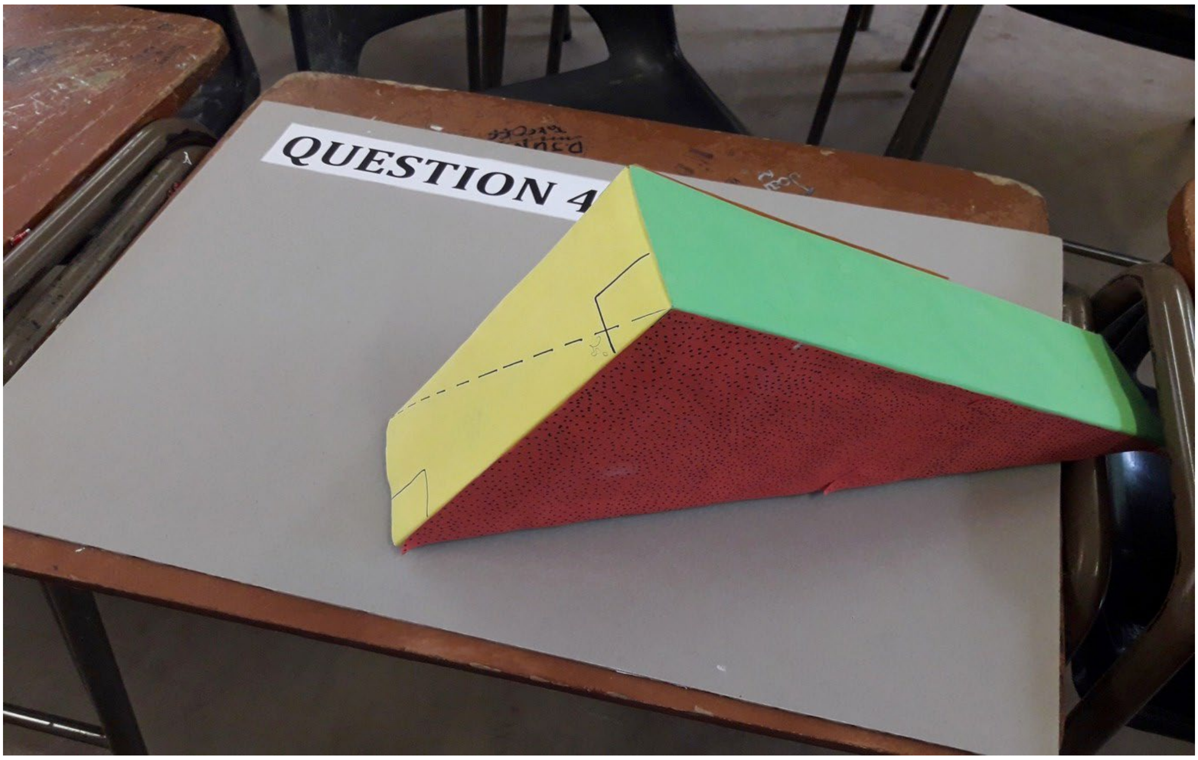


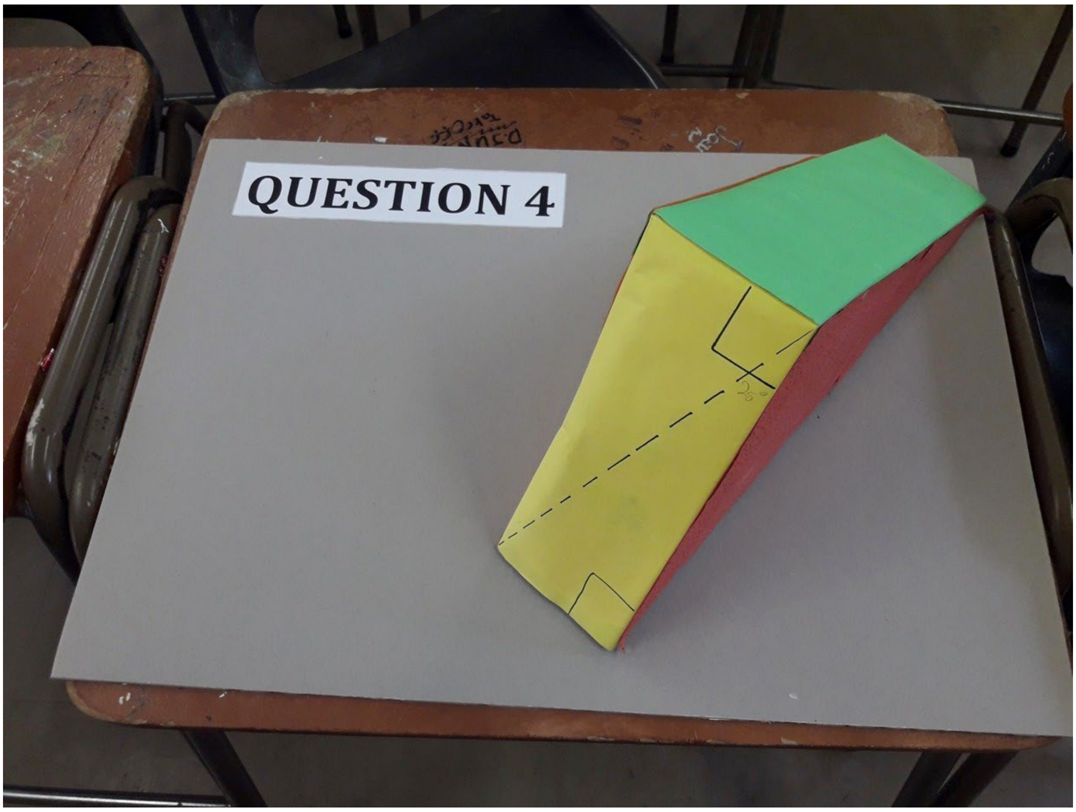


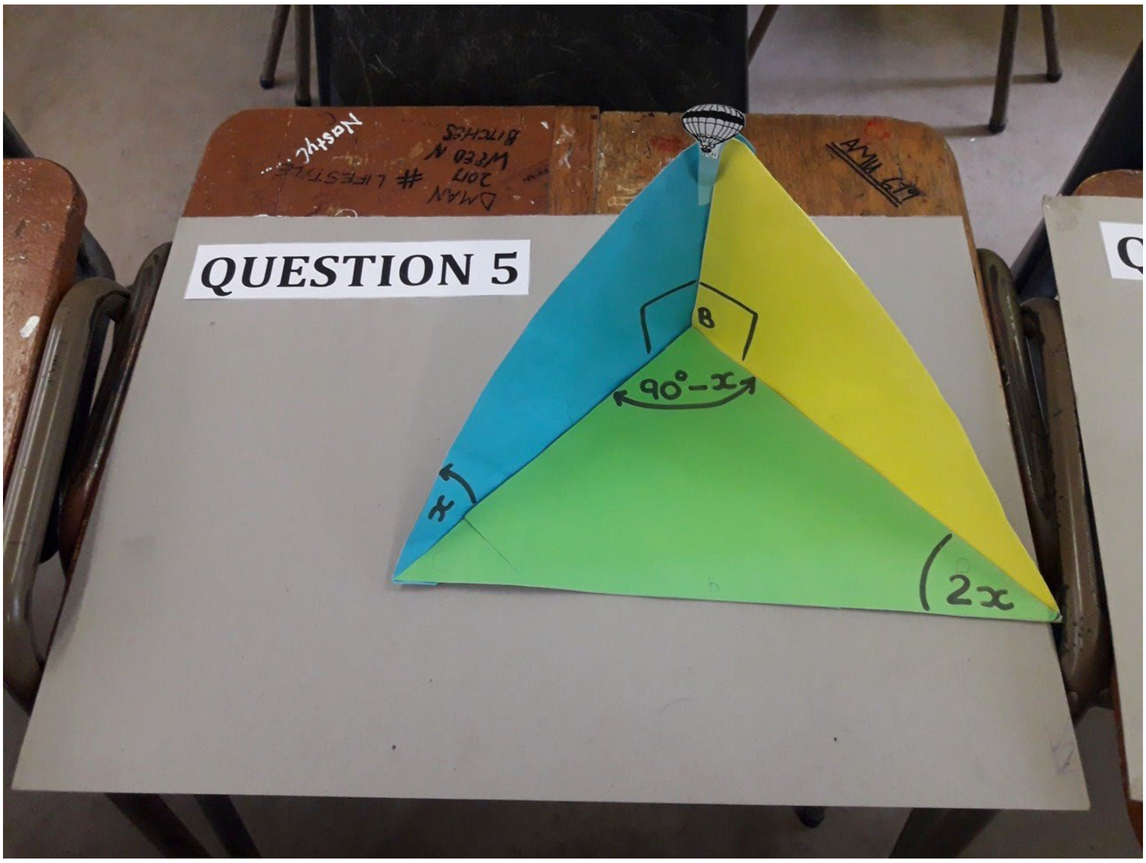

QUESTION 2

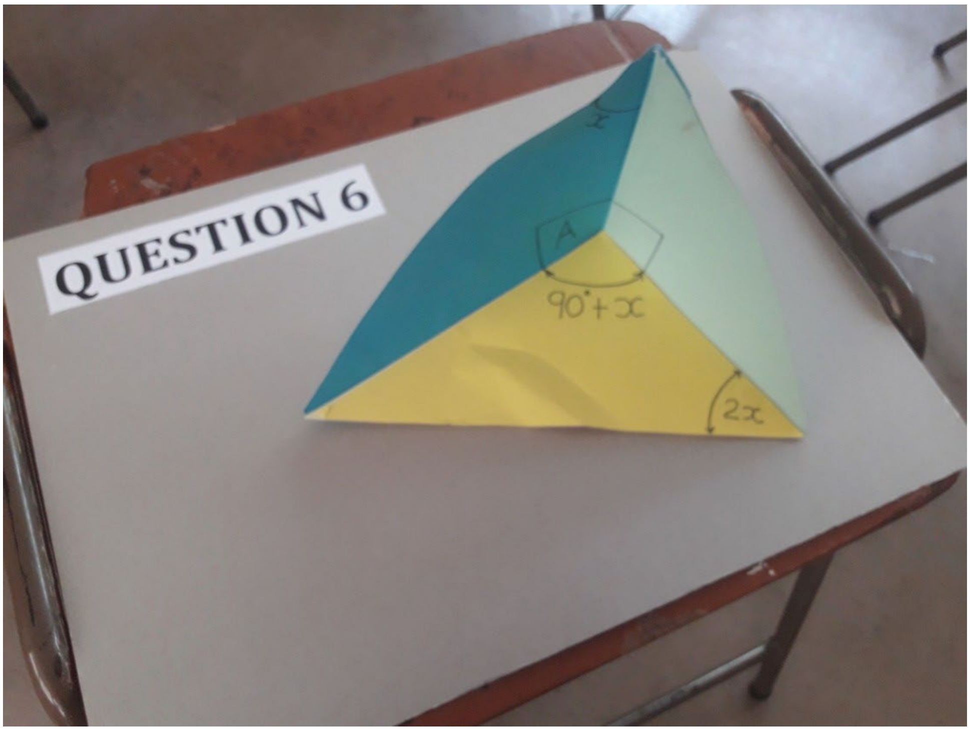

QUESTION 3
AEJ 2017


QUESTION 4


QUESTION 4


QUESTION 4


QUESTION 5
B
90°−x
x
2x


QUESTION 6
x
A
90°+x
2x

Appendix N
Questionnaire Semi structured interview

INTERVIEW SCHEDULE (Semi –Structured - Modified)

Date:______

Grade:_____

Name of Learner:____________________(Pseudonym):_____________

Topic: "The use of models to develop skills to solve 3D trigonometry problems: A case study of grade 12 learners"

For question 1 to 5 please CIRCLE your response

---

3. Manipulatives help me understand mathematics better
   A. Never
   B. Sometimes
   C. Usually
   D. Always
4. Manipulatives help me finish my work quicker
   A. Never
   B. Sometimes
   C. Usually
   D. Always
5. I enjoy using manipulatives in mathematics lessons
   A. Never
   B. Sometimes
   C. Usually
   D. Always
6. Would you like your teacher to use manipulatives?
   A. Never
   B. Sometimes
   C.Usially
   D. Always

7. Manipulatives make learning meaningful, a link from concrete to abstract.
   A. Never
   B. Sometimes
   C. Usually
   D. Always

8. Have you ever used artifacts/models before to learn trigonometry? Explain .
   ____
   ____
   ____
   ____

9. What trigonometric concepts/rules are needed in order to solve the QUESTION 1 in the activity?
   ____
   ____
   ____
   ____

10. Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 1 ?

   ____
   ____
   ____
   ____

11. Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?
   ____
   ____

____

____

12. Where you able to draw the triangles in the different planes?____

____

____

____

13. What where some of the challenges you experienced when answering this one?____

____

____

____

14. What trigonometric concepts/rules are needed in order to solve the QUESTION 2 in the activity?

____

____

____

____

15. Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 2 ?

____

____

____

____

16. Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?

____

____

____

____

17. What where some of the challenges you experienced when answering question two?____
____
____
____

18. What trigonometric concepts/rules are needed in order to solve the QUESTION 3 in the activity?
____
____
____
____

19. Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 3 ?
____
____
____
____

20. Could you explain how the sides and angles in your trigonometric concept could be found in the artifact?
____
____
____
____

21. What where some of the challenges you experienced when answering question three?____
____
____
____

22. What trigonometric concepts/rules are needed in order to solve the QUESTION 4 in the activity?

____

23. Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 4 ?

____

24. What where some of the challenges you experienced when answering question four?____

25. What trigonometric concepts/rules are needed in order to solve the QUESTION 5 in the activity?

____

26. Why did you decide to use the trigonometric concept/rules you wrote down in QUESTION 5?

____

27. Could you explain how the sides and angles in your trigonometric concept could be found in the artifact? ____

28. What where some of the challenges you experienced when answering question five?____

29. What trigonometric concepts/rules are needed in order to solve the QUESTION 6 in the activity?

____

30. Why did you decide to use the trigonometric concept/rules you wrote down QUESTION 6?

____

31. Could you explain how the sides and angles in your trigonometric concept/rules could be found in the artifact?

____

32. What where some of the challenges you experienced when answering question six?____

33. Was it easy to solve the problem when using models/artifacts ? Explain ____

34. What do you think of mathematics particularly three dimensional problems? Explain.____

35. Describe how the models/artifacts helped you learn during the lesson____

36. What do you think of manipulatives?____

37. How has manipulatives affected your learning of mathematics?____

38. What is your opinion of the performance of your teacher after the use of manipulatives?____

____

____

____

39. Are you now more confident of answering your three dimensional question in the NSC November Examination? Explain

____

____

____

____

Some Standard Probes for the interviewer

FOR CLARITY/SPECIFICITY

- **Can you be more specific?**
- **Can you tell me more about that?**

FOR COMPLETENESS:

- **Anything else?** • **Tell me more.**

OTHER PROBING TECHNIQUES:

- **Repeat the question**
- **Echo their response**
- **Pause a second**
- **Baiting**
- **What is your best estimate?**

OTHER PROBING TECHNIQUES

- **Which would be closer?**
- **Which answer comes closest to how you feel/ think?**
- **If you had to pick one answer, what would you choose?**
- **What do you think?**

Appendix O
Editor's Report

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Declaration of Editing of a Thesis:

The use of models to develop skills to solve 3D trigonometry problems:A Case Study of Grade 12 learners in a selected school in the Pinetown District .

I hereby declare that I carried out language editing of the above by Caresse Niranjan (Student No. 21452071).

I am a professional writer and editor with many years of experience. I specialise in Social Sciences and Humanities ' editing – but am adept at editing in many different subject areas.

Yours sincerely

Carolyn Turnbull-Jackson (D.Ed)

March 2022