THE ROLE OF PEDAGOGICAL CONTENT KNOWLEDGE IN PRESERVICE TEACHERS' CHOICE OF EXAMPLES TO CREATE OPPORTUNITIES FOR LEARNING ANALYTICAL GEOMETRY IN KWAZULU-NATAL MULTILINGUAL CLASSROOMS

A Thesis submitted to the Faculty of Arts and Design, Durban University of Technology, KwaZulu Natal in fulfilment of the requirements for the degree of Doctor of Philosophy

By

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NOVEMBER 2023

DECLARATION

I, Sibongile Zulu, hereby declare that this research report is my original work except as indicated in the text and the references. The research was conducted as part of the degree of Doctor of Philosophy at the Durban University of Technology, KwaZulu Natal. It has not been submitted substantially for any purpose or degree or examination in any other institution before

I affirm that:

- I have acknowledged all sources of information and used referencing to provide proper credit to the works of others.
- Any assistance I received in the preparation of this report, whether technical, financial, or otherwise, is acknowledged in the acknowledgment section of the report.
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Signed

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ABSTRACT

The study set to respond to calls for research in multilingual mathematics teacher training classrooms with the intention to advance the quality of mathematics preservice teachers (PSTs) content knowledge and improve mathematics education for school learners. In this qualitative study, the pedagogical content knowledge (PCK) of PSTs in one of the universities in the KwaZulu Natal Province in South Africa was explored. This was to understand how it guides their choice of examples for teaching analytical geometry in multilingual classrooms. Grade 11 analytical geometry lesson plans of 21 PSTs who had been given an analytical geometry test were analysed. The test was to investigate the PSTs content knowledge, which is a basis for PCK, and the lesson plans were analysed to examine the PSTs PCK from the way they structure their lesson objectives and choice of examples. Semi-structured verification interviews were conducted to follow up on both the test responses and lesson plan content, mainly the choice of examples. The domains and subdomains of the Shulman's PCK born Mathematics Teacher's Specialised Knowledge (MTSK) model and patterns of variation from the variation theory were used as the theoretical framework, which also guided the analytical framework. The study established that the PSTs had limited content knowledge, and PCK for teaching analytical geometry at Grade 11. The limited knowledge was seen in the way the PSTs relied on procedures for problem solving rather than conceptual understanding, and for this reason they chose procedural examples for teaching the topic. The study also found that the language of learning and teaching affects the PSTs ability to explain concepts in their problem solving, and this also affects their examples choice. Interview responses confirmed that the PSTs are not cognizant of the use of language in their example choice. Therefore, PSTs in multilingual teacher training must be explicitly taught to choose examples and consider learners in multilingual classrooms.

Key words: Pedagogical content knowledge, preservice teachers, multilingual mathematics classrooms, analytical geometry, exemplification, patterns of variation, object of learning, language of learning and teaching, opportunities for learning.

ACKNOWLEDGEMENTS

"The Lord is my strength and my shield; in him my heart trusts, and I am helped; my heart exults, and with my song I give thanks to him" – Psalm 28:7

I extend my deepest gratitude and appreciation to all those who have been instrumental in the conceptualisation, development and completion of my doctoral work, a milestone that has been both challenging and immensely rewarding. First and foremost, I express my sincere gratitude to my supervisor, Professor Deonarian Brijlall for his incisive and painstaking supervision. His guidance, support, and invaluable insights have been the cornerstone of my research. In the midst of other obligations, he was generous with his time and engaged with my work without a whimper. His invaluable advice challenged me but left room for my thoughts and innovations. I have grown in confidence and my research skills.

Heartfelt thanks to my husband, Thandwa Mthembu for always being available to give me emotional support, love and understanding in this tough journey. He was my sounding board and living journal at all times, especially when some concepts appeared incomprehensible. Thank you to my sister, Melissa Zulu, for inspiring, encouraging and supporting me. My mother, Elsie Zulu, has always wanted me to be my best self and made me believe I could achieve anything when I focus on it. For this, and the courage she imbued in me amongst many other values she instilled in me, I am eternally grateful. To my little brother, Sizwe Zulu, who brightens my life and days with his youthful banter and unbridled love, I am grateful for making my life joyful.

To my Head of Department, Dr Cynthia Mthiyane, I am very grateful for her support and allowing me time to take leave to focus on my research. I would also like to thank the preservice teachers who kindly availed themselves to participate in this study, even when their schedules were tight. I am greatly indebted to them.

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LIST OF ACRONYMS AND ABREVIATIONS

- **PST** – Pre-Service Teacher
- **PCK** – Pedagogical Content Knowledge
- **LoLT** – Language of Learning and Teaching
- **WIL** – Work Integrated Learning
- **CAPS** – Curriculum and Assessment Policy
- **CCK** – Common Content Knowledge
- **SCK** – Specialized Content Knowledge
- **KCT** – Knowledge of Content and Teaching
- **KCS** – Knowledge of Content and students
- **MTSK** – Mathematics Teachers' specialized Knowledge
- **KoT** – Knowledge of Topics
- **KSM** – Knowledge of the Structures of Mathematics
- **KPM** – Knowledge of Practices in Mathematics
- **KMT** – Knowledge of Mathematics Teaching
- **KFLM** – Knowledge of Features of Learning Mathematics
- **KMLS** – Knowledge of Mathematics Learning Standards

DEFINITIONS OF KEYWORDS

- **Exemplification (examples)** – Use of raw material for Illustration of concepts and principles; motivating; exposing possible variation and change, and for practising technique. This includes the use of diagrams, pictures, statements, symbols, and equations.
- **Multilingual classroom** – A classroom in which the LoLT (language of instruction) is not the same as that of the teacher and the learners, and more than one language presents the opportunity to be used.
- **Pedagogical Content Knowledge** - the knowledge of appropriate analogies, representations, examples, illustrations, and explanations that effectively create opportunities for learning.
- **Variation** – Exposing learners to different contexts in which a concept can be applied to facilitate their ability to generalise.
- **Opportunities for learning** - Experiences and situations in which learners can acquire knowledge and skills in context. In the context of this study, these opportunities may be created through exposure to, or experiencing variation in mathematics examples.
- **Contrast** – contrasting a set of examples for learners to see and identify what is common and unique across the set.
- **Fusion**- integrating different aspects of a concept to help learners see the relationships between the concepts.
- **Separation** - isolating critical aspects of a concept in examples to help learners focus on the specific aspects.
- **Generalisation** - Allowing learners to experience a range of examples that have the same critical features in order for them to generalise.
- **Misconceptions** – Misunderstanding or incorrect belief about a mathematical concept, procedure, or reasoning.
- **Errors** – Mistakes or inaccuracies in a mathematics procedure or task.

CHAPTER ONE: BACKROUND TO THE STUDY

1.1 Introduction

This is an introductory chapter that provides the background and an overview of the present research study by outlining the research problem, and the identified gap in the research that led to the conception of this study. The chapter uncovers the purpose and research questions that guided the study and summarises the context and the experiences of the researcher and participants during the study. In doing so, the reasons for specifically focusing on analytical geometry and the choice of examples in teaching the topic in multilingual classrooms, are explained with the intention of expounding the importance of exploring these concepts in mathematics teacher training. Finally, the chapter outlines the significance of the study by looking at the different ways in which it will make contributions to the mathematics education fraternity.

Experience in teaching mathematics in the South African multilingual high school classrooms and tertiary institutions triggered my interest in researching the teaching and learning of mathematics in multilingual settings. This is owing to the similarity in the challenges that these classrooms are faced with. More often than not, one is faced with having to research different ways to introduce a mathematical concept to make it more accessible and understandable by all learners or students in the classroom. This is even more challenging in a preservice teachers' (PSTs) classroom where they are not only learning mathematical content, but they also have to understand it to improve their ability to impart their knowledge of the concepts to their future learners.

The challenge here becomes more than the triple challenge of teaching in multilingual classrooms that was identified by Barwell (2009) who argued that teachers are faced with the challenge of having to maintain a balance between the Language of Learning and Teaching (LoLT), the language of mathematics, and the mathematics content. In multilingual teacher training classrooms, the challenge goes from being a triple challenge to a quartet challenge. In this quartet challenge, the fourth challenge is that the lecturer or teacher educator is also faced with having to develop and improve the PSTs' ability to impart their knowledge of mathematics language and content. This is in line with Wilkinson's (2018) argument, that the complexity of learning mathematics requires teachers to use language and non-linguistics ways of representing mathematical concepts, and this is a skill that must be acquired by PSTs. This argument resonates with the recognition that is made by research, which is that there is a need for teacher training to include facilitating the PSTs' understanding of how to address language diversity in mathematics classrooms (Eikset and Meaney, 2018).

The above-mentioned challenges show that there is a strong correlation between language and mathematics (Barwell, 2016). This is because, similar to many other disciplines, mathematics has its own language which Haliday (1975, cited in Wilkinson, 2019), referred to as the mathematics register, which refers to the ways in which mathematics is spoken and written. This makes it necessary for research to be conducted constantly on language and teaching of mathematics. This is especially significant, given the era which we live in, where people are exposed to different languages and dialects through social media, which could affect their understanding of the mathematics register.

The challenges faced in multilingual mathematics classrooms led to the conceptualisation of the study, which understands that mathematics teaching and learning employs examples as a pedagogical device. Hence, studying the choice of examples for teaching mathematics in multilingual classrooms is essential. Research in mathematics education for multilingual settings shows the importance of careful selection and use of examples in mathematics teaching. This is seen in an argument made by Essien (2021) that in a mathematics multilingual classroom, the teacher's choice and use of examples play an important role in what is to be taught, and what opportunities for learning can be created.

It goes without saying that the teacher's ability to choose and use examples effectively in a multilingual classroom is dependent on their Pedagogical Content Knowledge (PCK). In line with this argument, Rowland (2008) argued that the diversity of the teacher's knowledge has an impact on their choice of examples. PCK is diverse knowledge for teaching. It is the knowledge of appropriate analogies, representations, examples, illustrations, and explanations that effectively create opportunities for learning (Shulman, 1986). It is developed in teacher training, but further develops and improves through teaching experience (Brijlall, 2014). The significance of this knowledge in teaching led to the study seeking to probe into the PCK that PSTs have for choosing examples to facilitate learning in multilingual mathematics classrooms. Especially, since most PSTs in South African teacher training programmes, are also learning in multilingual classrooms.

The rationale behind the focus of this study being the choice of examples for teaching mathematics, and focusing on analytical geometry as a mathematics topic is provided in the next two subsections.

1.1.1 Role of examples in teaching

It has already been established that one cannot overrule the fact that language is a potent medium of communicating the concepts involved in teaching and learning. However, in mathematics teaching and learning, language works together with examples, which scholars in mathematics education argue, are a critical pedagogical tool in facilitating the learning process in mathematics (Chick, 2009; Zazkis and Leikin, 2008).

Researchers define examples in different ways. According to Zodik and Zaslavsky (2008), examples are a specific case of a larger class, from which reasoning and generalisation can be attained. Watson and Mason (2006, p. 3) define examples as "illustrations of concepts and principles". In this study, the use of this term resonates with how Liz, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006, p. 127) define it as. "… anything used as raw material for generalising, including intuiting relationships and inductive reasoning; illustrating concepts and principles; indicating a larger class; motivating; exposing possible variation and change, etc., and practising technique". To the above-mentioned definition, an addition that is applicable to this study is that examples can also be diagrams, pictures, symbols, statements, and equations that are used to enhance understanding.

According to Cayo, Codes, and Contreras (2023), selecting and using appropriate examples is considered the most complex and critical process in teaching mathematics. The teachers' ability to generate effective examples in mathematics is influenced by their knowledge of mathematics, the student's learning abilities, and PCK (Zodik and Zaslavsky, 2008). It is therefore crucial for research to focus on the knowledge that PSTs have for selecting and generating examples, especially for teaching in multilingual mathematics classrooms. This study explores the role of PCK in the PSTs' choice of examples for teaching multilingual analytical geometry classrooms.

1.1.2 Why analytical geometry?

The South African National Diagnostic Report (DBE 2022) highlighted that for five years, from 2018 to 2022, the overall pass percentage in mathematics was 55,8% on average. The data in Table 1.1 illustrates that when considering the rate of learners who achieved 40% and above, the average pass percentage for the past five years is 36,3%. Included in this small group are some of the learners who would be eligible to enrol for mathematics teacher education qualifications.

Table 1.1 The overall achievement rates in Mathematics from 2018 to 2022. (Source: www.education.gov.za

Year	No. wrote	No. achieved at 30% and above	% achieved at 30% and above	No. achieved at 40% and above	% achieved at 40% and above
2018	233 858	135 638	58,0	86 874	37,1
2019	222 034	121 179	54,6	77 751	35,0
2020	233 315	125 526	53,8	82 964	35,6
2021	259 143	149 177	57,6	97 561	37,6
2022	269 734	148 346	55,0	97 041	36,0

One of the topics that contribute more marks in the overall examination, and which has an effect on the mathematics pass percentage is geometry. More than 50% of the Grade 12 mathematics paper 2 comprises analytical and Euclidean geometry. For instance, the 2020 final National Senior Certificate (NSC) examination paper 2 included Euclidean geometry and analytical geometry contributing 49 and 40 marks, respectively that constitutes 89 marks of a total of 150 marks in the paper.

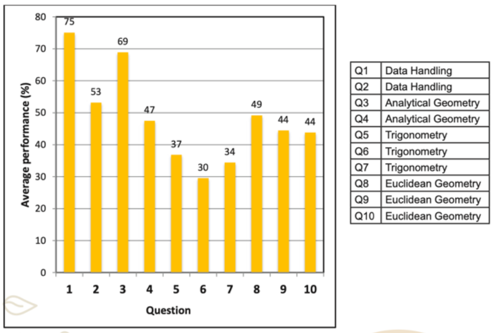
Euclidean geometry has received a lot of attention in mathematics education research. Therefore, this study only focuses on analytical geometry, which Vaisman (1997: 1) defines as "the study of geometry by algebraic means". According to the National Diagnostic Report

(DBE 2020), learners experienced a number of difficulties in analytical geometry. For instance, learners were unable to use coordinates to calculate the gradient, and they also did not understand the properties of quadrilaterals. The Diagnostic Report (DBE, 2019) shows that learners were challenged by the integration of concepts, which Setiadi, Mulyana, and Asih (2019) claim to be because analytical geometry concepts are difficult for learners to understand because of the gap between the abstract mathematical concepts and the real life.

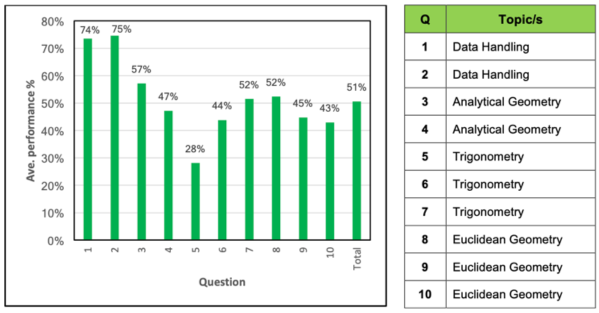
The average performance in paper 2 from the year 2019 to 2020 is provided in Table 1.2. These years include two years (2020 and 2021) during which South Africa was on lockdown due to the Coronavirus pandemic. So, Table 1.2 provides a perspective on the performance in analytical geometry during, before or after the lockdown. During all years shown, performance in analytical geometry had an average below 50%, but this is not the main reason for this study focusing on analytical geometry. Instead, it examines analytical geometry because it is one of the topics that links many topics in mathematics.

Table 1.2 The average performance in paper 2 in the years 2019 to 2020. (Source: www.education.gov.za)

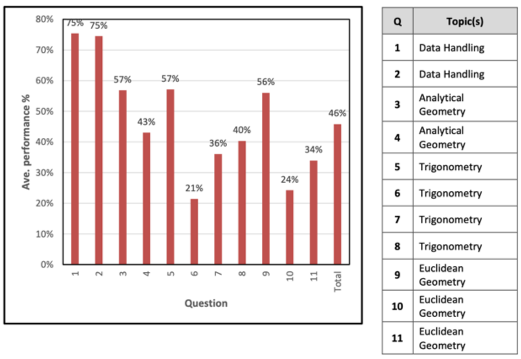
2019



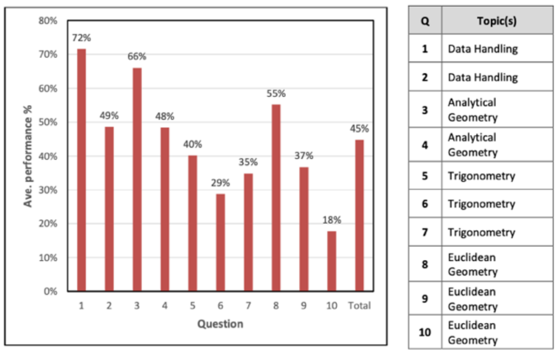
2020



2021



2022



The lack of understanding of concepts in analytical geometry does not only affect the pass rate but also the understanding of other topics in mathematics and in other subjects (Anif, Prayitno, and Sari, 2019). Analytical geometry promotes the development of geometric skills and reasoning in learners (Edwards, 1997), which is essential in mathematics as a subject, and in other fields. In support of this argument, Otumfuor and Carr (2017) assert that learners with good geometric skills are likely to perform well in mathematics, physics, and engineering. For this reason, some researchers have investigated learners' misconceptions in analytical geometry (Ozkan, Ozkan, and Karapicak, 2018), while others have studied learners' strategies in solving problems in analytical geometry (Sproule, 2005; Bansilal and Naidoo, 2012). The present study explores the knowledge that is required to choose effective examples for teaching analytical geometry in multilingual classrooms.

1.2 Context of the study

This is a qualitative study that employs the interpretative approach as it seeks to understand the participants' experiences from their own perspectives. The participants are PSTs who are enrolled in the Bachelor of Education Senior Phase (SP) and Further Education and Training Phase (FET) in one of the universities in the KwaZulu Natal province of South Africa. The participants are senior PSTs who are in their final year of studying mathematics as a specialisation in their training to become mathematics teachers. In this particular institution, the Bachelor of Education degree was revised and a new programme was introduced in 2020. This followed an old programme which had a completely different structure in terms of the courses offered. The most outstanding differences between the new programme and the old programme is that; (1) the latter offered annual courses while the former has semesterised courses, and (2) the new programme also introduced methodology courses, which the old one did not have, but had the aspects of methodology courses integrated in the content of the specialisation courses. The institution now has a structure that is similar to the structures of many other South African teacher training institutions that offer methodology courses as a direct means to develop PCK for specific specialisations.

The institution in which this study is conducted also focuses the first two semesters of the teacher training programme in closing the gap between high school mathematics and the mathematics content that is presented at a university level, which is required to produce competent mathematics teachers. Beyond matters of the teacher training program, the population of the students that enrol in the institution, and especially in the School of Education are mostly from rural KwaZulu Natal. This presents the PSTs' educators with more challenges, such as addressing the lack of exposure to technological tools which are essential for teaching and learning in higher education. This challenge is even worse in geometry classes where the PSTs' educators use Dynamic Geometric Software for teaching. Another challenge here is that of the LoLT of the institution, which is English, and may be a learning barrier for these PSTs who are predominantly IsiZulu-speaking.

1.3 Research gap and significance of the study.

There have been calls for more research to be done on teacher education in multilingual classrooms (Rangnes and Meaney, 2021). This study provides responses to these calls by exploring the knowledge of PSTs who are in multilingual mathematics teacher training programmes, being trained to teach in multilingual mathematics classrooms. More research on this is essential and would be of great benefit to the field of mathematics education and the South African learners who are situated in multilingual mathematics classrooms where many languages are used or have the potential to be used (Barwell, 2018).

PSTs are prepared to teach in multilingual classes through methodology courses and work integrated learning. The methodology courses expose them to factors that contribute to the hinderance of learning in these classrooms, and equip them with theories and practices to curb the challenges. Work integrated learning is there to expose them to the actual multilingual classroom settings where they can practice, apply their knowledge, and gain some experience. However, this exposure does not seem to completely help the mathematics understanding and performance of learners in the country as Table 1.1 indicates the poor performance in the five years from 2018 to 2022. This may be due to specific details such as the teachers' awareness of the types of examples they select and use in their teaching, which leads to an enquiry about weather teacher training course teaches PSTs about examples selection.

Many aspects of the mathematics content and PCK are covered in teacher training courses. Yet, little attention is given to how the PSTs must choose, structure, and use examples in their planning and in lessons to create more opportunities for learning in multilingual classrooms. This study examines the knowledge that PSTs in multilingual contexts use, when choosing examples for teaching analytical geometry in multilingual classrooms. It contributes to the small body of research that exists in the way analytical geometry content is taught and understood in the South African context. The findings will also inform teacher trainers about the aspects of analytical geometry content that need to be further developed in the process of developing PCK for PSTs.

Researchers have established that examples are an essential tool in the teaching and learning of mathematics, and this study does not only accentuate the need to train mathematics PSTs to choose examples but it also proposes a framework to be considered in developing their PCK and ability to choose examples. The framework is based on PCK and employs the patterns of variation from the variation theory. This is as Al-Murani, Kilhamn, Morgan, and Watson (2019) assert that the variation theory is applicable in structuring examples that promote and accelerate learning in mathematics. Therefore, the framework suggests that while developing PCK, the variation theory must be introduced to the PSTs by teaching them how the patterns of variation are applied in examples.

1.4 Rationale of the study and research questions

According to the variation theory (Lo, 2012), to understand a concept, learners must know what it is not and not only what it is. In simpler terms, the understanding of a concept does not come from being exposed to the same kind of examples of that concept, but from examples with different structures to show learners what a concept is not. In the case of mathematical concepts in geometry, for instance, learners cannot understand what parallel lines are without understanding what they are not. See example below in Figure 1.1

Figure 1.1 Example of parallel lines

For learners to understand which of the lines are parallel, they must understand that parallel lines do not look like they will meet or intersect at any point as they extend. In this case, the lines in A have the same distance between them from the beginning to the end while the distance between the lines in B becomes smaller as the lines are extended showing that they will eventually meet, and therefore the lines in B are not parallel.

According to Lo and Marton (2011), the variation theory can be used to plan lessons that promote learning. For this reason, the variation theory suites the theoretical framework of this study as it embarks on the PCK of PSTs in structuring and choosing examples, which is seen in their lesson plans.

PCK is one of the most essential facets of a teacher's professional knowledge (Torbeyns, Verbruggen and Depaepe, 2020). It is essential for PSTs to be equipped with PCK for them to cope with different situations that they are likely to come across, in multilingual mathematics classrooms (Setyaningrum, Mahmudi and Murdanu, 2018). That is, they need to know and understand their learners' backgrounds in order for them to choose and use examples that create opportunities for learning. This indicates that teacher training institutions must make it a goal to improve PSTs' PCK (Evens, Elen and Depaepe, 2017) in order to improve their ability to choose and use examples, which are a powerful tool in facilitating procedural and conceptual understanding in mathematics classrooms (binti Suffian, 2010).

This study seeks to investigate the role of PCK in the PST's choice of examples, and whether it informs the structure of the examples. It also explores the ways in which the chosen examples create opportunities for learning in multilingual analytical geometry classrooms. Given that the PSTs are in multilingual teacher training classrooms, training to teach in multilingual classrooms, the study scrutinises other elements that inform the types of examples the PSTs choose, and the impact of the LoLT and mathematics register in their choices.

It is important to echo that content knowledge is part of the basis for PCK. So, while the study will focus on the role of PCK in the PSTs' choice of examples, it will also explore their content knowledge, and understanding of the mathematical language. This is to justify the level of PCK that is observed in the PSTs and therefore understand the impact of their PCK on their choice of examples. To fulfil its purpose, the study will be guided by the following research questions:

- What pedagogical content knowledge do preservice teachers have for teaching analytical geometry in multilingual classroom?
- What informs the types of examples preservice teachers choose for teaching analytical geometry in multilingual classrooms?

- How does the preservice teachers' choice of examples in analytical geometry create opportunities for learning?

Question 1 explores the PCK that PSTs have for teaching analytical geometry in multilingual classrooms, and Question 2 seeks to understand the rationale behind the choice of examples they choose for these classrooms. Question 3 explores the ways in which the chosen examples create opportunities for learning analytical geometry in multilingual classrooms.

1.5 Conclusion

This chapter provided a glimpse into the study by outlining its background and inspirations, and the identified gaps in research. The chapter also emphasised the significance of this study, its purpose and the research questions that guided it while briefly providing a rationale for the focus on examples and analytical geometry.

This study comprises nine chapters, with Chapter 2 dedicated to the literature review. Chapter 3 details the theoretical framework that underpins the study, and Chapter 4 outlines the research methodology. Data presentation and analysis, guided by an analytical framework developed from the theoretical framework, are covered in three chapters: Chapters 5, 6, and 7. Specifically, Chapter 5 presents and analyses the PSTs' analytical geometry content knowledge; Chapter 6 examines the PSTs' example choices from their lesson plans; and Chapter 7 discusses the PSTs' interview responses regarding their content knowledge and example choices. The final chapters, 8 and 9, include a discussion of the findings in Chapter 8 and the conclusion, study limitations, and recommendations in Chapter 9.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

This chapter provides an overview of the literature that was explored and contributed to guiding the study. In exploring the preservice teachers' knowledge in choosing examples for teaching in multilingual analytical geometry classrooms, the study delved into literature that focuses on its key aspects. These aspects are organised as sections, and they are: mathematics PSTs' knowledge development, teaching and learning in multilingual mathematics classroom, examples in mathematics teaching and learning, and the teaching and learning of analytical geometry.

2.2 Mathematics PSTs' knowledge development

By mathematics PSTs' knowledge development, this study refers to the process of developing the PSTs' mathematical proficiency, and the ability to teach mathematics effectively. Mathematics teacher training has received a lot of attention in research because of the many challenges that learners and teachers experience with the subject. To list but a few, some of the challenges are: learners' poor performance, the lack of conceptual understanding, underperformance in university courses that are mathematical in nature, and the mathematics PSTs' lack of content knowledge. In an attempt to curb the different challenges around mathematics teaching and learning, a lot of attention has been given to how the mathematics PSTs are taught, and learn to teach the subject (Minor, Onwuegbuzie, Witcher, and James, 2002).

Some studies such as the one conducted by Bjerke and Solomon (2020), have also explored and examined the depth of content knowledge and its role in the mathematics teaching practice of PSTs. These studies are related to some others that have looked at how mathematics PSTs choose and use examples (Essien, 2021), which are dubbed a useful pedagogical tool in the teaching and learning of mathematics (Rowland 2008; Zazkis and Leikin, 2008). This makes examples another aspect that needs focusing on, in teacher training and research, especially, in multilingual teacher training classrooms, which this study contributes to in response to the calls for more research on teacher training in multilingual classrooms (Rangnes and Meaney, 2021).

It is evident and will be outlined in this literature review, that researchers in mathematics education have explored different ways to improve mathematics teacher training. However, the discussion around what actually takes place within the "black box", that is, what the PSTs are actually taught is minimal (Althauser, 2018). The discussion on exactly what mathematics PSTs should be taught is essential in ensuring that their practice aligns with theoretical ideas. This study hopes to promote these discussions by exposing the importance of having examples selection and generation as a subject of discussion in teacher training classrooms. This would also contribute to the ever-pursued change in mathematics classrooms. Concurring, Karisan, Macalalag, and Johnson (2019) further explain that to ensure lasting change in mathematics classrooms, there must be careful guidance when it comes to PSTs having to thoughtfully consider the outcomes of their pedagogical choices. These opportunities can be created by exposing PSTs to examples selection and structuring, which requires deep thinking about what the goal of the lesson is.

Teacher training institutions must also make it their focus to create opportunities for PSTs to reflect on their content and teaching knowledge through course work and practical work. This is in line with Jung, Stehr and He's (2019) claim that research shows that PSTs learning opportunities differ significantly according to the programmes' focus, courses, and work integrated learning. The most certain and similar aspects in mathematics teacher training are content and pedagogical knowledge development. This is due to the lack of mathematics content knowledge that PSTs express.

In a study that was conducted by Afifah, Khoiri, and Qomaria (2018), the findings revealed that mathematics PSTs do not only need development in mathematics but also in mathematical literacy. In this study, the PSTs admitted and indicated that they need to improve their mathematical literacy. They expressed that the connection between mathematical concepts and real-life problems in school mathematics is necessary in improving one's mathematical literacy (Afifah et al. 2018), and this is a skill they do not have. In another study, some PSTs vent about their bad experiences in the way they had to learn mathematics (Althauser, 2018). In Althauser's (2018) study, the PSTs also express how adamant they are, that their mathematics instructional practices will be nothing close to what they had experienced in their mathematics learning in school. One must admit that this claim by PSTs is what many in mathematics education would want to hear, especially since Karisan et al. (2019) still find that PSTs have difficulty in effectively reflecting on and adapting to practices that are different from what they may have experienced in school.

The lack of mathematical literacy skills in mathematics PSTs, highlights the gap in their knowledge and understanding of mathematics just as much as it highlights the need for better teachers and improvement in the teaching of the subject (Adler et al., 2005). This improvement could be achieved through adequate teacher training, which is not as straightforward as pointing it out is, especially in this day's society where we have learners and teachers who freely share their dislike for mathematics (Althauser, 2018). The dislike for mathematics and failure in the subject are becoming societal norms and some mathematics PSTs are starting to accept it, which Althauser (2018) emphasises has to be stopped, if the teaching and learning of the subject is to improve in future.

According to Bjerke and Solomon (2020), there is a strong correlation between the mathematics teachers' quality of mathematics knowledge and their ability to create opportunities for learners to learn the subject. For this reason, Gule and Celik (2018) highlighted and emphasised that content knowledge is an essential component for mathematics teaching. This statement coincides with a concern that was highlighted by some researchers for example (Lowrie and Jorgensen, 2016; Yet et al., 2021), that the attention that must be given to mathematics PSTs' content knowledge is being silenced by PCK in research. In this study, these two components are looked at together because the PSTs' ability to choose effective examples in mathematics is dependent on both mathematics content knowledge and pedagogical knowledge.

In an informative study that touches on both content knowledge and the ability to explain concepts, Bjerke and Eriksen (2016) measured the self-efficacy of PSTs in tutoring mathematics. They found that mathematics PSTs had more instrumental understanding (knowing how) and showed signs of a lack in relational understanding (knowing why) (Skemp, 1976). Concurring with this claim, another study that explored the types of explanations that PSTs use in their mathematics practice, which was conducted by Murtafiah, Sa'dijah, Candra, and As' ari, (2018), found that most PSTs can only rely on descriptive explanations in their mathematics teaching. This is basically the response to the “how” question, and not the “why” and “what”, which Hargie (2006) refers to as the reason-giving and interpretative explanations, respectively.

According to Norton (2019), mathematical content knowledge is a basis for mathematical pedagogical knowledge, and so just knowing how to find a solution does not guarantee that a teacher would have the necessary skills to use analogies, representations, illustrations, and explanations that effectively create opportunities for learning (Shulman 1986). In support of this argument, Araújo Filho and Gitirana (2022), claim that teachers need to understand the reasons behind teaching and learning content, and this is achieved by ensuring that their content knowledge intersects with their pedagogical knowledge. This is possible only when a teacher ensures that they do not only understand the “what” or “which” of the content they are teaching, but also the “why” (Araújo Filho and Gitirana, 2022).

In another study that explored the development of self-efficacy in teaching mathematics, Bjerke and Solomon (2020) looked at the PSTs’ perceptions of what they understand the role of content knowledge to be, in mathematics. They employed a notion presented by Bandura (1977) that self-efficacy is one’s assessment of their own abilities to successfully execute a particular function. In Norton’s (2019) terms, self-efficacy in mathematics teaching is one’s faith in the capability to successfully effect a positive learning outcome. Based on the role of examples in mathematics teaching and learning, one may argue that the ability to choose, generate, and use effective examples for mathematics teaching is another way of developing and improving self-efficacy. Thus, example selection and generation is an essential skill for PSTs to develop for their self-efficacy to improve.

There is a direct relationship between a teacher’s self-efficacy and their learners’ performance. A confident teacher is likely to produce confident learners in the subject. Self-efficacy is especially significant and needs to be developed in mathematics PSTs who often express their lack of confidence and doubt in the subject, which is related to self-efficacy in its teaching (Briley, 2012; Bjerke and Solomon, 2020). This is why Norton (2019) suggests that another way of describing self-efficacy is that it is a teacher's confidence in their own pedagogical content knowledge. Bjerke and Solomon (2020) further suggest that more research is needed to understand the development of self-efficacy in teaching mathematics.

In their study, Bjerke and Solomon (2020) found that PSTs find more opportunities to develop and improve their self-efficacy in their work integrated learning experiences. This is in line with the framework that they employed, which entails that there are four sources of experience that facilitate the development of efficacy beliefs and these are:

- *Mastery experiences* – the impact of having succeeded in previous performance of particular tasks.
- *Vicarious experiences* – the experience of seeing someone else successfully perform a particular task that they would have been contemplating.
- *Verbal persuasion* – having positive feedback from others enhances a person's belief of their capability to perform a task at a certain level.
- *Physiological and Affective response* – which are bodily responses such as anxiety or pleasure that could be a basis of efficacy information.

(Bjerke and Solomon, 2020)

My experience as a mathematics teacher brings me to the realisation that the most common sources of self-efficacy in my mathematics teaching were the mastery experience and vicarious experience. The first few years were mostly based on the vicarious experience, which was confidence that was mostly based on my high school and teacher training experiences. One also develops from verbal persuasion as they receive feedback from learners, which means that PSTs could also be gaining confidence through the feedback they get from learners, mentor teachers, and their educators, as they participate in work integrated learning (school practicum).

The relationship between self-efficacy and professional knowledge are directly proportional and this is why self-efficacy together with the teachers' professional knowledge contribute greatly to the quality of their instructions and the learners' understanding of concepts (Torbeyns et al., 2020). One of the ways to theoretically equip PSTs with professional knowledge to improve their self-efficacy, is to pay attention to the notion of diversity in the classroom. This would contribute to their success in conducting inclusive lessons (Caurcel Cara, Crisol Moya and Gallardo-Montes, 2021) which in turn develops self-efficacy through verbal persuasion because when learners learn, they tend to express their excitement in the process.

According to Athauser (2017), some mathematics PSTs have challenges and do not have sufficient skills for them to identify and address the different learning needs in the classroom. They also cannot maintain a balance between classroom management and clarifying the mathematics instructions (Athauser, 2017). Teachers would typically acquire a portion of this knowledge in teacher training, and the rest from experience as teachers and teacher support programmes. Research also shows how some teachers get to learn more about their learners' sources of errors in mathematics through professional learning communities and teachers' workshops, and only then do they develop a clear understanding of their learners (Chauraya and Brodie, 2018).

Teacher training institutions do focus on Shulman's (1986) fundamental components of knowledge for teaching, which are PCK, content knowledge, and curricular knowledge which will be further explored in the theoretical framework of this study. However, it is also important to note as Althauser (2018) states, that teachers of this day have so much more on their plates than just having to master teaching strategies. They deal with diverse learners, some of whom have negative views about mathematics as a subject. Owing to this, Evens et al. (2017) argue that the professional knowledge PSTs can acquire in teacher training institutions, is limited. Bosica, Pyper, and MacGregor (2021) also point out that the amount of knowledge that PSTs should learn about in mathematics teaching, is a lot compared to the amount of time they have in the teacher training programs. In the case of the PSTs in this study, this was worsened by the pandemic which saw education institutions moving from the usual way of teaching and learning to less contact and physical interactions. Therefore, one can conclude that there is already a need for research on how the PSTs who trained during this time can be supported to cover up for the time they missed out on.

In the limited time in teacher training, Lee et al. (2021), argue that mathematics PSTs' educators are faced with the challenge of having to train the PSTs to know and understand learners' sources of errors, misconceptions, and mathematics learning needs. At this point, the best teacher training institutions can do, is to theoretically expose the PSTs to the different researched learning needs, mathematics misconceptions learners have, learners' sources of errors, and selecting and using effective examples. According to Caurcel Cara et al. (2021), training mathematics PSTs on inclusive education can also boost their confidence and change their attitude towards mathematics. Although Lee et al. (2021) argue that preparing mathematics PSTs to implement instructions that are accessible to all learners in the classroom, it is a challenge because they do not have enough experience of being in an actual classroom. This argument resonates with Brijlall (2014) and Evens' et al. (2017) arguments, that there is only so much PSTs can learn in teacher training institutions because they will gain more knowledge through experience, research on their teaching, and participating in professional learning communities. Fortunately, Karisan et al. (2019) claim that there are promising research findings that suggest that teachers' classroom practice can be influenced through professional development workshops. Therefore, in addition to the knowledge gained in methodology courses and the teaching practicum, the PSTs will further develop and improve their knowledge through professional learning communities.

To make teacher training more impactful in developing the PSTs' knowledge, Bosica et al. (2021) suggest that the Problem Based Learning (PBL) approach might fast track the PSTs' knowledge development. This is because of its underlying social constructivist learning aspects which offer opportunities to build on the capacity for lifelong learning and professional development (Bosica et al., 2021). They highlight that the lack of the use of PBL in mathematics teacher training is notable, especially for secondary school mathematics. In their study, Bosica et al. (2021) found that incorporating PBL in mathematics teacher training instructions, benefits the PSTs by improving their knowledge for teaching and increasing awareness of PCK. In their findings, they also note that PBL also facilitates reflections in PSTs, and makes them aware of their beliefs and the need to develop and improve their critical thinking, which would in turn improve their teaching abilities. They emphasise that the PSTs use of resources for teaching is evolving and so should their training (Bosica et al., 2021).

In another exploration of ways to develop the PSTs' knowledge, van Es, Cashen, Barnhart, and Auger (2017) explores the pedagogies for learning to teach and cite Grossman, Compton, Igra, Ronfeldt, Shahan, and Williamson (2009) who identified the three key concepts that underlay pedagogies for teaching practice, that is, for teaching how to teach. These are the use of *representations of practice, decomposing practice,* and *approximations of practice.* The *representations of practice* include all available examples of the teaching practice which PSTs are often exposed to, such as the direct observation of a qualified teacher on practice, the professional videos that explicate technique, and more (Grossman et al., 2009). The two mentioned examples of the representation of practice are the most used in teacher training nowadays and therefore one can argue that beyond theoretical teaching and learning, they are the most relied on for further development of PSTs' ability to teach.

The *decomposition of practice* refers to identifying the integral components of the teaching practice that should be targeted and improved in PSTs training through instruction (Grossman et al., 2009). In alignment with the decomposition of practice, Rocha (2020) heightens that it is crucial to know what needs to be done to promote the PSTs' professional development from the very beginning of the teacher training program. This entails breaking down exactly what needs to be the first thing to develop in PSTs, and the structure and order of the components of knowledge must be outlined from the beginning. In doing this, teacher training would also identify the appropriate time to specifically focus on developing the PSTs ability to choose and structure examples to facilitate learning in mathematics.

The decomposition of practice in learning to teach creates more opportunities for PSTs to see and understand the elements of the teaching practice effectively. For example, focusing on lesson planning (Grossman et al., 2009), which this study is seeking to achieve by focusing on what knowledge PSTs rely on when they prepare their mathematics lessons and choose examples for effective teaching. This is as Lim, Son and Kim (2018) argue that PSTs find it challenging to conceptualise and build lesson plans. It is not difficult to see why this is the case because lesson planning alone is one aspect of teaching that also needs decompositions, especially in mathematics teaching.

A lot goes into lesson planning, and as much as content is almost always readily available nowadays, there is a lot more to consider. There is content structuring or sequencing and thinking about the recipients (learners) and the relevance of the examples to be used to enhance the facilitation of the concept understanding. One also needs to clearly articulate the goals of the lesson which is very important and helps in guiding the progression of the lesson. This is just an incomplete list, there are many aspects in lesson planning. Therefore, one can argue that the decomposition of practice would also be a means of addressing what takes place in the "black box", which is not given enough attention in research.

Approximating practice exposes the distance between what actually happens in practice and in the training of the PSTs (van Es et al., 2017). It goes without saying that what PSTs experience as they are still in training, will not be entirely similar to what actually happens in practice. This is for many reasons, such as the fact that the time they spend in training is not enough to expose them to everything and the settings they are exposed to might differ from what will be their actual context. For example, in teacher training, you might find that a PST does their work integrated learning in a well-resourced school and spends a month of practice there only for them to work as a qualified teacher in an under-resourced school. In sustenance, Ni Riordain, Ni Shuilleabhain, Prendergast, and Johnson (2021), also explain that the PSTs' experiences of teaching will be inhomogeneous because their classroom experience is dependent on their work integrated learning school placement context. They add that regardless of this, the strong relationship between content knowledge, especially mathematical content knowledge (MCK), and the classroom practices must be considered at all times (Ni Riordain et al., 2021).

Grossman et al. (2009) prescribed what many teacher training programs do nowadays to curb and close this gap. That is the approximation of practice, where certain aspects of the teaching practice are simulated through activities such as micro-teaching, which has an element of role-playing. This allows teacher training the opportunity to expose the PSTs to some of the aspects of teaching that might not be so overt in theory. One of the good examples of approximation of practice can be seen in a case study that was conducted by Mallart, Font and Diez (2018) where they studied the mathematics PSTs' difficulty in problem posing. Approximation of practice was used very well here as it helped in identifying problems that the PSTs may encounter when they go into service. One of the interesting findings in this study was that the PSTs found it difficult to pose problems with well-formulated statements that have no missing information. Their problems do not allow several solutions and flexibility in problem solving, and they are not engaging for the learners in terms of their context relevance (Mallart et al., 2018).

Van Es et al. (2017, p. 165) alert that for more than a decade now, research in teacher education has also focused on an important skill for teaching referred to as "teacher noticing", which is "the ability to attend and reason about teaching and learning". Teacher noticing is an important skill for PSTs, and also speaks to the focus of this study, which is exploring the PSTs' PCK in their choice of examples. Exploring 'teacher noticing' entails understanding the decisions that teachers make in the moment, which is dependent on the teachers' understanding of what their learners are thinking and doing. It also entails exploring the teachers' reasoning about the learners' ideas in order to make an informed choice and decision on how to proceed with that particular lesson (van Es et al., 2017). This is an important skill to observe about PSTs and develop further. This approach is central to what mathematics education reform has been advocating for, that is, teaching approaches that employ instruction that encourages learner-centeredness as teachers closely attend to learners' mathematical thinking and development.

Wilson, McChesney and Brown (2017) also point out that teacher 'noticing' is one professional learning process that could enhance expertise in PSTs who also have to learn how to be cognisant of the learners' different cultures in the teaching and learning of mathematics. This is an essential aspect in lesson planning and choosing of relevant examples, especially in multilingual classrooms. Van Es et al. (2017) propose the use of ambitious pedagogies for developing PSTs noticing. In support, Lebak (2022) also points out that researchers have focused their attention on ambitious pedagogies as an approach to develop pedagogical knowledge in PSTs. This is because ambitious pedagogies present teaching and learning as a process in which the teacher and learners authentically co-engage in real-life and intellectual problems to make meaning of mathematical concepts (Lebak, 2022). This approach is the opposite of the traditional method of teaching where the teacher deposits facts and the learners mostly receive the facts without engaging with the facts.

Employing ambitious instruction in teacher training, could also set an example from which the PSTs could learn and implement in their teaching. It involves adapting instruction to learners' needs and thinking by constantly modifying pedagogical actions to accommodate all learners

(Lebak, 2022). Ambitious pedagogy would also make a suitable approach in guiding PSTs in their selection, generation and use of examples. This claim emanates from Stroupe's (2016) argument that the approach is unique because it emphasises the use of learners' ideas to deepen learning instead of just correcting misconceptions. The approach could develop the PSTs' ability to generate examples to enhance learning in multilingual classrooms by considering the learners' thinking and using it to build on more knowledge instead of using examples to show methods and do corrections. With this approach, the PSTs could learn to consider the learners' context to select relevant examples that will promote engagements that facilitate meaning-making.

2.3 Teaching and learning in multilingual mathematics classrooms.

Teaching and learning mathematics in multilingual classrooms is a topic that has received a lot of attention in research. It has led to scholars recommending that more research be conducted in the teaching and learning of mathematics in multilingual PSTs' classrooms. These recommendations stem from attempts to finding solutions and curb the challenges faced in multilingual classrooms. Some researchers, however, have presented ideas which suggest that teaching in these classrooms also presents opportunities for teaching and learning mathematics. These views are discussed in this section where the starting point is exploring the definition of a multilingual classroom.

2.3.1 Multilingual classrooms

In Barwell's (2012: 317) terms, a multilingual classroom is "… any setting in which participants could draw on a repertoire of different languages". It is a classroom in which the teachers are teaching in a language that is neither their first language nor that of their learners' (Chikiwa and Schafer, 2018) and more than one language is present and can be used or present the potential to be used (Barwell 2003). Although, this does not necessarily imply that all the learners can actually speak the different presented languages (Setati, 2008; Trinick, 2015).

Sibanda (2021: 21-22) citing King (2018: 8) outlines the key and valid indicators of multilingualism as, "The extent to which there is interaction between linguistic communities, the degree of public acceptance of and support for linguistic diversity, and the way in which this ‘multilingual capital’ is part of the political and economic infrastructure, including in the all-important area of education”. Barwell (2003) outlines three common scenarios for multilingual classrooms; a monopolist, pluralist, and a globalist multilingual classroom which are defined as follows:

- Monopolist classroom - the dominant or majority language in the society is used for teaching and learning.
- Pluralist classroom – teaching and learning takes place through the use of numerous languages present in society.
- Globalist classroom – An internationally used language such as English is used for teaching and learning.

The LoLT in many schools in South Africa is English (Setati, Chitera, and Essien, 2009). This makes our classrooms more of globalist classrooms where the LoLT is not necessarily every learner’s and teacher’s home language. In fact, mathematics teaching and learning in most South African townships and rural classrooms usually takes place through the use of a combination of the languages that are spoken in a particular community (Chikiwa and Schafer, 2018).

According to Chikiwa and Schafer (2018), when a teacher orally uses the indigenous languages in their teaching of mathematics, they are codeswitching. Researchers define codeswitching as the deliberate use of different languages alternatively in the same conversation (Baker, 1993; Adler, 2001; Choudhury and Bose, 2011). For instance, when one uses a different language to substitute a word(s) or phrase that is in another language (Baker, 1993). This practice of codeswitching is very helpful for teachers in multilingual mathematics classrooms where they have to juggle between using LoLT and the learners’ first language to help them understand the mathematics language. In this respect, this section also explores the issue of language in mathematics.

2.3.1.1 Language in mathematics

According to Caligari, Norén, and Valero (2021), there is a positive correlation between proficiency in the LoLT and understanding of mathematical concepts. In the South African context, one might argue that learners who are proficient in the LoLT are likely to understand mathematical concepts better than their counterparts because mathematical concepts and terms are also communicated in the LoLT. However, mathematics has its own register, it has its own words and structures that create appropriate meanings for a particular function of the mathematics language, see Halliday (1975, cited in Edmonds-Wathen, Trinick, and Durand-Guerrier, 2016).

Mathematics is a way of communicating ideas through the use of symbols and specialised vocabulary (Pimm, 1987), which is presented in the LoLT. Pirie (1998) and Wilkinson (2019) expand that mathematics is communicated through the use of ordinary language, mathematical verbal language, visual representations, unspoken but shared assumptions and quasi-mathematical language. According to Planas, Morgan, and Schutte (2018), the purpose of mathematics education is to induct learners into the mathematical activity, that is, manipulating language and notations, and communicating mathematically instead of just working on mathematical problems.

In agreement with Planas et al. (2018), Wilkinson (2019) also noted that mathematics is not language free, and it greatly depends on language in both oral and written aspects. Therefore, the teaching and learning of mathematics takes place not only through the LoLT, but through the language of mathematics as well. The understanding of this language, which facilitates conceptual understanding in mathematics, is mostly facilitated through the use of examples. When teachers select examples, they also have to consider how the language is used in the examples to promote learning. This is in line with the purpose of this study, which is also to understand how language is used in the PSTs' choices of examples for multilingual classrooms.

According to Edmonds-Wathen et al. (2016), the issue of the role of language in the learners' acquisition of mathematical knowledge and skills began attracting attention in research more than five-decades ago. Since then, a lot of research in multilingual mathematics classrooms has been conducted (Planas, 2021), and researchers continue to identify gaps that still need attention in this area. For example, in their exploration of the lessons and directions from two decades of research in mathematics education and language, Planas et al. (2018) highlight that research is still needed in exploring how the new technologies influence the language of mathematics. We are all aware that the new technologies come with social platforms that have introduced new slangs in the way people speak, and so exploring the way these platforms and educational software influence language, is crucial in mathematics education.

Planas et al. (2018) also bring forth that of all the research that has been conducted in language in mathematics education, there is still little knowledge on how the proposed methodologies and findings could be applied in the professional development of mathematics teachers and PSTs. This also highlights the need for research on mathematics PSTs' teaching and learning of mathematics in their multilingual classrooms and training for teaching in multilingual classrooms. This is what drives the purpose of this study as it investigates the PSTs' knowledge in choosing examples for multilingual classrooms and how the chosen examples are meant to create opportunities for learning and how they ensure that the mathematical language in the examples is understandable and accessible to learners.

The point is to explore how the PSTs think about preparing to make the mathematical language accessible to learners whose home languages is not the LoLT. Edmon-Wathen et al. (2016) explain that the language of mathematics also evolves as the grammatical resources of the languages from which it was constructed, evolve. They further explain that this evolution also involves the introduction of the words that are used for new mathematical objects, their properties, functions, and relations to other objects. For example, the introduction of the word 'quadrilateral' (Edmon-Wathen et al., 2016), which is of Latin origin and is being used as an English mathematical term. The term requires that one understands the intricacies of the English multisyllabic word, reading to be able to recognise the word and also have the basic understanding of mathematical shapes and their properties to understand the word at a passive and active level (Sibanda, 2021). Sibanda (2021) continues to note that Science, Technology, Engineering, and Mathematics (STEM) are subjects that are unique languages on their own and learning to read them is not overtly and systematically taught, and teaching to read the language is not supported by research. This statement triggers the following questions:

- *Are the mathematics PSTs cognisant of the grammatical systems involved in the mathematics register as they prepare to teach in multilingual classrooms?*
- *Are PSTs taught to learn and teach the mathematics language?*

These questions are in line with a question posed by Sibanda (2021) asking, 'how well equipped a mathematics teacher is to handle the niceties of vocabulary, reading, and fluency in STEM teaching?'. The questions also highlight a gap that this study has identified in literature. It is notable that the field of language in mathematics has been covered broadly. However, more research is still needed to look at how the mathematics grammatical systems are considered in mathematics education to ensure that beyond facilitating successful learning, learners also learn to master the mathematics register which would in turn also facilitate conceptual understanding.

According to Pimm and Keynes (1994), the main activities involved in the teaching and learning of mathematics as a subject are, reading and writing; listening and discussing. Language is a key driver in all these activities, and for this reason Planas (2021) reports that research supports the adoption of translanguaging practices in the teaching and learning of mathematics. Translanguaging is defined by Otheguy, Garcia and Reid (2015), as a speaker's use of full linguistic repertoires without watchful adherence to the boundaries that are set politically and socially for the national languages in use. It can be seen as a free and creative use of language where teachers and learners make sense of their world through other languages and cultures (Planas, 2021). This study will not concern itself so much with matters of translanguaging as it will not focus much on the interactions between teachers/PSTs and learners, but more on the PSTs knowledge. One should point out that instead of delving into translanguaging, the study rather takes some interest in using 'multilingualism' and/or 'plurilingualism' where plurilingualism denotes one's proficiency in multiple languages (Mendoza, 2023).

Planas et al. (2018) explain that doing research that has to do with mathematics and language, entails looking at 'the language/s of the learners', 'the language of the teacher and the classroom' and 'the language of mathematics'. This study concerns itself more with the language of the teacher and classroom, and the language of mathematics. The present study is set to explore the knowledge that PSTs have and refer to as they plan lessons, choose examples, and the way the examples are structured to make mathematics more accessible in multilingual classrooms. This entails attempting to understand the reasoning behind the choice of examples that PSTs choose to teach analytical geometry.

2.3.2 Challenges in multilingual classrooms

Teachers and learners in multilingual classrooms are faced with many challenges, such as learners failing to understand the mathematical concepts presented to them by teachers in English, and teachers having to opt for code switching as a solution (Maluleke, 2019). Chikiwa and Schafer (2018) explained that these challenges also add an additional task for teachers teaching mathematics in the LoLT because they also have to constantly translate some of the concepts to the learners' home languages in an attempt to create more opportunities for the learners to learn. For example, teachers will often have to ask questions in two or more languages.

Another big challenge noted in teaching mathematics in multilingual classrooms is that, it makes it difficult for teachers to ignore the traditional approach of teaching and fully adopt mathematical problem-solving as a teaching strategy (Chirinda and Barmby, 2018). In support of this claim, Chikiwa and Schafer (2018) explain that there is a need for intense research on how codeswitching supports critical thinking in mathematics classrooms because the use of LoLT which is an unfamiliar language in the rural and township areas, encourages teachers to adopt rote learning (Setati and Adler, 2000). These teachers also tend to teach in a way that promotes safe talk and recalling and memorisation more than critical thinking and conceptual understanding. These are traditional classrooms where the mathematics teacher does most of the talking and depositing of information.

In these classrooms, learners and teachers are faced with many challenges (Barwell, 2012). According to Adler (2002), learners are faced with having to be competent in both English and mathematical language for them to be prepared for further education and employment. Setati et al. (2009) caution that many learners in multilingual classrooms are not fluent in the LOLT, and yet they must learn mathematics, which has aspects that are similar to learning a language, in a language that is not their home language. Learners whose home language is not the LoLT are faced with the challenge of having to simultaneously understand the linguistic structure of English and the structure of the mathematical language (Essien, 2013 and Halai, 2009).

In an attempt to help learners to understand mathematical concepts, teachers often use every day or informal language to teach mathematics, which will then be facilitated to a more mathematically structured language, and ultimately to academic mathematical language (Clarkson, 2009). This is a challenge because it is difficult for learners to move from the exploratory talk (everyday language) where learners often use their home languages, to discourse specific talk, which is the mathematics language (Adler, 2002 and Halai, 2004).

Another challenge is that some concepts which are linked to everyday life are difficult to explain in English, while they would be better conveyed in the teacher's home language because the teacher's knowledge of English is also not strong (Adler, 2002 and Clarkson, 2016). This is in line with Barwell's (2009) argument that teachers in multilingual classrooms are faced with having to strike a balance between the mathematics content, LoLT, and the mathematical language. Setati et al. (2009) add that the major task faced by the teachers in such classrooms is to teach both mathematics and English at the same time.

Adler (2002) identified three dilemmas in the teaching and learning of mathematics in multilingual classrooms. These are, the dilemma of code switching, dilemma of mediation, and the dilemma of transparency. The dilemma of codeswitching is about deciding whether to use the LoLT for learners to be fluent in , or to use their first language for learners to understand the mathematics (Adler, 2002). The dilemma of mediation is when a teacher is trying to focus on the learners' understanding of mathematics and their ability to correctly use the mathematical language (Adler, 2001). The dilemma of transparency is about whether to be explicit or implicit. Being explicit is focusing on ensuring that learners know and use the mathematical language correctly and being implicit is allowing learners to make use of informal language as long as the mathematical content is correct (Adler, 2002).

With all the noted challenges in the above literature, looking at example selection for multilingual classrooms, and the knowledge that PSTs have, is also a way of exploring different ways to improve teaching and learning in these classrooms.

2.3.3 Attenuating the challenges in multilingual classrooms

According to El Mouhayar (2021), research on multilingual mathematics classrooms emphasises the importance of perceiving learners' linguistic repertoires as a resource. Some researchers see languages as a resource in teaching multilingual mathematics classrooms (Caligari et al., 2021). This is because teaching in these classrooms, presents teachers with the opportunity to facilitate the learners' language development and conceptual understanding. In agreement with this notion, Pelaez, Calleros, Parra, and Zahner (2023) bring up Yosso's (2006: 43) concept of 'linguistic capital', which highlights the "intellectual and social skills learned through communication experiences in more than one language". This entails learning through more than one language, and using formal and informal ways of communicating through those different languages. Pelaez et al. (2023) propose that teachers should not only focus on the learners' learning pronunciations and memorising vocabulary. They must also make mathematics instruction focus on multiple ways to express meaning and use language as an asset by encouraging learners to speak languages of their own choice when dealing with mathematical concepts (Zahner, Calleros, and Pelaez, 2021).

Planas et al. (2018) accentuate that it is inevitable to use both non-mathematical (everyday language) and mathematical language in the teaching and learning of mathematics. In multilingual classrooms, the everyday language is often in the learners' home or choice of language, and it contributes to the effective progression of the mathematics lesson. Therefore, Planas et al. (2018) suggest that the connection and disconnection between everyday and mathematical language should be taken advantage of in helping learners make everyday or mathematical meanings. In this regard, Sibanda (2021) suggests that mathematics language should be taught and learnt simultaneously with the mathematics content, and not sequentially or separately. Although one would also have to consider what to foreground or put in the background.

Pimm (1991) suggested two ways a teacher can facilitate the learners' movement from informal spoken language to academic mathematical language. These are, encouraging learners to write down mathematically what they say in everyday language; and to get learners to practice speaking mathematically using formal language. This would allow learners to see and understand the relationship between everyday and academic mathematical language.

According to Cummins (2009), when learners are allowed to communicate in two or more languages during teaching and learning, they tend to perform better than their peers who are not allowed. This implies that, as much as language can be seen as a learning barrier for learners in multilingual classrooms as Robertson and Graven (2020) argue, it is also crucial to consider their different languages as a resource in meaning making. In support of this notion, Planas (2018) exemplified that the context of culture provides a potential for shared meaning and reconstructing texts in language. Chikiwa and Schafer (2018) added that teachers must also utilise the opportunity of using the learners' home languages to promote conceptual understanding and critical thinking through questioning. This argument is based on research that has unveiled the direct influence of language related factors on successful mathematics teaching and learning in South African schools.

Chikiwa and Schafer (2018) urge teachers in multilingual classrooms to not deprive learners of opportunities for meaningful engagements with mathematical concepts by asking learners questions of lower orders. They should rather use language to make meaning of complex mathematical problems. In this regard, Barwell (2018) proposed the notion of looking at language as a 'source of meaning' instead of the term 'resource'. This is based on the idea that meaning making happens through talk and interactions in multilingual classrooms, and during this process language is the main driving tool.

According to Moschkovich (1999), learners make use of conjectures and explanations to communicate and participate in mathematical discussions. In these discussions, examples are a fundamental tool used to communicate and explain mathematical discourse. This argument emphasizes the focus of this study. That is, as much as language is the medium through which teaching and learning mathematics takes place, mathematics is primarily learned through the engagement with examples (Watson and Mason, 2002), and therefore, it is crucial to explore how examples can be used with the aid of language to create opportunities for learning in multilingual classrooms.

The suggestion is that teachers must carefully select examples that will accommodate learners in multilingual classrooms. This is as Essien (2021) notes that in multilingual classrooms, teachers are faced with a challenge of having to choose and use examples that account for how language is used to facilitate productive discussions. At this point, it is difficult to find literature on the use of examples in multilingual mathematics classrooms. This study will be amongst the few studies to explore this area in mathematics education.

2.4 Examples in mathematics teaching and learning

Exemplification has been a pedagogical tool for ages and still continues to be advocated for as a useful tool in the teaching and learning of mathematics (Watson and Mason, 2002; Rowland, 2008; Zazkis and Leikin, 2008; Chick, 2009; Adler and Alshwaikh, 2019; Mwadzaangati, Takker, and Adler, 2022; Cayo et al., 2023). I can also confirm from my experience of teaching mathematics that examples are still a very powerful tool in facilitating learning, especially in multilingual classrooms. They are the main resource in the teaching and learning of mathematics (Cayo et al., 2023), and using them skilfully leads to effective teaching (Lee and Lee 2021).

According to Zaslavsky and Zodik (2007) examples are a major component of knowledge development in mathematics as they enhance the ability to work with generalisations, abstraction, and inductive reasoning. Liz et al. (2006) and Rowland (2008) explain that examples are an element of a larger concept which can be used to promote the ability to see relationships, and inductive reasoning. This makes examples an even more reliable resource in analytical geometry where concepts from different mathematics topics come together, and their relationship is essential in solving problems in the analytical geometry topic.

Examples are a useful tool in facilitating the understanding of mathematical concepts, theorems, and techniques (Cayo et al., 2023) as it is known that in mathematics teaching, one cannot only rely only on definitions to facilitate learners' ability to apply their knowledge of a concept (Figueiredo and Contreras, 2015). In this regard, examples help to develop concept formation through illustrations, or they could be used as demonstrations for problem solving procedures and approaches (Huntley, 2013). In multilingual classrooms, examples also help in facilitating meaning making. For instance, the best way to explain the meaning of the angle of inclination to learners in multilingual classrooms is to use diagrams as examples. This also allows the teacher to show the relationship between the angle of inclination and gradient as shown in the Figure 2.1.

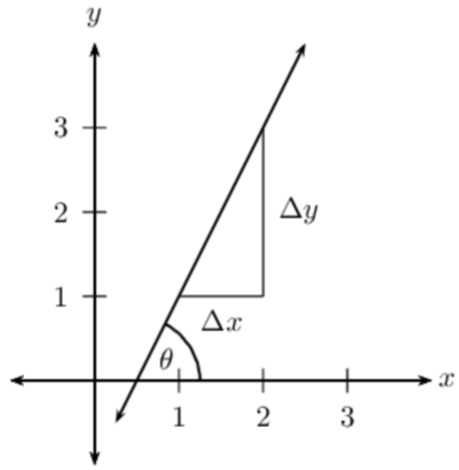


Figure 2.1 Example of the relationship between gradient and the angle of inclination

This example makes it easier for learners to observe that when the gradient changes, the angle of inclination also changes. Hence, binti Suffian (2010) refers to the use of examples as an effective method of teaching and learning mathematics because they can facilitate procedural and conceptual understanding in addition to their ability to enhance the recognition of relationships and generalizing. It is clear from the way scholars in mathematics education advocate for examples that a teacher must have knowledge of different examples presentations, that is, abstract, visual, and physical examples (Lee and Lee, 2021). They must also know to carefully select and use examples to promote learning, especially in multilingual mathematics classrooms.

Zodik and Zaslavsky (2008) warn that, when generating or choosing examples, teachers must bear in mind that some may promote or hinder learning. In addition, Cayo et al. (2023) emphasize that examples determine the learners' success in learning a concept, and that selecting and using appropriate examples is considered the most complex and critical process in teaching mathematics. They further add that the ability to generate effective examples by teachers in mathematics is affected by the knowledge of mathematics, the knowledge of the student's learning abilities, and PCK, which Zodik and Zaslavsky (2008) also noted. Moreover, Rowland (2008) also noted that the diversity of the teacher's knowledge has an impact on their choice of examples. These arguments are in line with the interest of this study, which is, exploring the role of PSTs' PCK in their choice of examples.

Different scholars have taken interest in categorising the different roles of examples, and the categories are related or rather connected. Karaagac (2004) defined two types of examples, these are; active examples and passive examples. Active examples are examples that can also be viewed as exercises where learners are supposed to apply their knowledge after learning from the passive examples, which are mainly to expand and explain the definition of a concept further. These two types of examples are similar to Olteanu's (2017) idea of the difference between a task and an example, where a task is what learners work on, and an example is used to enhance an explanation. Therefore a 'task' can be said to be an active example, and an 'example' in Olteanu's (2017) terms, can be seen as a Karaagac's (2004) notion of a passive example.

Watson and Chick (2011) categorised examples in terms of their roles as follows: (1) examples that expose the interconnections between concepts in the examples. (2) Examples that facilitate generalizations, which often have to do with finding similarities between examples. (3) Examples that facilitate abstraction. These are also linked to the categories that are presented by Cayo et al. (2023); Zodik and Zaslavsky (2007a, 2007b) where they categorise examples according to the mathematical skill they promote. That is, whether they promote the knowledge of a theorem, a method or procedure, or to illustrate a concept.

Cayo et al. (2023, p. 6) cite the following list of categories of examples by Figueiredo, Blanco, and Contreras (2007):

- Definitions – examples given to foreground the characteristics of a definitions to help learners grasp the definition, or the examples given after a definition.
- Representations – examples that are used to expose learners to a concept they are encountering for the first time. These are often presented as exercises or problems.
- Characteristics – These examples are presented after the explanation of a concept, and they are set to help learners surmount difficulties and clear misconceptions and doubts.

- Internal applications – examples that link the concept or topic at hand to other topics that were covered before or to topics that will be covered in future.
- External applications – these are application examples that are based on real life problems or scenarios, or examples that link mathematical problems to other areas of science.

One question was triggered as I read through the different categories. Could there be a category of examples that are used to show and explain the mathematical language, beyond the examples that are used to explain definitions?

The different examples in the teaching and learning of mathematics all play important roles in their own rights. Rowland (2008) identifies different types of content development examples, which are worked-out examples, exercise examples, and spontaneous examples. The worked-out and exercise examples are pre planned by the teacher, while the spontaneous examples come up during the lesson from the teacher to enhance the development of content, and from learners as they ask questions and make inputs (Zodik and Zaslavsky, 2008).

This study will only be able to analyse the worked-out examples and exercise examples as they are pre planned and would be noted on the lesson plans. Teachers use worked-out examples to introduce and explain how a particular problem is solved. These examples are the basis of teaching as they are a common instructional method because teachers use them to demonstrate problem-solving procedures (Mevarech and Kramarski, 2003). Exercise examples are more illustrative and assessment oriented (Essien 2021). All the explored examples classifications are connected therefore, this study does not necessarily consider the type of examples, but what the examples are meant to achieve in the lesson.

According to Lee and Lee (2021), regardless of the importance of examples in the teaching and learning of mathematics, there is a relative scarcity in research on the teachers' choice of examples. This study addresses this shortage by exploring the PSTs choices of examples, which is a good starting point to understand and attend to issues that affect their choices as PSTs. This will improve their choice and use of examples as in-service teachers.

Cayo et al. (2023) argue that there is a need for further studies on how examples are used in the classroom because the fact that examples are the main source of learning mathematics has also made them a major source of confusion for learners (Maso, 2011). This highlights the need for more research on the teachers' choices of examples and the knowledge base they draw from when choosing these examples, which is the main objective of this study. The study explores the PSTs' knowledge base for choosing examples because researchers have already established that both the choice or selection and the use of examples require a set of specialised knowledge from the teachers' side (Cayo et al., 2023). PCK is one form of knowledge that researchers (Suffian and Abdul, 2010) point to, which would of course need to coalesce with mathematical content knowledge (Zaslavsky, 2008) for teachers to skilfully choose and use examples.

In the exploration of ways in which teachers could learn to use examples, Mwadzaangati et al. (2022) suggest lesson study, which has to do with the analysis of lesson planning, teaching, and lesson reflection. Lesson study would be the best way for teachers to learn exemplification and understand it as a teaching practice because it would provide the opportunity for them to scrutinise the examples they choose and use, and how effective they are in lessons. A suitable strategy employed by Mwadzaangati et al. (2022) was to study the way teachers analyse and discuss what varies and remains the same in their example spaces. This aligns with the approach that is used in this study because after analysing the examples, the PSTs are approached to discuss their thoughts and ideas about their examples' choices. Therefore, one can claim that from this experience, the group of participating PSTs also started thinking differently about their examples.

2.5 The teaching and learning of analytical geometry.

In the South African context, geometry, which is a spatial domain of mathematics (Kholid et al., 2022) has two components in the secondary school curriculum. That is, Euclidean geometry and analytical geometry. Euclidean geometry is associated with constructing proofs, promoting the ability to think logically, developing spatial intuition about the real world, and promoting the ability to read and interpret mathematical arguments (Van Putten, Howie and Stols, 2010). Analytical geometry, also known as coordinate geometry, on the other hand, is the use of algebraic symbols, methods, manipulations, and equations to solve geometry problems

(Vaisman, 1997). Analytical geometry is an intersection of geometry and algebra (Bossé, Bayaga, Lynch-Davis, and DeMarte, 2021).

According to Bansilal and Naidoo (2012), analytical geometry is an important component of mathematics that also facilitates the connection between other aspects of mathematics such as, measurements, space, shape, trigonometry. It also involves key concepts of calculus (Campuzano and Crisanto, 2022). Jones (2003) has also posited that some aspects of analytical geometry can help learners in linking different areas of mathematics, such as the relationships between graphs of functions and graphical representations of data in statistics. Khalil et al. (2018) argue that the basis of analytical geometry is mainly about identifying relations between two or more variables graphically.

The most researched of the two geometry components is Euclidean geometry with the van Hiele levels of geometric thinking being used to investigate how learners understand it. Very little has been done though in regards to Analytical geometry (Bossé et al., 2021), and the van Hiele levels cannot be applied on it (Abdullah and Zakaria 2013). However, algebra, which is part of analytical geometry has also been investigated extensively using the Structure of the Observed Learning Outcome (SOLO) framework (Bossé et al., 2021).

The attention analytical geometry has mostly received is about its teaching and learning through dynamic geometry software with the most commonly studied software being GeoGebra (Khalil et al., 2018; Campuzano and Crisanto, 2022; Kholid et al., 2022), but there hasn't been a framework that has been introduced to assess how learners learn and experience it. For this reason, Bossé et al. (2021) use both the van Hiele levels and the SOLO frameworks to analyse the learners' understanding of analytical geometry, whereby the van Hiele levels are used for the geometry aspect and the SOLO framework for the algebra part, and any other mathematics component that may be involved.

Developing a framework to examine the teaching and learning of analytical geometry would be beneficial in the field of mathematics education. This is because understanding the topic also helps learners understand the real world, like noticing that photographs around them are merely made up of geometric objects (Jones, 2003; Khalil et al., 2018).

According to Mogari, Kriek, and Atagana (2016), teachers find it difficult to teach analytical geometry while learners also find it difficult to learn the topic. In a study that worked with mathematics PSTs, Asmarani, Zahroh, and Dewanti (2022) also noted that the PSTs' ability to understand analytical geometry concepts is low. These PSTs are likely to be amongst the teachers who find the topic difficult to teach in future. So, it is important to understand their level of content knowledge in this topic in order to close gaps where possible and prepare for future PSTs. In exploring the PSTs' PCK for choosing examples for teaching analytical geometry in multilingual classrooms, this study also uncovers the depth of the PSTs understanding of analytical geometry concepts.

The PSTs' low ability to understand analytical geometry concepts may be a result of the challenges they experienced as learners in the topic. Learners have difficulties in understanding analytical geometry concepts (Noto, Hartono, and Sundawan, 2016) because of their inability to connect and see relationships between concepts (Anif et al., 2019; Fourrie and Machaba, 2020). Gargrish, Mantri, and Kaur (2020) have also noted that many researchers have discovered that learners have a problem with linking the geometric concepts that they learn in the classroom to the real world. Unfortunately, these challenges are taken on to teacher training as we see them in PSTs.

However, not all hope is lost as Nurhasanah, Kusumah, Sabandar, and Suryadi (2018) suggest that this topic can be used to develop horizon knowledge in PSTs. Horizon knowledge is a subdomain of the subject matter knowledge domain, and it is the ability to see how mathematics topics and their concepts are related (Ball, Thames, and Phelps, 2008). This knowledge is further discussed in the next chapter, which is the theoretical framework of this study. Nurhasanah et al. (2018) suggest that the concept of parallel coordinates has a significant role in developing the abstraction process in learning analytical geometry, and it could also be seen as a way of developing horizon knowledge in PSTs. This is because the concepts of analytical geometry link various concepts from other topics in mathematics.

To curb the challenges faced in the teaching and learning of this topic, Ozkan (2018) suggests that teachers must identify learners' errors and misconceptions in analytical geometry in order for them to use effective instructional strategies. Yildiz and Baltacis (2016) argue that contextual teaching and learning could improve learners' understanding of concepts in geometry, especially in analytical geometry. Contextual teaching and learning is an approach that promotes the connection between mathematical concepts and real world or everyday experiences (Mentari and Syarifuddin, 2020). It would be of great benefit for learners in multilingual mathematics classrooms as it would encourage learners to engage with the mathematics register when solving problems. It would also encourage them to learn to represent problems mathematically as Campuzano and Crisanto (2022) argue that the key to learners learning and understanding complex problems in analytical geometry is to teach them to master mathematical representation.

In one of the earlier studies, Bansilal and Naidoo (2012) suggested that teachers must provide learning opportunities in analytical geometry by allowing learners to engage more in conversion activities instead of just focusing on treatment activities. In Bansilal and Naidoo's (2012) terms, treatment activities are those that involve the movement from one semiotic representation to another within the same area. Conversion activities would be about moving from one semiotic representation to another (in different areas) but conserving the meaning of the concept (Bansilal and Naidoo, 2012).

This could be a challenge for some teachers as Adolphus (2011) noted that most mathematics teachers have an extremely poor foundation in geometry, which is the reason behind Barrantes and Blanco's (2006) argument that, it is important to focus on initial teacher training in geometry in order to improve the teaching and learning of the topic in schools. In their study in which they investigated the incorporation of GeoGebra in Project-based learning (Geo-PjBL) in a mathematics PSTs' analytical geometry course, Kholid et al. (2022) noted that the Geo-PjBL has the potential to facilitate the understanding of analytical geometry in mathematics PSTs. They add that teacher training institutions should consider employing this strategy because it has more benefits to it, such as honing the PSTs ICT skills.

Campuzano and Crisanto (2022) also found that learners who are taught analytical geometry through the Wolfram Alpha software significantly performed much better academically than their counterparts. This could be the case for many learners considering the exposure to technology that learners of this day have, and how they may find learning easier through these technological tools. For this reason, Khalil et al. (2018) suggested that teachers should use GeoGebra to assist learners in the development and understanding of concepts in analytical geometry by ensuring that they explore and engage with the concepts through the tool. Technological tools are making teaching and learning much easier, but a teacher needs content knowledge in order to effectively teach using technological tools. In the use of technology for teaching, examples are still used to explain, illustrate, and demonstrate through the tools. Therefore, this study will always have a place in literature as it focuses on the PSTs' PCK, and timeless examples in the teaching and learning of mathematics.

The explored literature shows that there is a gap on literature about teaching and learning analytical geometry in multilingual classrooms. Research on this is important because analytical geometry connects different topics in mathematics, and understanding its terms and concepts would promote understanding in mathematics as a subject.

In this chapter, the literature review covered the key aspects of this study. The explored literature guides the development of the whole study. In the next chapter, a holistic overview of the theoretical framework is provided. The framework adds to the explored literature and also guides the development of the analytical framework of the study.

CHAPTER THREE: THEORETICAL FRAMEWORK

3.1 Introduction

According to Wood et al. (1991), there is a significant relationship between the teacher's knowledge for teaching, and the knowledge of learners learning. In a study that explores the relationship between the teacher's knowledge and the learners' performance, König and Pflanzl (2016) found that higher teacher knowledge is directly related to the quality of instruction, and learning in the classroom. Hence, it is important for a teacher to be able to specify learning goals for a lesson, be able to align them with their instructions, and see the relationship between their teaching and learners' learning (Morris, Hiebert and Spitzer, 2009). This study explores the PSTs' knowledge for teaching, and the role it plays in their instruction, which in mathematics, instruction is mostly through examples. The theoretical framework of the study exploits the symbiotic relationship between the PSTs' PCK and their ability to create conditions that are necessary for learning, which are analysed using some aspects of the variation theory.

It is standard to expect that anything that has to do with geometry must involve the van Hiele's theory of geometric thinking (Yudianto, Sugiarti, and Trapsilasiwi, 2018). This study, however, does not employ van Hiele's theory because it does not seek to evaluate learners' thinking, but the conditions that lead to learning, which are often created by teachers, in this case, PSTs. Therefore, the variation theory is more suitable for this study as Kullberg, Kempe, and Marton (2017, p. 560) state that the theory "…spells out the conditions of learning…".

3.2 Pedagogical Content Knowledge

Reflecting on the knowledge that develops in the teacher's mind throughout their teaching experience, Shulman (1986) introduced three categories of knowledge which he posited to be exclusive to teachers (Cayo et al., 2023). These are subject matter knowledge, curricular knowledge, and PCK. Shulman later expanded the three categories to seven categories: subject matter knowledge (content knowledge), general pedagogical content knowledge, curricula (curriculum) knowledge, pedagogical content knowledge, knowledge of educational contexts, knowledge of learners, and the "knowledge of educational aims, purposes, and values, and their philosophical and historical foundations" (Cayo et al., 2023, p. 2). Of these forms of teacher knowledge, PCK attracted a lot more attention and was defined as the ability to represent ideas, use analogies, examples, illustrations and demonstrations (Shulman, 1986) to promote learning in the classroom. Shulman (2005, p. 11) later defined and described it as a "special amalgam of subject and pedagogy that is uniquely the province of teachers".

According to Shulman (1986) a teacher with PCK understands what makes learning specific topics of a subject easy or difficult, and understands that their learners of different ages, and coming from different backgrounds have preconceptions about certain topics. Bansilal, Mkhwanazi, and Brijlall (2014) added that there is a strong correlation between PCK and what transpires in the classroom. That is, whether opportunities for learning are created or not. Put in simpler terms, the knowledge of content and the knowledge of enacting the content with the understanding of the educational context are the basis for PCK.

In recent years, all the forms of knowledge are receiving attention with some scholars modifying them to suit specific subjects and contexts. One of the major modifications that set the education field abuzz is Technological Pedagogical Content Knowledge (TPCK) which is commonly known as TPACK. This entails the integration of PCK and Technological Content Knowledge (TCK) which is the ability to represent concepts and teaching material using technology (Mishra and Koehler, 2006). Many studies have explored TPACK (Chai, Koh, and Tsai 2013; Listiawan et al., 2018), and researchers in mathematics education have explored its application in mathematics teaching and learning. This has led to a new framework called Technological Pedagogical Mathematical Knowledge (TPMK), which directly links technological knowledge, mathematical knowledge, and pedagogical content knowledge (Wijaya et al. 2021).

This study is mostly interested in examining the chosen examples to understand the knowledge base that mathematics PSTs draw from to create opportunities for learning in multilingual classrooms. Investigating their knowledge of integrating technology in their chosen examples is another area, researchers can probe into further, because to understand the knowledge that is required to structure or choose appropriate examples for multilingual classrooms might not have to do much with the knowledge of integrating technology in teaching. Therefore, the most applicable modifications of PCK and from which the framework of this study will draw are the models that were introduced for teaching mathematics by Ball et al. (2008) and Carrillo-Yañez, Climent, Montes, Contreras, Flores-Medrano, Escudero-Ávi 1la, Vasco, Rojas, Flores, Aguilar-González, and Ribeiro (2018).

Ball et al. (2008) devised a model that has sub-domains for subject matter knowledge and PCK. The subdomains were inspired by their analysis of the demands that come with mathematics teaching. They subdivided subject matter knowledge into; Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK). The CCK is mathematical knowledge and skill that is used outside teaching, and the SCK is knowledge and skill that is specifically for teaching (Ball et al., 2008). The subdomains of PCK are the Knowledge of Content and Students (KCS), and the Knowledge of Content and teaching (KCT). KCS is the teacher's knowledge about learners and mathematics, and KCT is the knowledge about teaching and mathematics (Ball et al., 2008). They then provisionally added Curricular knowledge as an additional sub-domain of PCK, and Horizon knowledge as an additional sub-domain for subject matter knowledge. The diagram in Figure 3.1 shows the model of mathematical knowledge for teaching as developed by Ball et al. (2008).

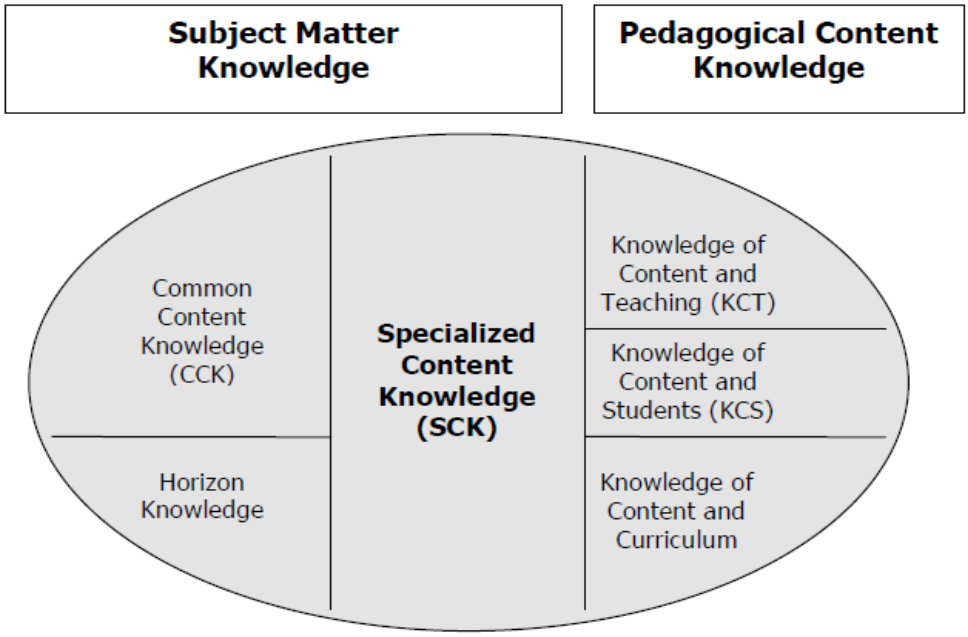


Figure 3.1 Domains for mathematical knowledge for teaching as developed by Ball et al. (2008)

The two additional subdomain, curricular knowledge and horizon knowledge are the knowledge of Content and Curriculum, and the teacher's awareness of the connections and relationships between mathematical topics and the curriculum (Ball et al., 2008), respectively. This model was influential in the development of the Mathematics Teachers' Specialized Knowledge (MTSK) model which was proposed by Carrillo-Yañez et al. (2018). These scholars mostly appreciated that the model that was introduced by Ball et al. (2008) recognised the one type of knowledge that is more exclusive to teachers, that is, the SCK. Although, they argue that the model has some shortcomings in a sense that the subdomains might overlap when the framework has to be employed for analysis.

Therefore, in their model, Carrillo-Yañez et al. (2018) propose a framework that does not only explore the knowledge that is used by teachers for teaching, but one that also examines the knowledge that is needed. This framework is more invested in the analysis of the knowledge that is employed by teachers in their teaching, which makes it applicable in this study as it explores the role of PCK in the PSTs' choice of examples. The model also refers to subdomains that are related to those of Ball et al. (2008). In their introduction of the MTSK model, Carrillo-Yañez et al. (2018) explain that the specific knowledge that is needed by mathematics teachers includes understanding meanings, knowledge of definitions and properties of the different topics within the subject and their connections, knowledge of teaching and facilitating the understanding of mathematics, and the knowledge and understanding of how learners learn mathematics.

The above explanation confirms that a teacher cannot only rely on content knowledge without the knowledge for teaching, and other forms of knowledge that are needed for teaching. In line with this explanation Ball (1991) argued that content knowledge is crucial for teaching, but it is not entirely sufficient to facilitate learning. In support of this argument, Chick and Harris (2007) added that subject matter knowledge alone is not sufficient for teaching and creating opportunities for learners to learn in a classroom and this is exposed by the necessity and definition of PCK. According to Shulman (1986), a teacher with PCK understands what makes learning specific topics of a subject easy or difficult, and understands that their learners of different ages, and coming from different backgrounds have preconceptions about certain topics. Therefore, according to Carrillo-Yañez et al. (2018), the specificity of the knowledge that is needed for teaching mathematics affects both SCK which in this case is specialised mathematics knowledge, and PCK, and so each one of these cannot be presented as a subdomain of either. Hence the introduction of the model in Figure 3.2, which has different subdomains.

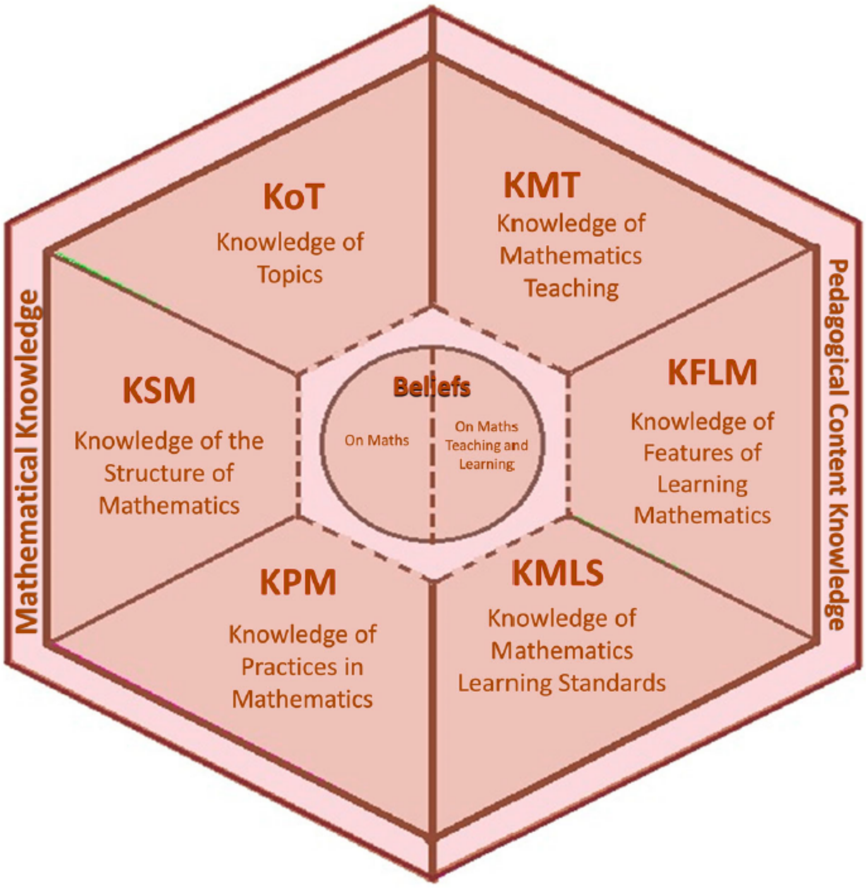


Figure 3.2 The MTSK model (Carrillo-Yañez et al. 2018)

The model has two domains, the Mathematical Knowledge, and the Pedagogical Content Knowledge. Each one of them has three subdomains and one shared or common subdomain as shown in Figure 3.2. The common subdomain is "Beliefs", which explores both beliefs on mathematics as a subject and beliefs on the teaching and learning of the subject (Carrillo-Yañez et al., 2018). The noncommon subdomains are further explained as follows:

3.2.1 Mathematical Knowledge

• Knowledge of Topics (KoT)

This is about the "what and in what way" (Carrillo-Yañez et al., 2018, p. 242) the teacher knows the mathematics topics. It entails looking at whether a teacher has instrumental or relational understanding (Skemp, 1976) of the mathematics content by checking for example, their knowledge of concepts, procedures, theorems, rules, facts, and their meanings. Carrillo-Yañez et al. (2018) present the categories of the KoT in Table 3.1:

Table 3.1 Categories of KoT (Carrillo-Yañez et al., 2018).

Procedures	How to do something? When to do something? Why something is done this way? Characteristics of the result
Definitions, properties and foundations	
Registers of representation	
Phenomenology and applications	

These categories provide a sense of what KoT is about. They show that in addition to knowing the 'what and in what way', teachers must know the solutions to the problems that their learners are presented with and know how to help learners in understanding what they are doing and why (Zakaryan and Ribeiro 2019). According to Zakaryan and Ribeiro (2019), KoT also allows teachers to understand the mathematics content that is presented in textbooks and other curriculum resources.

• Knowledge of the Structure of Mathematics (KSM)

KSM is the knowledge of the connections and relationships between mathematical concepts. Carrillo-Yañez et al. (2018) recognise that there should be the knowledge of inter and intra conceptual connections in mathematics teaching. They explain that the intra and inter conceptual connections are already covered in the KoT. Intra conceptual connections are corresponding the definition, properties, and foundations category. The inter conceptual connections correspond with the phenomenology and applications category (Carrillo-Yañez et al., 2018). Table 3.2 shows the categories of the KSM subdomains.

Table 3.2 Categories of KSM (Carrillo-Yañez et al., 2018)

Connections based on simplification
Connections based on increased complexity
Auxiliary connections
Transverse connections

According to Ferretti, Martignone, and Rodríguez-Muñiz (2021), KSM is about knowing the connections between mathematics topics and concepts which promote the recognition of structures in mathematics topics. This knowledge is essential in the teaching of analytical geometry, where the teacher must be able to recognise the different concepts that structure the topic.

• Knowledge of Practices in Mathematics (KPM)

According to Carrillo-Yañez et al. (2018, p. 244), "In the MTSK model, Knowledge of Practices in Mathematics focuses specifically on means of production and mathematical functioning, leaving aside structuring and organisation as part of Knowledge of the Structure of Mathematics." The categories of this subdomain are not yet tabulated because they are under study. Ferretti et al. (2021) describe KPM as the knowledge of how the KSM relationships are established, knowing the aspects of mathematical communication, reasoning in proofs, and having ways to represent, argue, generalise, and explore mathematics.

3.2.2 Pedagogical Content Knowledge

The recognition of PCK in the field of teaching is said to be a critical advancement because it goes beyond subject matter knowledge. PCK implies a specialized form of subject matter knowledge, it can also be referred to as the 'subject matter knowledge for teaching' (Scheiner, Montes, Godino, Carrillo, and Pino-Fan, 2019). One cannot resist but notice the different approach Carrillo-Yañez et al. (2018) are using to engage with the concept of PCK. They are not looking at it as mostly or only about the knowledge of the different aspects of teaching, but they also bring in the understanding of the learning process that a teacher must also have. This special aspect of this model is the main reason for its applicability to this study as it explores the way in which the examples that PSTs choose create opportunities for learning. This entails understanding the features of learning.

• Knowledge of Mathematics Teaching (KMT)

KMT allows teachers to facilitate the development of conceptual and procedural understanding in learners (Ferretti et al., 2021). This knowledge is normally based on theories and research in mathematics education, and at times on the teacher's experience and reflections on their teaching practice. The categories of KMT in Table 3.3 indicate that this knowledge includes knowing the relevant teaching resources and material that one can use to promote conceptual and procedural understanding in mathematics (Carrillo-Yañez et al., 2018).

Table 3.3 Categories of KMT (Carrillo-Yañez et al., 2018).

Theories of mathematics teaching
Teaching resources (physical and digital)
Strategies, techniques, tasks and examples

KMT is also about the awareness of the potential that different teaching strategies for specific mathematics content, and activities have in promoting learning. It also entails being aware of the limitations they present which could lead to the learning process being hindered (Carrillo-Yañez et al., 2018).

• Knowledge of Features of learning Mathematics (KFLM)

This subdomain focuses on the teacher's knowledge and understanding of the learning process and their awareness of the possible difficulties and obstacles in the learners' learning process (Ferretti et al., 2021). This is the knowledge of the features that facilitate learning mathematics. According to Carrillo-Yañez et al. (2018), the KFLM subdomain focuses mainly on the object of learning, which is the mathematics content rather than the learners themselves. So, it focuses on the key aspects of teaching that promote learning, such as knowing and applying mathematics learning theories and the way learners interact with mathematics. See Table 3.4 for the categories of KFLM.

Table 3.4 Categories of KFLM (Carrillo-Yañez et al., 2018)

Theories of mathematical learning
Strengths and weaknesses in learning mathematics
Ways pupils interact with mathematical content
Emotional aspects of learning mathematics

KFLM also entails that a teacher must be aware of how learners think and build on their knowledge when they are solving mathematical problems. This includes knowing the different learning styles that learners draw from for successful learning. This subdomain shows the importance of the teacher's knowledge of the learners they teach.

• Knowledge of Mathematics Learning Standards (KMLS)

According to Ferretti et al. (2021), KMLS is about the teacher's understanding of what learners should or can learn and achieve in mathematics topics in addition to knowing about the formally prescribed standards in the curriculum. This means the teacher must know the level of their learners' ability to understand and construct mathematical knowledge, which determines their suitability and ability to learn mathematical concepts (Carrillo-Yañez et al., 2018). The categories of this subdomain are outlined in Table 3.5.

Table 3.5 Categories of KMLS (Carrillo-Yañez et al., 2018).

Expected learning outcomes
Expected level of conceptual or procedural development
Sequencing of topics

For a teacher to be able to design or understand the designed standards of learning they must have mathematical knowledge. KoT, KSM and KPM provide a knowledge base for a teacher to know how to sequence topics, to plan for the expected level of conceptual and procedural development in learners, and to achieve the expected learning outcomes. This shows that mathematical knowledge and PCK are interdependent in the teacher's knowledge for teaching. The MTSK model is relevant for this study because it explores both the knowledge used and needed by mathematics teachers for teaching, and according to Carrillo-Yañez et al. (2018), the model could assist teacher training institutions in organising the necessary training needs for PSTs. This is in line with the study investigating and analysing the PSTs' knowledge in their choice of examples. Employing the MTSK in this study is driven by the notion that a teacher cannot only rely on content knowledge without the knowledge for teaching (Chick and Harris, 2007), and more.

According to researchers in mathematics education, there is a strong correlation between PCK and classroom interactions (Bansilal, Mkhwanazi, and Brijlall, 2014; Carrillo-Yañez et al., 2018). The classroom interactions involve different activities which facilitate the learning process, and in these interactions, the teacher must have the knowledge of mathematics content, pedagogy, and the features of learning the content in order to create opportunities for learning in the classroom interactions. Considering the teacher's ability to create opportunities for learning is important because the main goal of teaching is to facilitate learning. In this respect, the variation theory becomes a handy tool in observing how opportunities for learning are created by PSTs through their chosen examples for teaching analytical geometry.

This study interlaces the MTSK model with some aspects of the variation theory. However, it is important to note that as one explores the domains and subdomains of the MTSK model, there are some aspects of it that seem to draw from concepts of the variation theory. For instance, the KFLM subdomain focuses on the features of learning in the object of learning. In this study, the object of learning is explored as far as the intended, and beyond this, the patterns of variation are employed to analyse the possibility of the intended object of learning being achieved through the chosen examples. These concepts are explained further in the next section.

3.3 Variation Theory

The variation theory originates from the phenomenography tradition, which grew out of empirical research studies (Runesson, 2005; Bussey, Orgill, and Crippen, 2013; Cheng, 2016). It was introduced by Ference Marton and colleagues in their work which proposed that learning is a result of the discernment of the variation of some aspects of a concept over the invariance in the background (Al-Murani et al., 2019).

The variation theory is a learning theory that defines learning as the development of capabilities to experience a concept differently compared to the prior experience of the same concept. According to Marton and Pang (2006), the variation theory points out the necessity of variation as a component in teaching in order for learners to see what is to be learned. The variation theory entails learners being able to recognize and make meaning of the critical features of the concept that 'is to be learned', which is referred to as the object of learning (Bussey et al., 2013). The critical features of an object of learning are the aspects of a concept one attends to, and to which meaning is ascribed, and therefore discerned (Bussey et al., 2013). These are the features of the object of learning that facilitate the difference in the way a learner understands a concept (Bussey et al., 2013).

According to Kullberg et al. (2017), the learners' discerning of critical features or seeing a concept differently greatly depends on the chosen and used examples by teachers (Kullberg et al., 2017). Kullberg et al. (2017) expound that the teachers' use of variance and invariance within examples when teaching mathematics can help learners understand concepts. To do this in mathematics teaching, the patterns of variation, which are; contrast, generalisation, separation, and fusion (Marton et al., 2004) must be incorporated in the example space. The patterns of variation are explained or defined as follows:

- Contrast is about comparing features of a concept with what it is not (Bussey et al., 2013). Baskoro (2021) adds that contrast shows that for a learner to experience something or learn a concept, they must experience something else with which they can compare.
- Generalisation is about comparing similar features of the object of learning. For a learner to
  discern the critical features of the object of learning, they must experience many examples from which they can generalise (Bussey et al., 2013).
- Separation is explained by Marton et al. (2004) as varying some aspects of the critical features of the object of learning while other features remain invariant. According to Bussey et al. (2013), separation allows learners to make meaning from experiencing different features of the object of learning, regardless of their importance. In Baskoro's (2021) terms, separation hints that in order for learners to experience critical features of a concept and be able to separate them from others, those critical features must be variant while the others are in variant.
- Fusion is when learners experience variation by being exposed to several features of the object of learning which they are to discern simultaneously (Marton et al., 2004).

Adler and Ronda (2015) revised the patterns to similarity, contrast, and fusion. They argue that generalization can be facilitated through similarity and contrast, and that incorporating variance and invariance in the use of examples makes up for separation. While fusion is kept as the third pattern of variation and defined as having more than one variant and invariant aspect in the object of learning (Adler and Ronda, 2015). This study adopts the four unrevised patterns of variation because without generalization in the patterns of variation suggested by Adler and Ronder (2015), this study, could easily fail to explore and expose the importance of examples.

This is owing to Zodik and Zaslavsky (2008) emphasizing that examples are items of larger concepts, from which a learner can reason and generalize in mathematics.

In support of the notion held in this study, Al-Murani et al. (2019) add that the variation theory emphasizes that through contrasting and generalizing in examples, fusion is enabled to help learners conceptualize complex concepts which have various features. For this reason, this study will observe and recognize generalization as a pattern of variation to be analysed in the examples.

According to Kullberg et al. (2017), variation in examples can facilitate the discerning of the critical features of the object of learning which can be observed from three different perspectives (Bussey et al., 2013). (1) What learners should learn about a concept according to the teacher, policy documents, and textbooks, which is referred to as the intended object of learning. (2) What is possible for learners to learn about a concept during a lesson, that is the enacted object of learning, and is seen from the observers' or researcher's perspective. (3) What learners learn, which is the lived object of learning, and is seen from a learner's perspective (Bussey et al., 2013). See the schematic view of the objects on learning in Figure 3.3

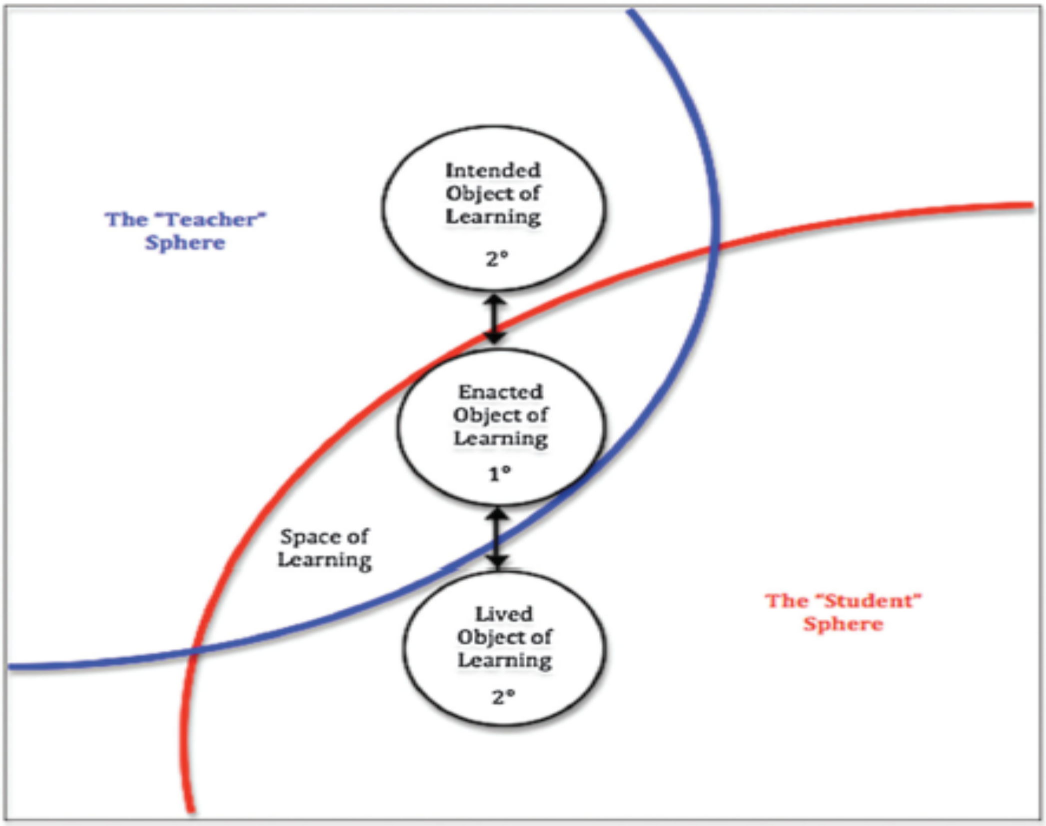


Figure 3.3 Modified representation of the objects of learning within the variation theory from Rundgren and Tibell (2009): (Bussey et al., 2013)

The framework of this study focuses on the teacher's sphere and how the teacher intends to create the 'space of learning' that has opportunities for learning. As already mentioned, the methodology of this study goes as far as exploring the intended object of learning, where examples in lesson plans are analysed and the PSTs are interviewed for their perspectives. This study mainly draws from the patterns of variation. They are used as a tool to analyse the structure of the PSTs' chosen examples to understand if they create opportunities for discerning critical features of the intended object of learning.

According to Bragg (2017), it takes detailed considerations and experience to make it possible for a teacher to create opportunities for learning, and create the "spaces of learning" (Marton et al., 2004, p. 20) through learning focused activities and examples. To explore the knowledge and considerations that mathematics PSTs must make when creating 'spaces for learning' and choosing examples that create opportunities for learning, this study merges the patterns of variation with the domains of the PCK born MTSK model.

3.4 Integration of PCK and Variation theory

Having explored the key concepts of PCK and its modified models, and the key elements of the variation theory, this section provides an overview of the merger framework of this study. According to Scheiner et al. (2019), in mathematics, PCK involves knowing about mathematics, knowing about teaching, and having the knowledge of instruction designing, such as generation, choosing or selecting, and using examples to facilitate learning. To do this, the teacher needs to know their learners and their learning context. Therefore, in the definition of PCK, most of the subdomains of the MTSK model are included. Accordingly, this study employs the domains and subdomains of the MTSK model in exploring the role of PCK in the PSTs' choice of examples.

According to Al-Murani et al. (2019), the variation theory is suitable for examining and analysing examples in mathematics because they are the main pedagogical tool and through examples, critical feature are varied through representations, numerical content, and orientation. The effective use of examples as a pedagogical tool is depended on mathematical knowledge and PCK. The merger framework is outlined in the schematic view in Figure 3.3, which shows how the mathematics teacher's knowledge informs the intended object of learning. This indicates the existing relationship between PCK and the variation theory. However, because the study mainly employs the patterns of variation in the variation theory, the proposed merger framework if referred to as the PCK-PV framework.

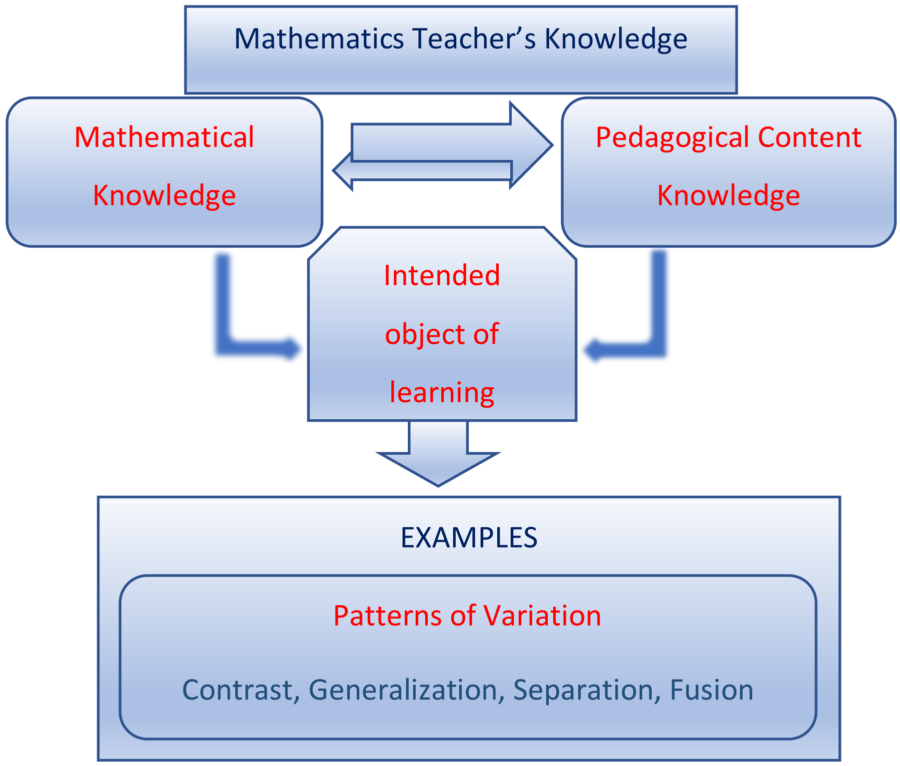


Figure 3.4 The PCK-PV framework

The framework shows that mathematical knowledge and PCK are interdependent in teaching mathematics. One must retain both mathematical knowledge and PCK to teach mathematics effectively. The arrows between mathematical knowledge and PCK on the PCK-PV framework indicate that mathematical knowledge has an impact in the development of PCK (big arrow), while PCK also play a role in improving mathematical content (small arrow), but it does not develop it if there is none.

Both forms of knowledge are responsible for the way the teacher plans and outlines a lesson, including the selection of relevant examples for the class they will be teaching. In this framework, the knowledge of a teacher is also responsible for their ability to choose examples, and structure them using the patterns of variation to create opportunities for learning. Understandably, PSTs may not be thinking about the patterns of variation if they were not exposed to them. However, this framework is still applicable in analysing the examples that the PSTs choose, and it will further inform teacher training on the importance of exposing PSTs to the variation theory.

Some studies have explored the application of the variation theory in teacher training and focused on the lived experiences of the PSTs as they were learning to plan, evaluate, and revise lessons using the variation theory. These studies found that it is possible for mathematics PSTs to actually learn and apply the principles of the variation theory in their teaching practices (Martensson and Ekdahl, 2021). In addition to these studies and with the intention of adding a new framework to analyse the PSTs' choice examples, this framework focuses on the knowledge applied in choosing and structuring examples for creating opportunities for learning.

This chapter explored the guiding framework of this study and how it is applicable in the analysis of the collected data. The framework also guides the discussion of the findings made from data analysis. In the next chapter, the methodology introduces the sampling method and participants of the study, the data collection tools, and how the collected data was analysed.

CHAPTER FOUR: RESEARCH METHODOLOGY

4.1 Introduction

This chapter outlines the research methodology of this study by introducing the research design, setting, sampling approach, and the methods used to recruit the participants. The chapter further discusses the outcomes of the pilot study and how they directed the data collection tools, and it also outlines the analytical framework that the study employed. Finally, the chapter discusses the trustworthiness, validity and reliability, and the ethical considerations made in the course of the research study.

4.2 Research design

This qualitative study employs interpretative paradigm as it seeks to understand and describe the experiences of the participants involved (Chilisa, 2012), and explore how they make meaning out of their experiences (Miller, Chan and Farmer, 2018). According to Castleberry and Nolen (2018) qualitative research methods allow researchers to explore the beliefs, values, and motives of the participants behaviour. The main aim of qualitative research is to get better insight of a phenomenon through the experiences of those who are directly involved, or have already experienced the phenomenon. It recognises the views of the participants and that they can only be understood in their context. In its descriptive approach, qualitative research allows researchers to build and provide a holistic picture in a natural phenomenon (Castleberry and Nolen, 2018).

This study seeks to understand the subjective world of the participants' experiences (Kivunja and Kuyini, 2017) by analysing and interpreting the role of PCK in the PSTs' choice of examples for teaching analytical geometry in multilingual classrooms. The study takes note of the characteristics of an interpretative study as outlined by Morgan (2007) that it entails the following:

- Accepting that there is an inevitable interaction between the researcher and the participants.
- Acknowledging and accepting the importance of context for knowledge and knowing.
- Believing that values must be made explicit because the knowledge created by the findings can be value laden.

These characteristics are considered in the data collection, analysis, discussion of findings, and recommendations that come from the study. This is to ensure trustworthiness, validity and reliability in the findings of the study, which is also discussed later in this chapter.

According to Castleberry and Nolen (2018), there are five steps to dealing with qualitative data. These are; compiling, disassembling, reassembling, interpreting, and concluding, and they are described as follows:

- Compiling – Using research tools that align with the research questions to collect usable data as a way to find meaningful answers to the research questions. In this step, all data is organised and transcribed if there are interviews, observations, or focus groups.
- Disassembling – This is where data is separated. Here, the data is taken apart to create meaningful groupings. This can be done through the use of an analytical framework, coding, or thematising.
- Reassembling - The codes that were identified when disassembling, are now put together in the context of the present study to create themes.
- Interpreting – In this step the researcher does an analysis and interprets the data to make sense of the information that comes out of the data. Interpretation also happens in the above first three steps.
- Concluding – This then becomes a response to the research questions and talks to the purpose of this study. Recommendations and suggestions may come up based on the first four steps.

(Castleberry and Nolen, 2018)

The methodology of this study follows these steps as a guide. The first step was addressed from the introduction to the data analysis chapters, where disassembling and reassembling was also done. The interpretation of the data findings and conclusion are expanded on in Chapters Eight and Nine, respectively.

4.3 Research setting

The study is conducted on mathematics PSTs from one of the universities in the KwaZulu Natal (KZN) province in South Africa. The province is predominantly populated by the Nguni speaking Africans, mostly isiZulu speakers as indicated in Figure 4.1, which shows the population distribution in KZN. Most of the classrooms in the province are multilingual as Barwell (2012) puts it that, these are classrooms in which teachers and learners draw on different languages.

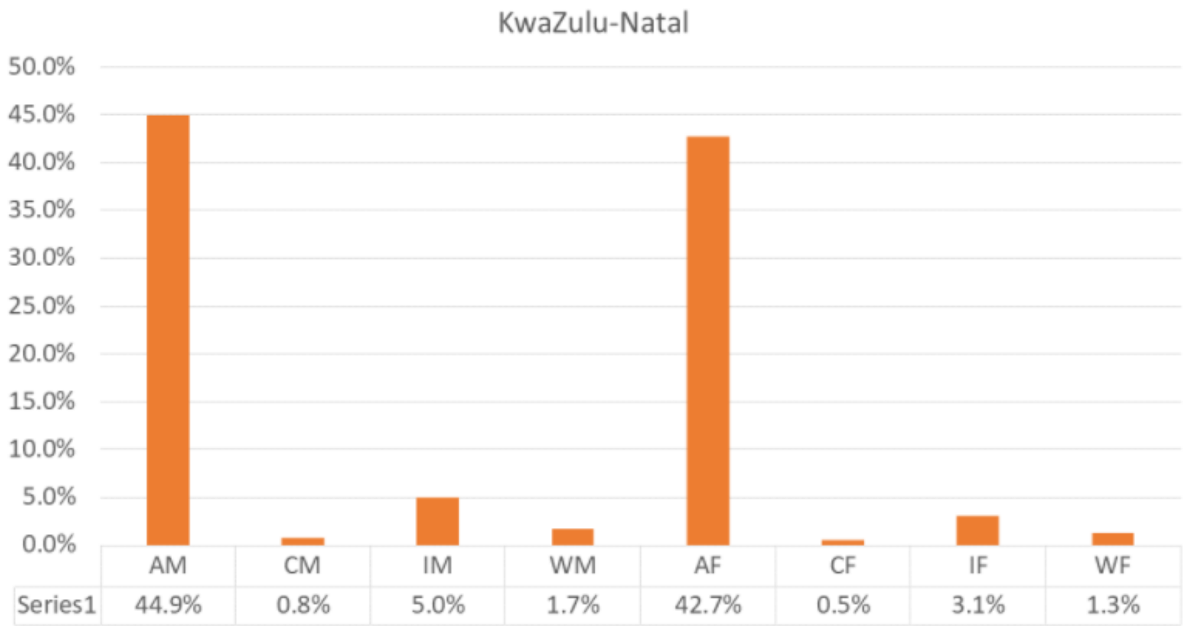


Figure 4.1 KZN population distribution. Adapted from *http://www.workinfo.mobi/index.php/articles/item/2142-national-and-regional-economically-active-population-profile-qlfs-q2-2021*
Original Data Source - StatsSA 19 August 2021

Key A =African, C=Coloured, I=Indian, W=White, M=Male, F=Female

Most of the PSTs who are enrolled in this particular institution are from the rural parts of KZN. They do their Work Integrated Learning (WIL) and start their work as teachers in multilingual classrooms, while they are also training in multilingual classrooms. In these classrooms the educators or lecturers do not necessarily speak the same language (Setati, 2008; Trinick, 2015) as the PSTs.

The study sampled PSTs who are in their final year of taking mathematics as a specialisation in the Bachelor of Education in Senior phase and further education and training. The institution in which the PSTs are enrolled revised the Bachelor of Education program into a differently structured program. The revised program introduced methodology courses, while the old one had the aspects of methodology courses covered in the specialisation courses. The institution also focuses the first year of the teacher training program in bridging the gap that may exist in the PSTs' content knowledge in preparation for the movement to university level content.

At their level of study, the sampled PSTs have been exposed to different teaching and learning theories, and strategies. Therefore, it is possible to observe the professional knowledge they would have acquired from all the theoretical course work. These PSTs have also had the opportunity to teach analytical geometry in Grade 11 during their WIL or teaching experience. The rationale for choosing to work with PSTs who would have taught Grade 11 classes is that analytical geometry would have been introduced in Grade 10. This allows me to observe the PSTs' PCK and its role in their choices of examples to connect the learners' prior knowledge to new concepts, and to create opportunities for learning and understanding the topic.

4.4 Sampling

The participants were sampled using the purposive non-probability sampling method where they were sampled based on the judgement of the researcher, and not randomly (Etikan and Bala, 2017). The advantage of this sampling method is that it involves the selection of individuals or groups of individuals that are proficient and well-informed about the concept that is being studied (Etikan, Musa, and Alkassim, 2016). Figure 4.2 provides an overview of the sampling design of this study.

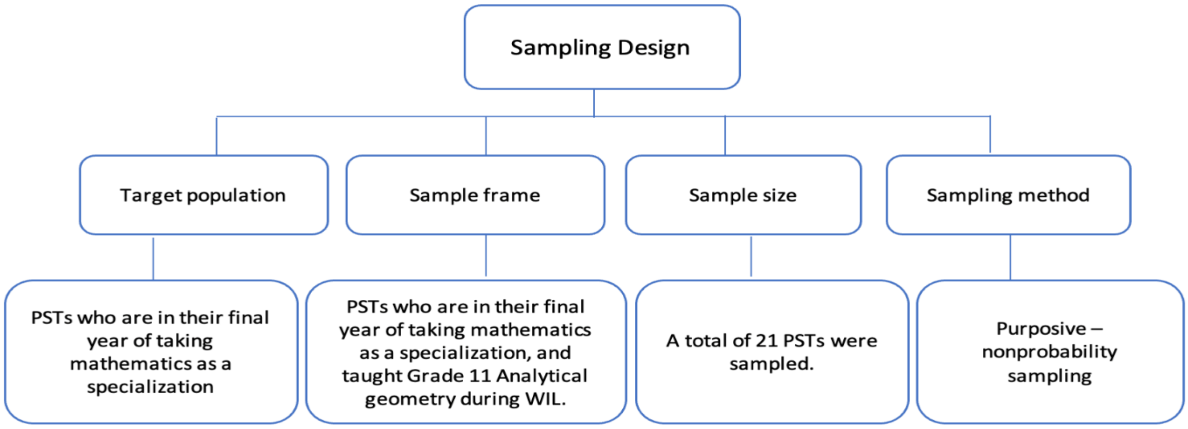


Figure 4.2 Overview of the sampling design

In this study, the participants were selected based on their subject specialization, year of study, and the grade and topic they would have taught during their WIL. All 57 mathematics PSTs who are in their final year of taking mathematics as a specialization in the Bachelor of Education in one university were approached and asked to participate in the study via email. They were also approached on campus during their lectures because there was hardly any response to the emails. The turnout of the PSTs who gave consent to participating in the study was 23 while the study had proposed to work with 15 PSTs. However, the data collection process led to more than 15 participants being sampled because saturation was not reached with 15 PSTs. Saturation being a point where all key issues are exhausted from the collected data, which highlights the saturation of the categories that comprise the theoretical construct (Hennink and Kaiser, 2022). In this case, saturation was based on the analysis of the chosen examples in the lesson plans. This is further discussed in Chapter 6.

Ultimately, the sample size of this study was 21 participants, some of which had conditions for their participation, which will be discussed in the data collection section. According to Hennink and Kaiser (2022), qualitative research sample sizes are guided by the adequacy of data, and the sample size is more about the data being able to provide sufficient information to fulfil the purpose of the study. It took 21 PSTs for the study to reach data saturation, where categories of common trends in the PSTs lesson plans were established.

4.4.1 Inclusion and exclusion criteria

Robinson (2014) advises that outlining a clear inclusion and exclusion criteria to define a population, and ensuring that the criteria are specific makes the target population more homogenous. As already explained in the research setting, the study was conducted in one of the universities in KZN focusing only on PSTs who specialise in mathematics. The inclusion and exclusion criteria that was employed in the study are outlined in Table 4.1.

Table 4.1 The inclusion and exclusion criteria

Inclusion	**Exclusion**
✓ **Final (4th) year preservice teachers**	× 1st, 2nd and 3rd years preservice teachers
✓ **Specializing in mathematics**	× Not specialising in mathematics.
✓ **Taught Grade 11 analytical geometry during WIL.**	× Did not teach Grade 11 analytical geometry

4.5 Recruitment process

It is common knowledge that recruiting participants for qualitative research has its own struggles, and the process is more difficult when the population is of young adults who are moving in and out of different lectures in their training programmes (Bonisteel, Shulman, Newhook, Guttmann, Smith, and Chafe, 2021). The population of this study is a group of young adults (preservice teachers) who are in training for mathematics teaching. For the same reason as noted by Bonisteel et al. (2021), one of the difficulties the study faced in recruiting these young adults is getting hold of them on email and, or in person. It helped the recruitment process that universities are now back to contact classes and I used this opportunity to approach the PSTs during one of their lectures where most of them were in attendance.

According to Bonisteel et al. (2021), the participant recruitment process occurs over four phases. (1) Recruitment plan development. (2) Recruitment plan implementation. (3) Maintaining participant engagement after data collection. (4) Reflecting on the overall recruitment activities after completion. In this study the recruitment plan was outlined during the proposal phase and was executed after obtaining ethical and institutional approvals which are needed before the potential participants are contacted (Bonisteel et al., 2021)

When the PSTs were approached and invited to participate in the study, a brief explanation of what the study is about and how data would be collected was provided in addition to the information being provided in the letters of information. The consent forms which were approved by the institution's ethics committee were distributed together with the information letters for the PSTs to complete a give back upon consent.

After the PSTs sent their signed consent forms, the analysis of their lesson plans commenced until data saturation was determined. Thereafter, the participants whose lesson plans were analysed were invited for interviews. Sixteen of the participating PSTs were available for interviews.

4.6 Pilot study

According to In (2017: 1), a pilot study is a way of validating "the feasibility of a study by assessing the inclusion and exclusion of participants". A pilot study was conducted to confirm the feasibility of the study in terms of the research questions and relevance of the data collection tools in obtaining sufficient data from which conclusions can be drawn. Seven files were sampled and the lesson plans in analytical geometry were analysed using the proposed analytical framework. After the analysis of the lesson plans, the 7 participants whose lesson plans were analysed were interviewed for follow up questions. The preliminary analysis of the lesson plans and files demonstrated that useful data could be collected through document analysis. Furthermore, the seven participants who were interviewed understood the interview questions, and some of their responses exposed the need for one additional question to the interview questions.

On asking the interviewed participants 'How they chose the examples they used, and what they had to consider' as the first question, many responded by starting with revealing where they adapted the examples from. This response led to my follow-up question being why they used that platform, textbook, or document. Some of their responses further revealed that they chose the examples from textbooks and platforms that provide solutions to the problems/examples. Therefore, a new question was added as the first question. The question is, “What platforms or documents/textbooks did you use to select your examples for teaching analytical geometry, and why?”

The pilot study demonstrated that the participants understood the interview questions, and the proposed framework for analysing the lesson plans is practical and applicable. The analysis of the small data obtained from the pilot study showed that the research methodology and tools are relevant for the study, and would be able to direct the study towards addressing the main research questions.

Data collection commenced with the use of modified tools following the pilot study. The participants were from a different cohort, as data collection began more than six months later due to the ethical clearance application process.

4.7 Data collection

Data collection was in a form of interviews, task analysis, and documents analysis. The National Curriculum and Assessment Policy Statement (CAPS) document was consulted to analyze the chosen examples, and the alignment of the lesson plan to the prescribed curriculum. Prior to analyzing the lesson plans, a diagnostic analysis of the overall PSTs performance on analytical geometry test was conducted to scrutinize their content knowledge of the topic. The lesson plans were then analyzed to examine the PSTs PCK and to identify the participants who would be interviewed. Interviews with the PSTs were conducted during their free time to follow-up and gather more information about their chosen examples, their intentions, and to interpret and analyze their knowledge. The interviews were semi-structured to allow me the flexibility to restate, prompt, and to challenge the interviewee if the initial prompt was misunderstood (Segal et al., 2006).

4.7.1 Mathematics Tasks analysis

Stein, Smith, Henningsen, and Silver (2009) enlighten that tasks will not always be created such that they are equal or of the same levels and it is important to note that tasks of different levels will stimulate different levels and kinds of learners' thinking. When a task requires learners to perform a memorized procedure, little opportunity is provided to develop learners thinking. Whereas, tasks that require conceptual thinking incite learners to make connections between concepts and create opportunities for learners' thinking to develop (Smith and Stein, 1998). According to Stein and Lane (1996), when one intends to develop the capacity to think, reason, and solve problems, it is important to consider high level and cognitively complex tasks.

The 23 PSTs who gave consent to participate in the study were given a test on analytical geometry (refer to Appendix A for the test). The test content covered aspects of analytical geometry from Grades 10 to 12 as per the CAPS document. This is the minimum content knowledge that the PSTs must have in the topic. All the test scripts were analysed. However, the analysis and presentation of data had to focus on the 21 PSTs whose lesson plans were analysed. Out of the 21, only 9 of the PSTs had scored more than 50%. The test was analysed to provide a diagnostic report of their performance and understanding of concepts. This was done by employing the levels of cognitive demands for tasks to understand which levels the PSTs could and couldn't solve.

Beyond the test analysis, the study also analysed the examples that the PSTs gave to learners as tasks. A task here is treated in terms of Olteanu's (2017) definition, which states that a task is the example(s) that learners work on. These tasks were also analysed using the levels of cognitive demand, but specifically focusing on the levels that are described and prescribed in the CAPS document. All the used levels of cognitive demand of tasks as ordered by Smith and Stein (1998), and prescribed in the CAPS document are outlined in the analytical framework.

4.7.2 Document analysis

According to Bowen (2009), document analysis is about examining and interpreting data to elicit meaning. Bowen (2009: 27) defines it as "a systemic procedure for reviewing or evaluating documents”. In this study, lesson plans were analyzed to examine the PSTs’ choices of examples in analytical geometry for multilingual classrooms, and how these examples could possibly create opportunities for learning. Kutsyuruba (2017) suggested several terms to use interchangeably for document analysis, some of these are ‘document evaluation’ and ‘content analysis of documents’. These terms also come up in this study as it also analyzed the content of the lesson plan to investigate the PSTs’ PCK. When examining the PSTs’ knowledge, I referred to CAPS document to analyze the alignment of the content in the lesson plans to the curriculum prescription for the grade in the CAPS document. This was then followed by interviews with each of the selected PSTs.

4.7.3 Interviews

According to McGrath, Palmgren, and Liljedahl (2019), qualitative research interviews are meant to assist the researcher in understanding the subjective perspectives of the participants rather than making generalisations about a larger group of people. The interviews phase in this study came after the PSTs’ lesson plans and examples were analysed. They were then invited for interviews to get their individual perspectives about their choices of examples. See Appendix D for the interview tool.

DeMarrais (2004) advised that the purpose of an interview is to gain in-depth knowledge from the participant, about a concept or an experience. The interviews is this study were semi-structured, which Segal et al. (2006) explained that this type of interviews allow the interviewer the flexibility to restate, prompt, and to challenge the interviewee if the initial prompt was misunderstood. This characteristic of semi-structured interviews, allowed me to investigate the reasons behind the chosen examples in the intended object of learning. Some of the PSTs were giving very short and less informative responses, but I was able to rephrase or ask follow-up questions until detailed responses were provided.

The interviews took place over a period of 4 weeks. Only 16 PSTs out of the 21 were available for interviews, with some of them requesting that I give them the interview questions in writing so that they could respond comfortably in their own time. This was a setback because they were not available for more questions after I had read their responses. Of these 16 PSTs, 3 of them did not finalise the interview date, nor did they withdraw from the study. The interviews were in person and dependent on the participants' availability and ability to make it to the venue. They were intended to be about 20 minutes long, but the shortest was 9 minutes, and the longest was 28 minutes and they were all audio recorded.

4.7.4 Audio recording and transcribing

Researchers explain that in qualitative research, recording devises are used to capture responses and utterances from participants during interviews and observations (Bell, 2014; Stuckey, 2014). This way, the researcher can focus on the participants' responses and follow-up questions without being too distracted by taking notes (Stuckey, 2014). In this study, an audio recording device was used to enhance the notes collected during the interviews. After data collection, the recordings were transcribed for analysis as a second step to data analysis after having analyzed the lesson plans. The interviews transcription was done by the researcher to ensure deep and intimate understanding of the data such as having better understanding of the meaning of the terms used in context (Castleberry and Nolen, 2018).

4.8 Experiences of the researcher and participants during the study.

The context of the study also contributed to some of the experiences of the researcher and participants. One of the experiences that are worth sharing because of its relevance to the focus of the study is that of language use. Knowing that the LoLT in the institution is English made it reasonable to have all my data collection tools and engagements with the PSTs in English. However, it was difficult to get some of the PSTs to respond to interview questions without rephrasing them in IsiZulu. Also, most of the PSTs responded to questions in IsiZulu making it difficult to transcribe the data as it also needed to be translated. Some of the PSTs requested that I give them the interviews in writing so that they could have enough time to think in their own comfort zones. This made the process difficult for me because, when given responses in writing, it was not easy for me to follow up on some of them. This experience alone, also makes one wonder about the engagements some of these PSTs have in multilingual classrooms where English is the medium of instruction. It also highlights the challenges that these PSTs are faced with in their classrooms as not all of their educators can speak IsiZulu. The way in which the LoLT interferes with the PSTs content knowledge, PCK, and ability to choose examples is touched on in the data analysis chapters.

4.9 Trustworthiness

To ensure trustworthiness in this study, a table with descriptors is used for the analysis process as noted by researchers that tables are useful in qualitative research as they support data analysis and ensure trustworthiness (Cloutier and Ravasi, 2021). Beyond using the table to strengthen trustworthiness by following the set descriptors, the study employs some of the strategies that Hadi and Closs (2016) recommend as a way of ensuring trustworthiness. The strategies are; triangulation, reflexivity and member check. Where triangulation is about ensuring that the study uses at least two data collection tools (Hadi and Closs, 2016), which in this study data was collected through task analysis, lesson plan analysis, and interviews.

Hadi and Closs (2016) define reflexivity as a strategy used to reduce bias of the researcher, and will be observed by ensuring that I make my point of view clear and ensure that it does not interfere with the participant's experiences. This also helped with the member checking strategy, which is a process in which the researcher follows up with the participants to ensure that findings are correctly reflected and the participants' lived experienced are reflected as they are (Kornbluh, 2015) without the influence of the researcher. According to Kornbluh (2015), one of the advantages of member checks is that the researcher gets the opportunity to deeply understand the data. After analysing the lesson plans, I had the opportunity to share and confirm the observations with the participants during the interviews. They also had the opportunity to express their views and give further insights about their experiences.

4.10 Validity and Reliability

Noble and Smith (2015) explain that validity is a measurement of whether the collected data is accurately represented by the findings, while reliability looks at how consistent the analytical procedures are, and whether personal and research method biases are accounted for as they may influence the findings. To ensure validity in this study, collected data was transcribed using accurate technologies to ensure that transcription is clear and precise. The chosen/selected examples in the lesson plans were captured as they were without being altered. To ensure reliability in this study, audio recording was used instead of videotaping during the interview as this would have influence the way the participants responded. Data collection tools which were revised based on the findings of the pilot study were used to also ensure reliability of the data, and the same analytical framework was used to analyse all the collected data to ensure the reliability of the results.

4.11 Ethical considerations

Prior data collection, permission to conduct this research was obtained from the gatekeepers (Appendix E) and ethics committee at the institution of study (Appendix G). This was followed by requesting consent from the participants (Appendix C). The following ethical principles were applied as adapted from Varkey (2021) in accordance to their relevance to this study.

- Informed consent

The participant, as Varkey (2021, p. 19) explains it, "(i) must be competent to understand and decide, (ii) receives a full disclosure, (iii) comprehends the disclosure, (iv) acts voluntarily, and (v) consents to the proposed action." The participants of this study are young adults (senior PSTs) who are competent and capable to comprehend the information they were provided about this research. They are also competent enough to be able to voluntarily decide to participate and not to participate.

- Truth telling

This is crucial for the researcher-participant relationship to ensure that the participant trusts the researcher. The participant must be given all the information about the research, and they must also know that they can opt to not participate at any point of the research (Varkey, 2021). All the participants received the letter of information (Appendix B) detailing the purpose of this study, its significance, and more. The letter also provided information on what is meant by being a participant of the study. That is, they were informed that giving consent to participate meant allowing the researcher to study their lesson plans and ask for follow-up interviews with them if necessary. The letters also clearly stated that upon providing consent, the participants could still withdraw from participating at any point without penalty.

- Confidentiality

According to Varkey (2021), the researcher must not disclose confidential information that is provided by the participant to any other party without authorisation by the participant. The participants' personal information was and still is kept confidential at all times. Their identity (name and surname), the name of the university in which they are training will be always kept confidential and anonymous in all academic writing about the research. To conceal their identity, pseudonyms are used, or participants are referred to as 'PST#', where the # is a number. The participants' individual privacy will be maintained in all published and written work resulting from the research. All research data is securely stored with the Durban University of Technology in a password protected folder within the university cloud and in locked cabins within the university (In the case of hard copies). These are only accessible to the researcher and the supervisor. The data will be destroyed (deleted and shredded) between 3-5 years after completion of the study.

- Beneficence

This is the researcher's obligation to act for the benefit of the participant by ensuring that there are no conditions that will cause harm, and protect and defend their rights (Varkey, 2021). It was explained in the information letter that participating in this study would not advantage or disadvantage the participants anyhow. The participants' rights were defended by respecting their opinions and decisions to provide more or less information. Conditions that may have caused harm were eliminated by ensuring that the environment in which the participants were interviewed was safe. This was achieved by using one of the university office spaces which was also private enough to protect the participants' identity. The data collected from their lesson plans and interviews will not be shared with anyone else, or used for anything else other than this research.

- Nonmaleficence

Varkey (2021) states that it is the researcher's obligation not to harm the participant. That is, not offending and not depriving the participants of any benefits. This research will provide opportunities for PSTs and teacher training institutions to reflect on the knowledge necessary to promote learning in multilingual analytical geometry classrooms, and skills that must be facilitated in mathematics teacher training. All PSTs will benefit from the outputs of this study because the results will be shared with participants in a results discussion/presentation session that will be held when the study has concluded. An academic paper will also be published with an accredited journal based on this Doctoral study to be accessed by all PSTs, teacher training institutions, and all other interested persons and bodies.

- Justice

This entails that the researcher much be equitable, fair, and treat people appropriately (Varkey, 2021). All senior PSTs who are in their final year of taking mathematics as a specialisation were eligible to participate in this study, and they were all invited to participate. The inclusion and exclusion criteria of this study did not discriminate individuals according to their personal traits or academic abilities. This is why all the lesson plans of the participants who had given consent were analysed until saturation was reached.

4.12 Data analysis

The collected data was analysed using the merger PCK-PV framework. The sequence of data analysis is shown in Figure 4.3. Data analysis was in three phases, where phase one explored the PSTs' content knowledge by analysing their test responses. The second phase begins to explore the PSTs' PCK by analysing the contents of their lesson plans and focusing on their lesson objectives and chosen examples, and their alignment with the recommendations in the CAPS document. The third phase is the analysis of the interview responses with the PSTs. This is a further probe into their PCK in order to make informed conclusions. The analysis of the interviews employed thematization in addition to the analytical framework. These phases are each discussed in separate data presentation and analysis chapters.

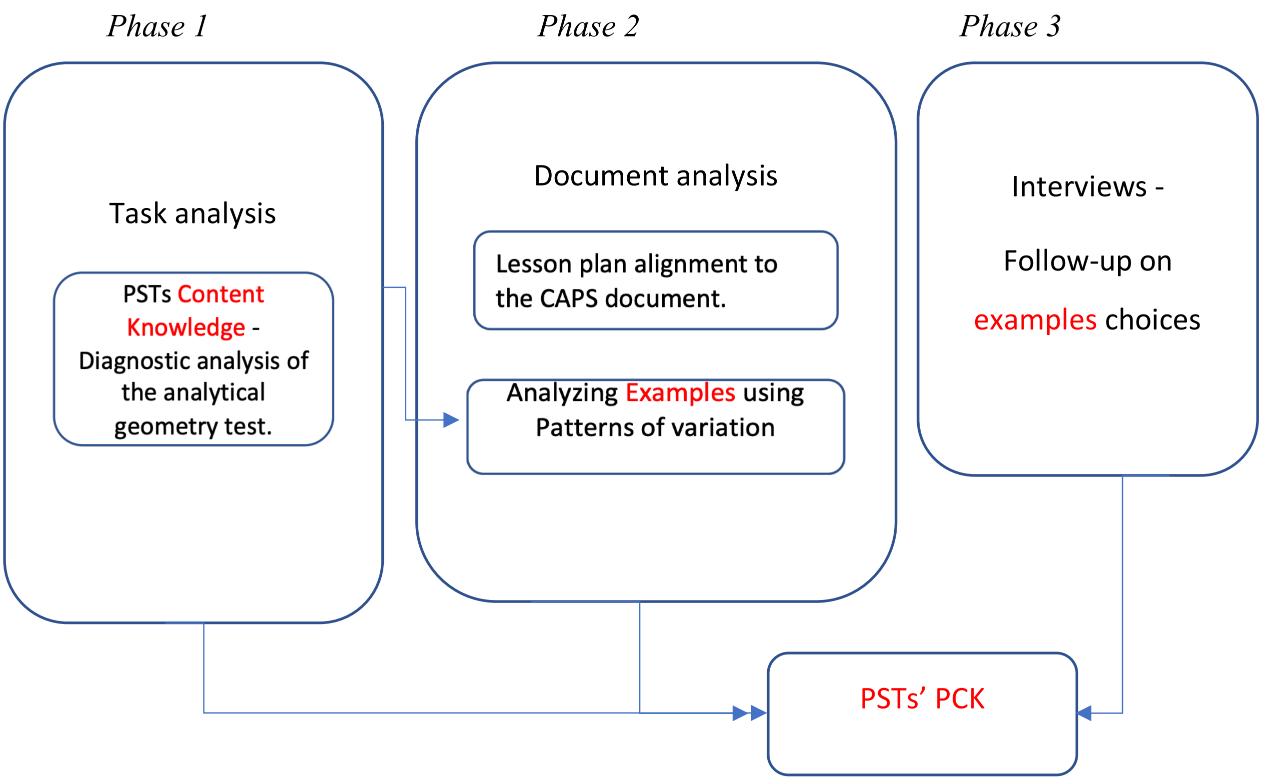


Figure 4.3 The sequence of data analysis

The connecting arrow between Phase 1 which is task analysis and the analysis of examples in Phase 2 indicates that the levels of cognitive demands of the PSTs' chosen examples are also analysed. This is because examples are considered as tasks that learners work on (Olteanu, 2017). Ultimately, all the phases are connected and indicate the PSTs' PCK. The highlighted terms are the key concepts of this study. For the detailed analytical framework refer to Tables 4.2.1, 4.2.2, and 4.2.3 in the next subsections. The interviews thematising process is also explained in subsection 4.10.2.

4.12.1 The analytical framework

The diagnostic analysis of the PSTs' responses to the different questions in the analytical geometry test was done using the descriptors in Table 4.2.1 and Table 4.2.2. The descriptors in Table 4.2.1 were used to describe the types of questions in the test. However, exploring the types of questions that PSTs would have been responding to and their performance is not enough as it does not give a clear idea of what being able to answer the questions entails in terms of content knowledge. Therefore, the mathematics questions' cognitive demand levels (Smith and Stein, 1998) in Table 4.2.2 were used to analyze levels of cognitive demand in the test questions. Table 4.2.3 provides an outline of the descriptors that were used to analyses the PSTs' PCK.

Table 4.2.1 Types of questions in the assessments

Type of questions	Content knowledge	Descriptor
1. Calculations	1.1 Substitutions and algebraic manipulations 1.2 Application of geometric properties. 1.3 Integration of geometric concepts with other mathematical concepts.	1.1 Questions that require the basic use of formulae and equations. For example, finding the distance between two points or manipulating the distance formular to find the coordinates of a point. 1.2 Knowing and applying geometric properties in calculations. For examples, the product of the gradients of perpendicular lines is negative one. 1.3 Being able to integrate different mathematical concepts to solve problems. For example, finding the angle of inclination of a line, which is also used to find the gradient of the line.
2. Explanation	2.1 understanding of complex concepts and language involved.	2.1 Questions that require an explanation or understanding a scenario for application in complex problems.

Table 4.2.2 Cognitive demand levels. Adapted from Smith and Stein (1998)

Cognitive demand levels	Descriptors
1. Lower-level demand – Memorization	Questions that require the reproduction of previously learned concepts or memorizing definitions and formulae.
2. Lower-level demand - Procedures without connection to concepts.	Tasks that require the use of algorithms or procedures which would have been previously learned in a lesson.
3. Higher-level demand - procedures with connections	Questions that focus on the learners' ability to use procedures for developing deeper understanding of concepts. They can be represented in different ways such as visual diagrams, problem situations, and symbols.
4. Higher-level demand - Doing mathematics.	Questions that require nonalgorithmic and complex thinking where learners are required to explore mathematical concepts and their relationships. Responding to these questions demands self-regulation of cognitive processes.

Table 4.2.3 Descriptors for lesson plan content and examples

The intended object of learning	
Teacher's space	***Descriptors***
CAPS document + Lesson Plans	• The CAPS document outlines the main objectives of teaching analytical geometry in Grade 11 as: "Use a Cartesian co-ordinate system to derive and apply: • The equation of a line through two given points. • The equation of a line through one point and parallel or perpendicular to a given line; and • The inclination of a line." • The CAPS document sets the time needed to complete the topic. • The CAPS document also suggests the cognitive demands for the examples chosen. For instance, an example to enhance knowledge (K), routine procedure (R), complex procedure (C) or problem solving (P) • Each lesson plan must have lesson objectives that align with the ones listed on the CAPS document. • The examples chosen by the preservice teacher will also reflect each of the cognitive demands.

Lesson plan content and interview responses.	To further understand the PSTs PCK, the MTSK model is used to analyse their mathematical knowledge and knowledge for teaching. Some of the PSTs responses were also analysed to understand the type of knowledge they revealed the PST to have.		
	MTSK Model		
	Mathematical knowledge	KoT	Knowledge of the analytical geometry topic. “Knowing the what and in what way” (Carrillo-Yanez et al., 2018). There is evidence of explaining the ‘what’, ‘why’, and ‘how’ aspects of concepts.
		KSM	Showing the knowledge of the inter and intra connections in the concepts. Able to show and clearly indicated the relationships between concepts within the topic/subject and to other topics/subjects.
		KPM	Evidence of being able to demonstrate clear understanding of how analytical geometry concepts are used in practical applications. That is, how they are used to perform mathematical operations and solve real world problems.
	Pedagogical Content Knowledge	KMT	The PST is aware of the issues around the topic. Knows the theories, teaching strategies and resources, and misconceptions that learners may have in this topic. There is evidence of being able to address the misconceptions and use relevant examples for the Grade and age group.

<table>
<tr><td></td><td></td><td>KFLM</td><td>There is evidence of adopting theories of learning and uses different strategies/ aids/ activities to create opportunities for learning. The examples are also set and structured in a way that provides enough opportunities for learners to grasp the critical features of the concept.</td></tr>
<tr><td></td><td></td><td>KMLS</td><td>Learning outcomes of the lesson are in line with the CAPS document outcomes. The level at which content is pitched matches the national standard as prescribed in the CAPS document.</td></tr>
<tr><td colspan="4"></td></tr>
<tr><td>Patterns of variation in the chosen examples.</td><td colspan="3">The examples that PSTs choose for teaching analytical geometry in multilingual classrooms were analysed using the patterns of variation to explore the opportunities of learning that they possibly created.

• Contrast: An example in which some features of the object of learning vary while some are kept the same. For example, calculating the gradient of a line and using lines that have a negative and a positive slope.
• Generalization: Examples in which similar instances of the object of learning can be compared. E.g., using lines of different lengths, thickness, and steepness, but going the same direction to calculate the gradient.
• Separation: Examples in which the feature of interest in the object of learning varies while all other features are kept the same. E.g., calculating the equations of different tangent lines to the circle.
• Fusion: Examples that simultaneously expose learners to the variation of a number of features of the object of learning.</td></tr>
</table>

4.12.2 Thematic analysis

The analytical framework above is suitable for analysing most of the aspects of the collected data, but the interview responses are not completely accounted for. Therefore, the study employs thematic analysis to ensure that all the responses are analysed to capture all the information that was provided by the participants. This is in line with Lochmiller's (2021) view that when using thematic analysis there is a presumption that the recorded data provides an accurate reflection of the real experiences of the participants in that context. Thematic analysis allows the researcher to achieve an understanding of the participants' experiences from the patterns of meaning identified in the data (Sundler, Lindberg, Nilsson, and Palmér, 2019). This characteristic of thematic analysis is in line with the research paradigm employed in this study, that is the interpretive paradigm, which is about seeking to explore, interpret, and understand the participants' experiences (Rahi, 2017).

According to Braun and Clarke (2006: 82) "a theme captures something important about the data in relation to the research question and represents some level of patterned responses or meaning within the data set." Castleberry and Nolen (2018) and Lochmiller (2021) expound that thematic analysis is centrally driven by the research questions and depends on the key features that the research is enquiring about to create a clear interpretation of the data.

In this study, the process of identifying themes started with the coding of data whereby the raw data was converted into usable data by identifying concepts that were connected in some way (Castleberry and Nolen, 2018). Codes are about the jotting down ideas or key words, and they are the basis of themes (Braun and Clarke, 2006; Saldaña, 2021). Similarities and differences between terms and phrases in the PSTs' interview responses were jotted down and put together as codes, and the recurring patterns of meaning were identified as subthemes and further organised as themes (Lochmiller, 2021). These themes comprise key concepts and terms from the main research questions because the interview questions were set based on them. Table 4.2.4 provides a glimpse of the concepts, terms and phrases that were identified as codes, and led to the subthemes and themes of this study.

Table 4.2.4 Summary of the thematisation process

Theme	**Subthemes**	**Codes**
The source and choice of examples	Teaching resources for choosing examples.	• Textbooks • Past examination question papers
	Uses of teaching resources.	• Order of examples • Timing/when to use the examples.
	Knowledge base for choice of examples.	• Learners' grade and content • Learners' understanding
The purpose of selected examples	Content knowledge development	• Topic understanding • Explaining terms/ Correct use of formulate
	Formative assessment	• Check if learners understand. • Class activities/task/exercise/Prepare for test
Use of language in the selected examples	Language and mathematics notations	• Showing formulae from textbooks • Explain terms/show symbols/codeswitching

The research setting and design of this study was explained in this chapter together with the sampling design, data collection methods, and the analytical framework that guided the data analysis process. The next three chapters present the three phases of data analysis as shown in Figure. 4.3. Each phase makes use of the relevant descriptors and shown in Tables 4.2.1, 4.2.2 and 4.2.3, and the identified themes in Table 4.2.4 are presented and further discussed in Chapter Seven. The next chapter, which is Chapter Five presents and analyses the content knowledge test results to provide an overview of these PSTs' content knowledge

CHAPTER FIVE: PRESENTATION AND ANALYSIS OF DATA - CONTENT KNOWLEDGE

5.1 Introduction

This chapter explores and presents the data collected on PSTs' content knowledge, which is a crucial component of PCK. The data was collected through an analytical geometry test that the PSTs wrote. The test covered all aspects of analytical geometry as per the prescription by the South African CAPS documents for mathematics in the senior phase (Grades 7 – 9) and further education and training phase (Grades 10 – 12).

The collected data from the test allowed the study to probe into the PSTs' knowledge of mathematical concepts that are interdependent in solving analytical geometry problems. This was done by analyzing the different concepts that are involved in each question of the test, the types of questions on the test and their cognitive demands. For this section, phase 1 (Table 4.2.1 and 4.2.2) of the analytical framework were implemented. The descriptors are presented again in Tables 5.1 and 5.2.

Table 5.1 Types of questions (Table 4.2.1 of the analytical framework)

Type of questions	Content knowledge	Descriptor
3. Calculations	3.1 Substitutions and algebraic manipulations	*1.4 Questions that require the basic use of formulae and equations. For example, finding the distance between two points or manipulating the distance formula to find the coordinates of a point.*
	1.5 Application of geometric properties.	*1.2 Knowing and applying geometric properties in calculations. For examples, the product of the gradients of perpendicular lines is negative one.*
	1.6 Integration of geometric concepts with other mathematical concepts.	*1.3 Being able to integrate different mathematical concepts to solve problems. For example, finding the angle of inclination of a line, which is also used to find the gradient of the line.*
2 Explanation	2.1 understanding of complex concepts and language involved.	*2.1 Questions that require an explanation or understanding a scenario for application in complex problems.*

Looking at the types of questions in the test evoked the idea of looking at the cognitive demands of the questions. To do this, the study employed the mathematics questions' cognitive demand levels by Smith and Stein (1998).

Table 5.2 Cognitive demand levels. Adapted from Smith and Stein (1998) (Table 4.2.2 of the analytical framework)

Cognitive demand levels	***Descriptors***
1. **Lower-level demand** (Memorisation)	*Questions that require the reproduction of previously learned concepts or memorizing definitions and formulae.*
2. **Lower-level demand** (Procedures without connection to concepts)	*Tasks that require the use of algorithms or procedures which would have been previously learned in a lesson.*
3. **Higher-level demand** (Procedures with connections)	*Questions that focus on the learners' ability to use procedures for developing deeper understanding of concepts. They can be represented in different ways such as visual diagrams, problem situations, and symbols.*
4. **Higher-level demand** (Doing mathematics)	*Questions that require nonalgorithmic and complex thinking where learners are required to explore mathematical concepts and their relationships. Responding to these questions demands self-regulation of cognitive processes.*

Furthermore, this chapter provides a diagnostic analysis of the PSTs responses to the test questions. Their performance on each question and sub-questions is presented together with their common misconceptions and errors. This is supplemented by an analysis of the PSTs' responses to the follow- up interviews which were intended to further probe into their key problems and thinking in the content area.

5.2 Analytical geometry test concepts and questions

Table 5.3 provides an overview of the different concepts that are required to solve each question on the test. It also provides an indication of the types of questions and their cognitive demands. The test had three major questions, each one with its sub questions. See appendix A for the test questions and marks allocation per question.

Table 5.3 Content knowledge for each test question and their cognitive demands

Question	Concepts	Type of question	Cognitive demand
1.1	Gradient of a straight line.	Calculation: substitution	Lower-level demand: Procedures without connections.
1.2	Equation of a straight line. $y_1(\mathrm{x} - x_1)$	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections
1.3	Angle of inclination, CAST diagram and angle of a straight line, and working with trigonometric ratios.	Calculation: integration of concepts.	Higher-level demand: Procedures with connections.
1.4	Concept understanding of perpendicular bisector, relationship between the gradients of perpendicular lines, and manipulating the gradient formula and equation of a straight line.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: procedures with connection
1.5	Properties of a kite, distance formula, and midpoint formula.	Calculation and explanation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.

1.6	Area of a triangle, properties of triangles, and distance formula	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
2.1.1	Equation of a circle – substituting the center and radius.	No calculation: substitution	Lower-level demand: memorization
2.1.2	Properties of a circle and understanding the cartesian plane.	No calculation: substitution	Lower-level demand: memorization
2.1.3	Distance formula and understanding the equation of a circle.	Calculation: substitution	Lower-level demand: Procedures without connections.
2.1.4	Finding intercepts using equation of a circle or manipulating distance formula.	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections.
2.1.5	Relationship between a tangent line and radius of a circle and finding gradient and equation of a line.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
2.2.1	Manipulating the equation of a circle.	Calculation: algebraic manipulation	Lower-level demand: Procedures without connections.
2.2.2	Calculating distance.	Calculation: substitution	Lower-level demand: Procedures without connections.

2.2.3	Using equations to draw circles on the cartesian plane and understanding the relationship between circles.	Explanation	Higher-level demand: Procedures with connections.
3.1	Equation of a circle, midpoint formula and distance formula.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: Procedures with connections.
3.2	Gradient of a line, relationship between perpendicular lines, and relationship between tangent to a circle and radius.	Calculation: application of geometric properties.	Higher-level demand: Procedures with connections.
3.3	The y-intercepts of a circle.	Calculation: algebraic manipulation and application of geometric properties	Higher-level demand: Procedures with connections.
3.4	Using angle of inclination to find angle between two lines. Angles of a straight line and a triangle. Using trig ratios.	Calculation: application of geometric properties and integration of concepts.	Higher-level demand: doing mathematics.

As already shown and defined in literature, analytical geometry integrates concepts from different topics in mathematics. This is also evident in the different concepts that are required to answer the test questions as outline in the Table 5.1. These concepts also reveal another important property of analytical geometry that shows just how mathematics is connected conceptually at different levels of education. Analytical geometry shows explicitly how one needs to know basic concepts that are appropriate for General Education and Training (GET), which consists of two phases, namely the foundation phase (Grades R to 3) and the intermediate phase (Grades 4 to 6). This makes a good example in mathematics education to encourage both learners and teachers not to treat mathematics topics and content levels (grades) as mutually exclusive.

5.3 Test scores

The PSTs' test scores and performance in each question are presented on table 5.5, where a breakdown of the scores per question for 20 of the 21 sampled PSTs is provided. This is to provide an idea of the analytical geometry content knowledge of the participating PSTs. From the list of all participating PSTs, 21 were sampled after analyzing their lesson plans. Of the 21, 9 of them scored above 50% and the others had less. Therefore, to present the test scores, four categories shown in Table 5.4 were created as criteria for selecting and picking from the 12 PSTs who scored less than 50%.

Table 5.4 Categories of the 20 PSTs' test scores

Number of PSTs	**Category**		
6	0%	$< mark \leq$	40%
5	40%	$< mark <$	50%
4	50%	$\leq mark <$	70%
5		$mark \geq$	70%

.

Given that the categories of PSTs who scored 50% and more had more PSTs in the higher category, the selection of the PSTs in the lower categories followed the same pattern. For the PSTs who scored less than 50%, which was a total of 12, 11 was picked with 6 in the lowest category. This makes a total of 20 PSTs whose scores are presented in Table 5.5.

Table 5.5 test scores

QUESTION	PST_1	PST_2	PST_3	PST_4	PST_5	PST_6	PST_7	PST_8	PST_9	PST_{10}	PST_{11}	PST_{12}	PST_{13}	PST_{14}	PST_{15}	PST_{16}	PST_{17}	PST_{18}	PST_{19}	PST_{20}
1.1 (2)	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1.2 (3)	2	3	3	3	3	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
1.3 (3)	2	1	3	1	3	1	0	2	1	1	3	1	1	3	0	1	0	3	3	1
1.4 (5)	0	2	5	1	5	5	5	5	5	5	5	5	5	5	5	5	0	5	5	5
1.5 (3)	0	0	1	0	2	2	1	1	1	2	2	1	1	3	1	1	0	1	1	0
1.6 (5)	0	0	2	1	5	5	0	1	2	5	0	1	1	1	2	0	0	1	1	1
2.1.1 (1)	1	0	1	1	1	1	0	1	1	1	1	1	0	0	1	0	0	0	1	1
2.1.2 (2)	2	0	2	2	0	0	0	2	1	2	0	0	0	2	2	2	0	0	0	0
2.1.3 (2)	2	0	2	2	2	2	0	2	2	2	2	0	0	2	2	2	0	0	0	0
2.1.4 (3)	0	0	3	3	3	3	0	3	3	3	3	3	1	0	3	0	0	0	0	0
2.1.5 (4)	0	0	4	2	4	4	0	4	0	4	4	1	0	0	4	0	0	0	0	0

2.2.1 (4)	2	4	0	4	4	2	1	2	4	0	4	4	0	2	4	0	0	2	4	0
2.2.2 (2)	0	2	0	2	2	0	0	2	2	0	2	2	0	0	2	0	0	2	2	0
2.2.3 (2)	0	0	0	2	1	0	0	0	0	0	0	0	0	0	2	0	0	0	1	0
3.1 (4)	4	1	4	4	4	3	2	4	4	1	4	0	1	3	4	1	0	4	0	0
3.2 (4)	2	4	4	4	4	4	4	4	1	4	2	3	3	0	4	4	0	2	4	0
3.3 (4)	4	0	4	0	0	0	0	3	0	0	4	0	0	0	4	0	0	0	0	0
3.4 (7)	0	0	7	0	0	0	0	7	0	0	3	0	0	0	7	0	0	0	0	0
TOTAL (60)	25	19	47	34	45	35	18	48	32	35	44	27	18	26	52	21	5	25	27	13

The table captures the marks scored by the twenty PSTs in each question on the test. The red zeros indicate that the question was not attempted at all, while the black zeros indicate that the PSTs' response to the question were completely incorrect. Marks were allocated for particular steps on the solutions, and it is important to note that the PSTs were not given information sheets with formulae when they wrote the test. During the analysis of their tests, it was noted that most of them were not affected by not having the formulae provided because they remembered the formulae. Especially, the formulae for Grade 10 concepts.

5.4 Diagnostic analysis test

The graph in Figure 5.1 captures the overall performance of the PSTs in each sub- question. Each bar indicates the percentage of the PSTs who scored marks on a particular sub-question.

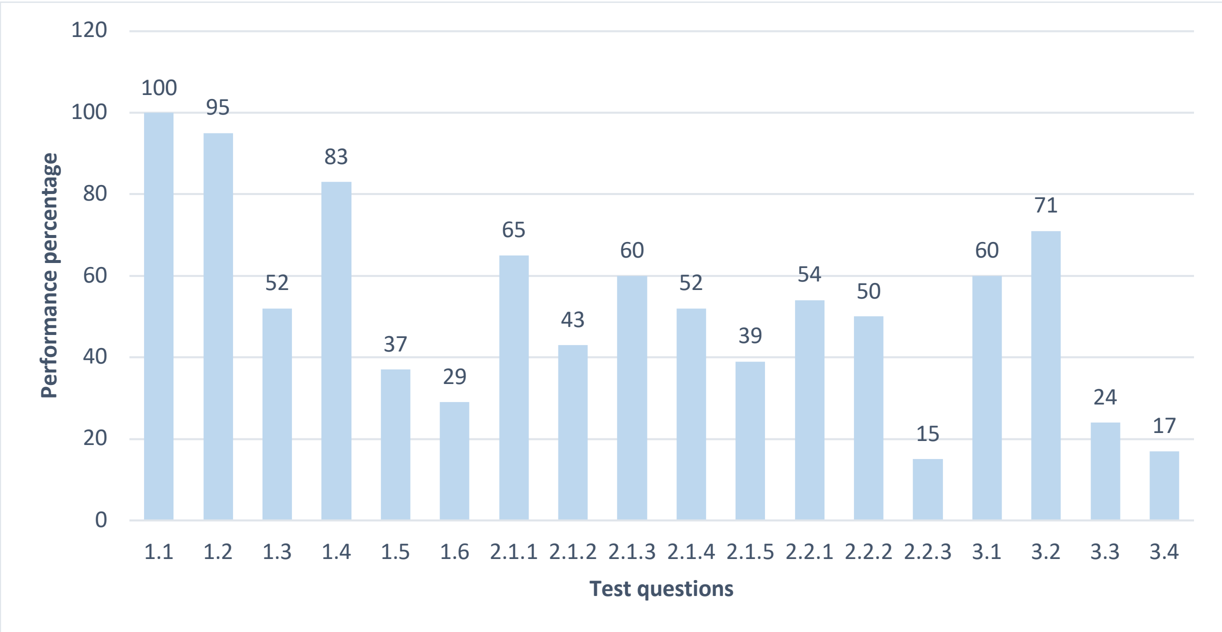


Figure 5.1 The PSTs' average performance percentage per sub-question

Please note that the performance percentage captures all the scored marks from zero to the total mark per sub- question.

5.4.1 Sub-question average performance

The graph shows that the most well answered questions were the calculation questions which simply require substitution and algebraic manipulations. These are questions of the lower-level demand where procedures are done without connections. This is evident in the performance of the PSTs in questions 1.1 and 1.2, which were simply about calculating the gradient and finding the equation of a line.

Question 2.1.2 is the most poorly responded to question of lower-level demand which barely needed any calculation, but simply observing the diagram and drawing from the knowledge of the coordinate system. See Figure 5.2 for reference.

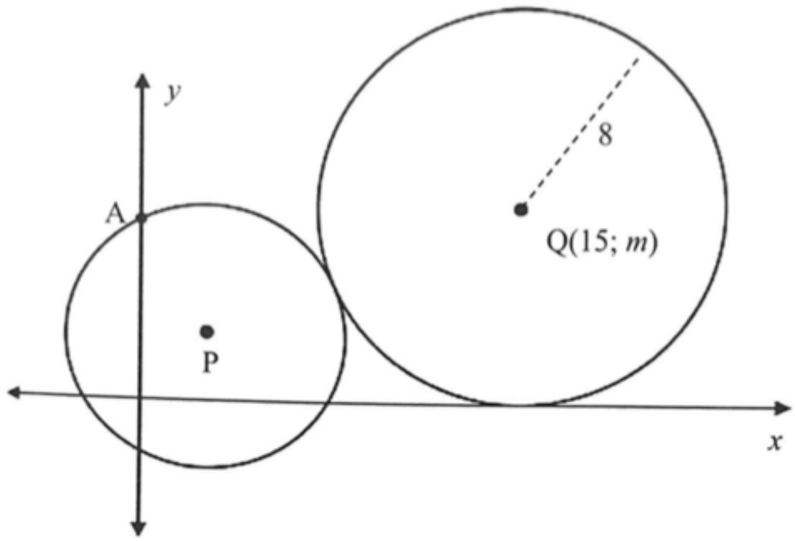
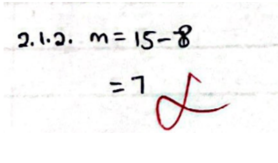
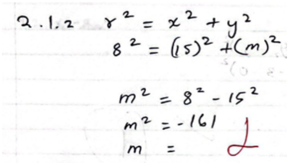


Figure 5.2 Diagram used in question 2.

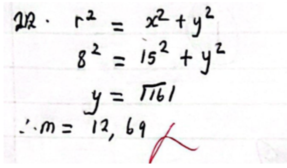
The question was "*Determine the value of m"* where the main statement explains that the larger circle touches both the x-axis and the smaller circle. A number of the PSTs left blank spaces or responded incorrectly to this question. Snippet 5.1 shows some of the PSTs responses to the question.



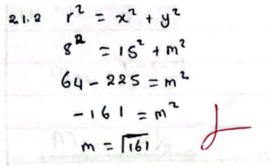
PST12



PST13



PST18

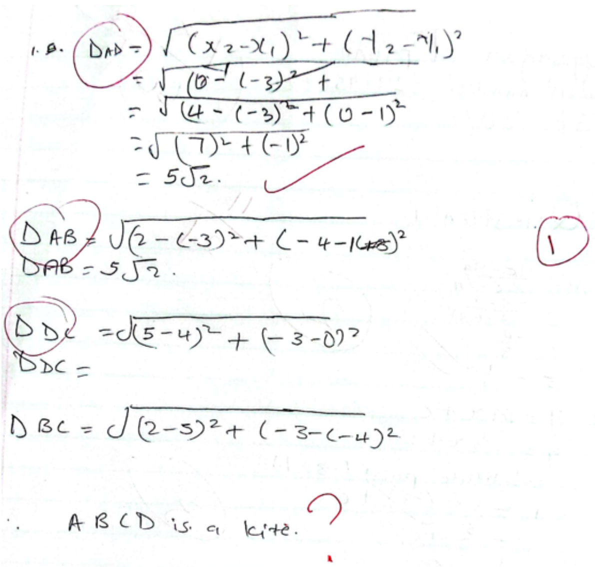


PST19

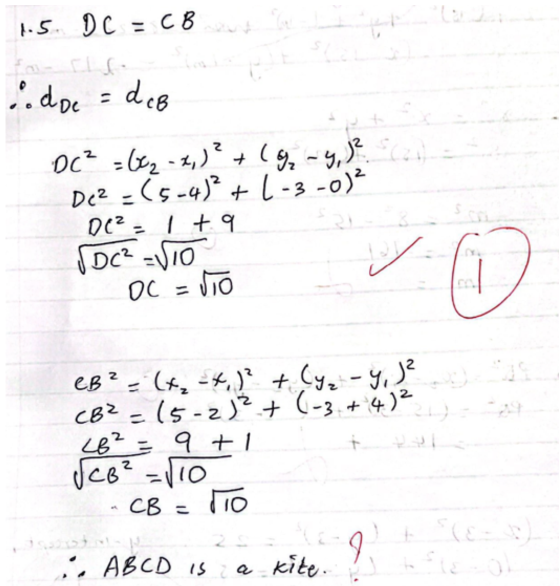
Snippet 5.1 Sample responses to sub-question 2.1.2

PST12 subtracted the radius from the x value of the coordinates of the center while PSTs 13, 18, and 19 treated the circle as one with the center at the origin. These PSTs ignored or did not understand the provided information that would have assisted in responding to this question.

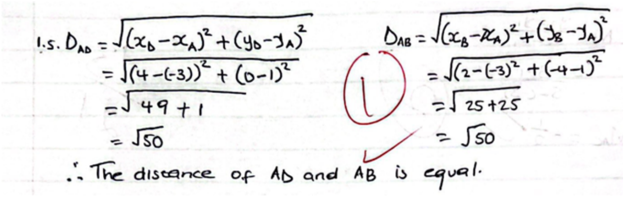
The PSTs performed poorly with average percentages of 37% and 15% in questions 1.5 and 2.2.3, respectively. These questions required some form of explanation that is backed up by simple calculations. The performance was low because the PSTs seemed to avoid any explanation, but simply provided calculations, which was not so helpful especially for question 1.5 where they had to make conclusions based on information that is provided in the preceding question. See responses to sub-question 1.5 in Snippet 5.2.



PST7



PST13

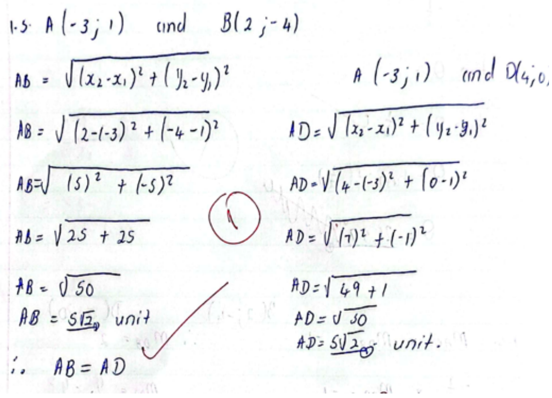


1.5. $D_{AD} = \sqrt{(x_D - x_A)^2 + (y_D - y_A)^2}$
$= \sqrt{(4-(-3))^2 + (0-1)^2}$
$= \sqrt{49+1}$
$= \sqrt{50}$

$D_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$
$= \sqrt{(2-(-3))^2 + (-4-1)^2}$
$= \sqrt{25+25}$
$= \sqrt{50}$

∴ The distance of AD and AB is equal.

PST12



1.5 A(-3; 1) and B(2; -4)

$AB = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
$AB = \sqrt{(2-(-3))^2 + (-4-1)^2}$
$AB = \sqrt{(5)^2 + (-5)^2}$
$AB = \sqrt{25+25}$
$AB = \sqrt{50}$
$AB = 5\sqrt{2}$ unit

A(-3; 1) and D(4; 0)

$AD = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
$AD = \sqrt{(4-(-3))^2 + (0-1)^2}$
$AD = \sqrt{(7)^2 + (-1)^2}$
$AD = \sqrt{49+1}$
$AD = \sqrt{50}$
$AD = 5\sqrt{2}$ unit.

∴ AB = AD

PST9

Snippet 5.2 Sample responses to sub-question 1.5

Most of the PSTs responded in a similar manner, just responding as if proving that the adjacent sides are equal was enough to deduce that the shape is a kite. The PSTs did not use the information that was provided in the subsequent question which was that AC is a perpendicular bisector of DB. They also did not provide much explanation for why they say ABCD is a kite. This is the reason for most of the PSTs (19 out of 20) not getting a total mark of 3 for this question.

The PSTs also performed poorly in question 3.3 where the average performance percentage was 24%. On this question, the PSTs were supposed to determine the coordinates of V, the y-intercept of the circle.

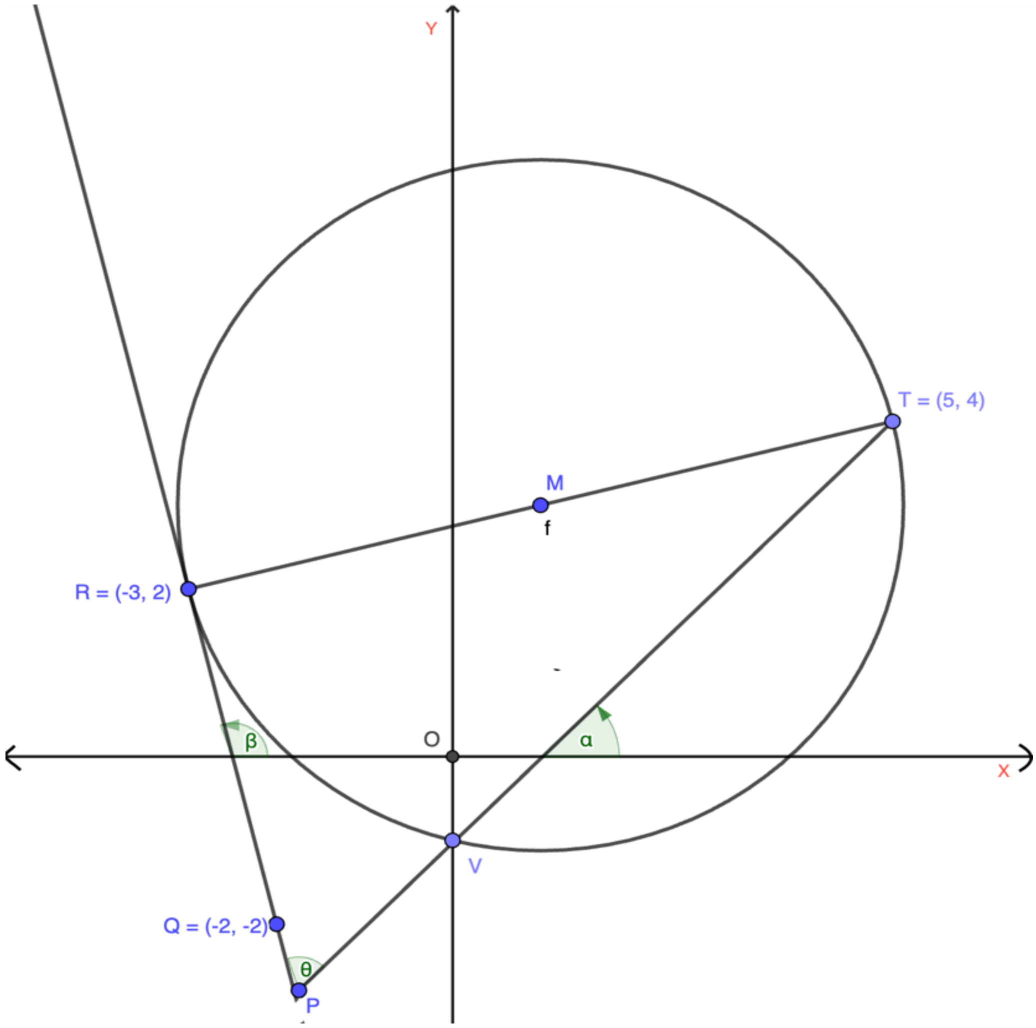
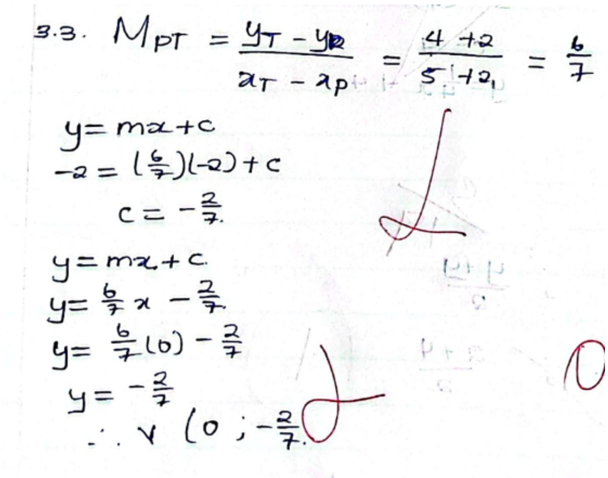
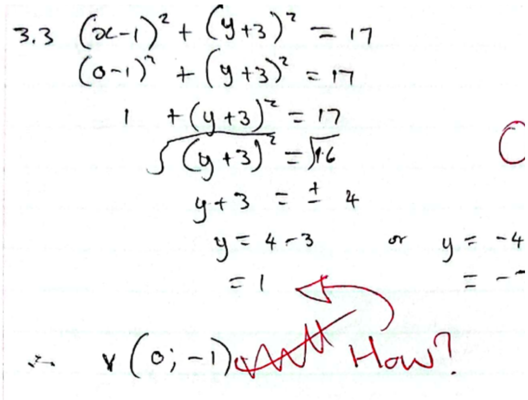


Figure 5.3 Diagram used in question 3

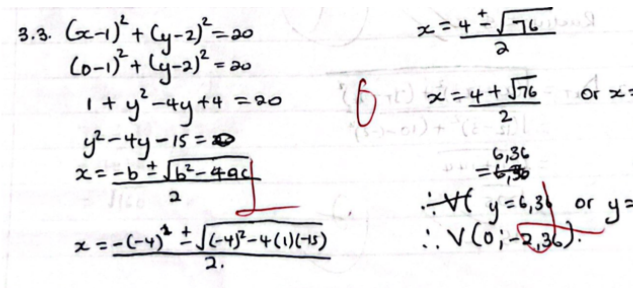
Even though a fair number of the PSTs were able to find the equation of the circle which was required in sub-question 3.1 for which the average performance was 60%, some of the PSTs ran out of time and could not answer sub-question 3.3 and others had incorrect answers to the sub-question. However, PSTs 6 and 13 shown in Snippet 5.3 got the equation of the circle incorrect.

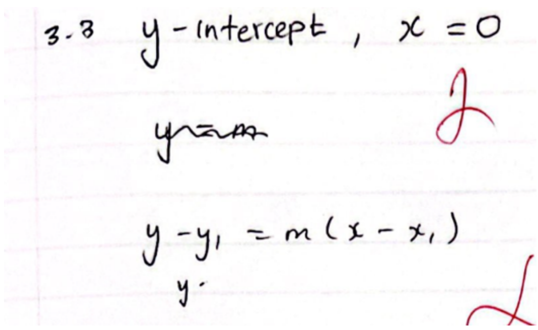




PST6

PST2





PST12

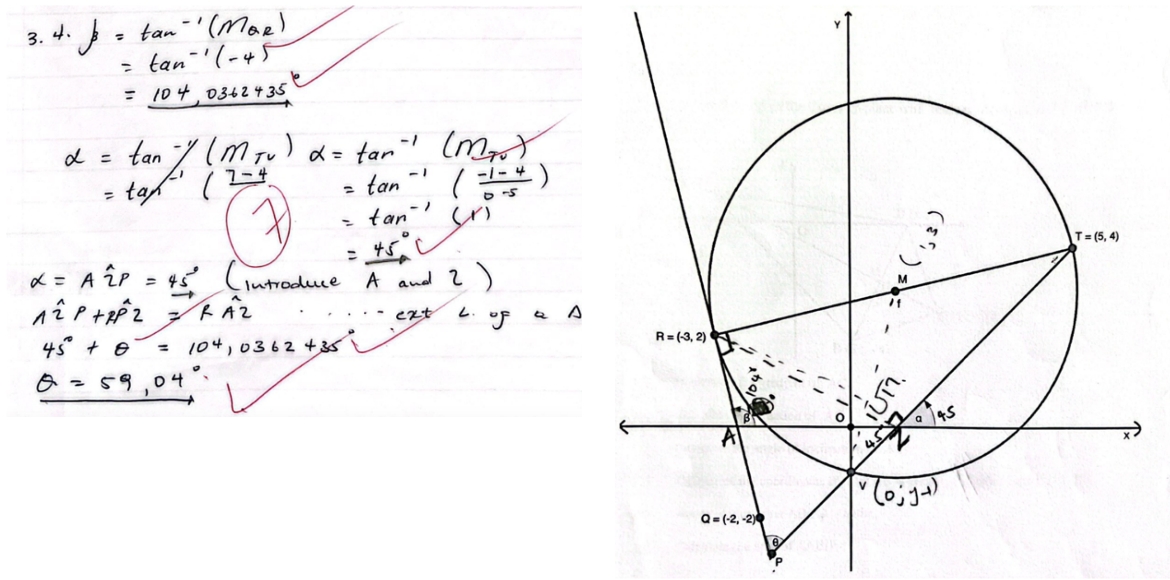
PST13

Snippet 5.3 Sample responses to sub-question 3.3

PSTs 2 and 12 were amongst a number of PSTs who tried to avoid using the equation of the circle by trying to find the equation of line PT whose y-intercept is also V. PST_2 went as far as using the coordinates of Q in the place of P.

The observation that was made here was that the PSTs had difficulties in solving problems of higher-level demand where they must use procedures with connections. This was shown by the performance of the PSTs in question 3.4, which 55% of the 20 PSTs did not attempt and most of the rest scored zero marks. Only 3 out of the 20 PSTs scored the total of 7 marks by using the angles of inclination of the two lines to find the angle formed where they intersect. The

PSTs' ability to connect mathematics concepts and use their relationships to solve this problem can be in the method used by PST_8 in Snippet 5.4.



Snippet 5.4 Response of PST_8 to sub-question 3.4.

The PST was approached for a follow-up interview to probe into their reason for introducing the line RZ to the diagram. See Excerpt 5.1 for their response.

Interviewer: what purpose did line RZ serve in this solution method?

PST_8: It was before I found the right way to do it because I thought the line was going to create a 90^0 angle, but it didn't work.

Interviewer: why do you think it would not work?

PST_8: Because Z is not on the circumference.

Excerpt 5.1: Interview engagement with PST_8

The PST's ability to connect mathematical concepts was also shown in their ability to realize that they could not just assume that any angle that is subtended by a diameter is *90^0*. This shows their ability to regulate their cognitive processes.

5.4.2 The PSTs' common misconceptions and problematic concepts

The test responses revealed some misconceptions and difficulties that the PSTs have. The most observed error and misconception was identifying the height of a triangle in sub-question 1.6. This misconception accounted for the 29% average performance in the sub-questions with most of the PSTs scoring up to 2 marks out of a total of 5. These marks were mainly for the area formula and finding the length of one of the sides of the triangle. Figure 5.4 shows the diagram that was used for this question.

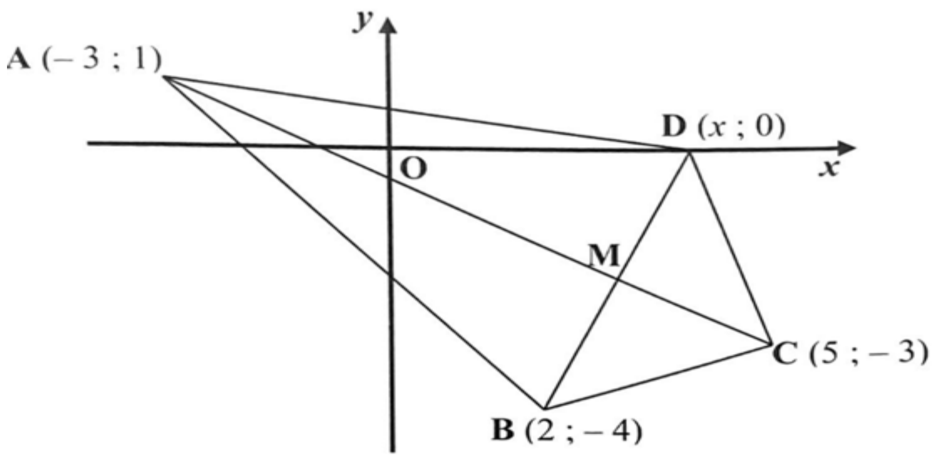
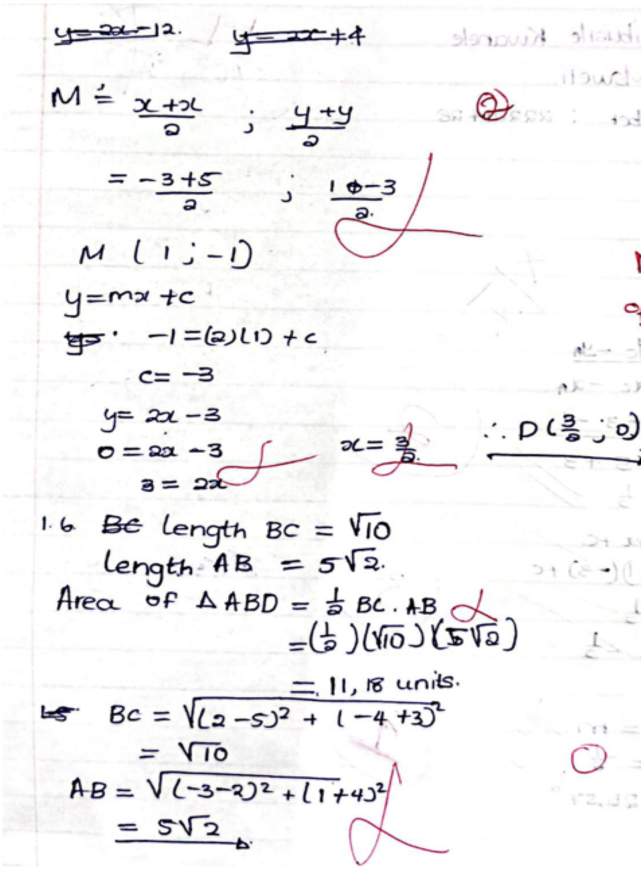


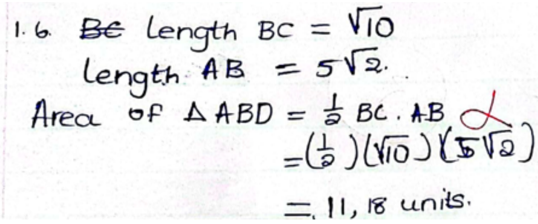
Figure 5.4 Diagram used in question 1

In sub-question 1.6 the PSTs had to calculate the area of triangle ABD. The coordinates of D were calculated in sub-question 1.4 which also stated that AC is a perpendicular bisector of BD. This given information also exposed a misconception that some of the PSTs such as PST_1 had, that M is the midpoint of both AC and BD. This misconception led to PST_1 failing to determine the coordinates of D as shown in Snippet 5.5

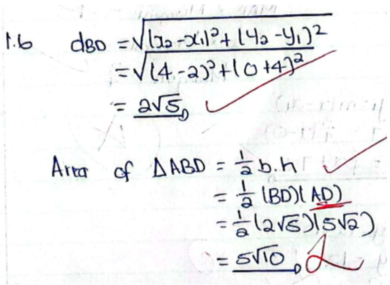


Snippet 5.5 Response of PST$_1$ to sub-question 1.6

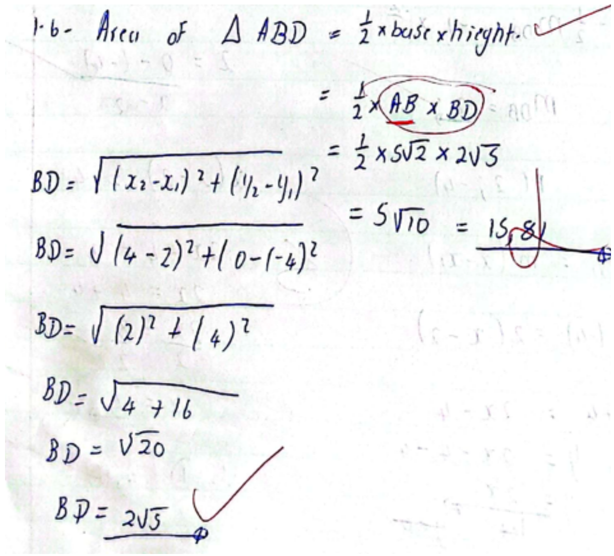
This also exposed the lack of knowledge of the properties of a kite, which will be discussed in the next section. Just like most of the PSTs failed to identify the height of the triangle when calculating its area, this was also the case for PST$_1$, but this PST also failed to identify the base of the triangle. They used BC as a base, and AB as the height. The fact that BC is not one of the sides of triangle ABD was not considered. This misconception is also observed in Snippet 5.6 which presents the response by PST$_2$.



PST$_2$



PST$_3$



PST$_9$

Snippet 5.6 Sample responses to sub-question 1.6

PST$_1$ and PST$_2$ were invited for follow-up interviews to investigate the reasons behind this misconception. An interesting response was provided by PST$_2$ as in Excerpt 5.2.

> *Interviewer: Why did you use sides AB and BC to calculate the area of triangle ABD?*
>
> *PST$_1$: It was a mistake. I see why it is wrong.*
>
> *PST$_2$: I thought BC is a base because it looks like the side the triangle is sitting on AB is the height because it is steep and higher.*

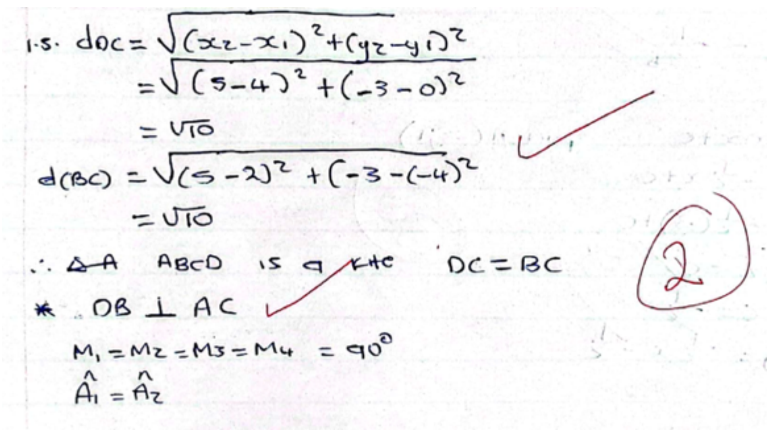
Excerpt 5.2 Interview engagements with PST$_1$ and PST$_2$

PST$_1$ responded in the same manner as most of the PSTs who were interviewed about the test. It seems that they become embarrassed and give very short responses in an attempt to hide their misconceptions or lack of knowledge all together. The responses are dismissive and often accompanied by a chuckle. PST$_2$ was carrying over a misconception from the way the concept of a base of a triangle was explained in high school. The PST worked with the positioning of the triangle more instead of working with the given geometric properties. This observation matches the findings made by Krajcevski and Sears (2019) in a study that explored the *"common Visual Representations as a Source for Misconceptions of Preservice Teachers in a Geometry Connection Course"*. The findings indicated that the PSTs have difficulties in conceptualizing the definition of an altitude and the other properties of triangles.

The challenge seems to be in understanding what height is because it is clearly taken as the distance from the base to top. This is also seen in the responses that were provided by PSTs 3 and 9 who considered side AB or side AD as the height of the triangle. Here, the PSTs clearly confused height to be the longest side of a triangle. They do not associate the height of a triangle with the altitude. This misconception is also noted in the NSC Diagnostic Report (DBE 2020), which was of the same year on which this group of PSTs would have been in Grade 12. The diagnostic report states that the candidates substituted the length of one of the sides of a triangle as the perpendicular height.

This misconception may still be lingering around in the third year of teacher training because for this group of PSTs, the first two years of their teacher training was mostly online learning due to the Corona virus pandemic regulations and lockdown. Unfortunately, the analytical geometry topic was covered in the first and second year of training in this institution. Due to the lack of content knowledge in PSTs, Alex (2019) argued that mathematics PSTs must be taught mathematics as though they are learners in order to curb the vicious cycle of having teachers who do not know what to teach, and learners who will not learn what they are meant to learn.

Another noted misconception was that the PSTs seemed to not know the properties of a kite. This can also be observed in the mark distribution on table 5.2 and the 37% average performance in question 1.5. The PSTs could not score the total mark because they could only show that adjacent sides are equal. They were not able to deduce that the figure is a kite using given information, which shows that they did not know the properties of a kite. PST_{11} whose response to sub-question 1.5 is shown in Snippet 5.7 went as far as making assumptions about angles whose information was not given, and for which they did not calculate any values.



PST$_{11}$

Snippet 5.7 Response of PST$_{11}$ to sub-question 1.5.

The made observation is in line with the findings made in the NSC Diagnostic Report (DBE, 2017) that learners do not understand the properties of quadrilaterals. This should not be the case for PSTs because they are taught this content again in teacher training. Just as Alex (2019) found it, the observations made in this study raise concerns about the PSTs own inability to develop and acquire content knowledge for teaching, and this is beyond the curriculum constraints in the institutions of training. Ascribed to this, and all the identified misconceptions and errors of the PSTs, this study confers with the findings made by Krajcevski and Sears (2019) which evinced that PSTs have difficulties in developing and acquiring their own workable knowledge of basic geometric concepts and their properties.

5.4.3 The use of language

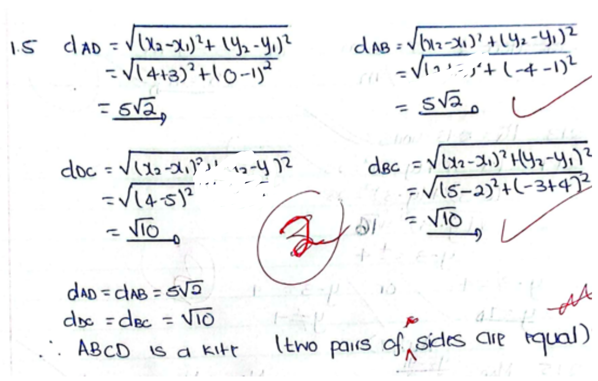
The PSTs lack of understanding the mathematics language and the LoLT can already be seen in their misconception about the height of a triangle. However, this was also noted in questions that required an explanation response like sub- questions 1.5 and 2.2.3. The average performance percentage for these questions is below 50% because the PSTs did not know or have the best way to express their thoughts in words. Most of the PSTs just did calculations and avoided any explanation in words.

Some of the expected responses to this sub-question were:

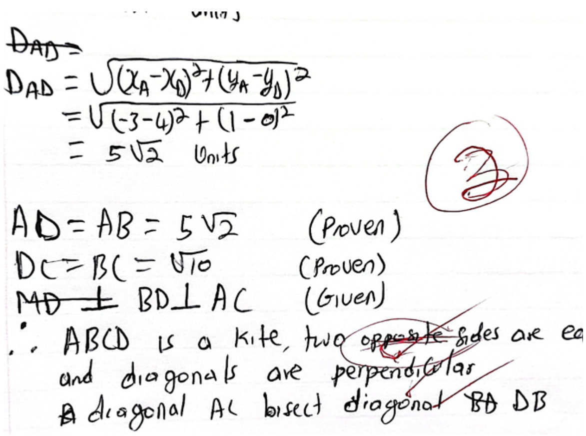
- The diagonals are perpendicular to each other.

- AC Bisects BD – a kite is symmetrical about the main diagonal.
- AB = AD and CB = CD – a kite has two pairs of adjacent equal sides.

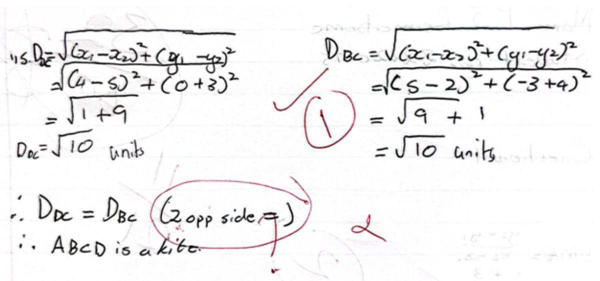
Only one PST out of the 20 got the three marks for this question. The rest of the PSTs could not explain the properties of a kite. Most of them definitely knew about the adjacent sides being equal but could not put it in correct words. They just wrote ABCD is a kite without giving reasons. Those who tried to explain, like in the instances that are presented in Snippet 5.8, either left the word *adjacent* out or used the word *opposite* when explaining about the two pairs of adjacent sides that are equal.



PST$_{16}$



PST$_{10}$



PST$_{18}$

Snippet 5.8 Sample of word responses which were provided for sub-question 1.5.

Some PSTs indicated that they forgot the word *adjacent*, while others said they did not know the word. PST$_{10}$ had the following to say in the interview. See Excerpt 5.3.

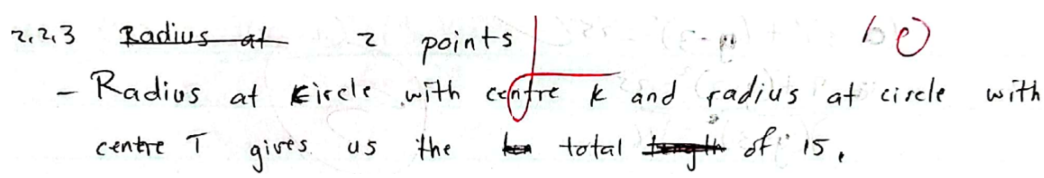
Interviewer: which sides are opposite on this diagram, and why?

PST$_{10}$: AD and AB because they are next to each other and kind of face each other like this… (does a v shape demonstration with hands)

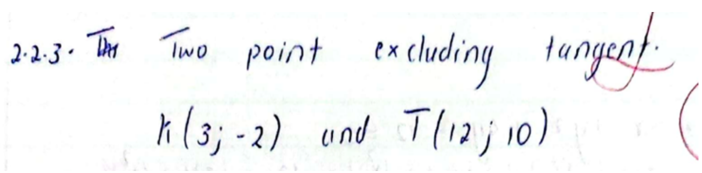
Excerpt 5.3 Interview engagement with PST$_{10}$

This response also showed that the PST did not know the term *adjacent* and misunderstood what is meant by opposite sides in mathematics. They used *"next to each other"* which means adjacent to explain the term *opposite.* The PST did not understand that in mathematics terms, two sides of a polygon are opposite if they do not share a common endpoint. This brings the issue of PSTs understanding of language for teaching and learning mathematics to light. In this regard, Sugimoto (2018) emphasized that there is so much to learn when it comes to factors that shape PSTs development and understanding of language in mathematics during their teacher training.

The issue of the use of the LoLT is also noted in sub- question 2.2.3 where the PSTs had to motivate for a very short calculation for determining the number of points the two circles intersect on. Only 4 out of the 20 PSTs got a mark for this question. The rest either did not attempt it or had incorrect answers, but this is not the only problem here. The other problem is in the way the PSTs explained mathematical concepts. PST$_6$ in the response that is shown in Snippet 5.9 repeated "radius at a circle" twice in one sentence which shows that there is no mistake, but that is the way they think of it. This makes it sound like the radius is not a part of a circle and may create misconceptions.


2.2.3 ~~Radius at~~ 2 points
- Radius at circle with centre K and radius at circle with centre T gives us the ~~ten~~ total ~~length~~ of 15.

PST$_6$


2.2.3. The Two point excluding tangent.
K(3; -2) and T(12; 10)

PST$_9$

Snippet 5.9 Sample responses to sub-question 2.2.3.

The response of PST$_9$ also exposed the lack of understanding of mathematics language. They seem to have attached the word *"intersect"* in the context of circles to a tangent because they mention it even though there was no mention of it in this question. The PST's response to a follow-up question, presented in Excerpt 5.4, also highlighted the issue of understanding language for teaching mathematics in PSTs.

Interviewer: Why did you mention the tangent here?

PST$_9$: Because a tangent can also intersect a circle.

Interviewer: But the question is just about the two circles…

PST$_9$: I was making sure.

Excerpt 5.4 Interview engagements with PST$_9$

This response revealed a need for training PSTs not only to develop content and ability to teach, but also to read and interpret questions and instructions. This is a skill they also need as teachers to impart it to their learners in multilingual classrooms.

Having explored the PSTs' content knowledge in this chapter, the next chapter looks at the PSTs PCK for teaching analytical geometry by exploring their ability to clearly set lesson objectives, and their choices of examples for teaching the topic. The next chapter presents phase 2 of the analysis process where the PSTs' selected examples will be analyzed using the patterns of variation, and levels of cognitive demands to understand the PSTs' ability to create opportunities for learning in multilingual mathematics classrooms.

CHAPTER SIX: PRESENTATION AND ANALYSIS OF DATA - CHOICE OF EXAMPLES

6.1 Introduction

In this chapter the examples that were chosen by the PSTs for teaching analytical geometry are presented and explored to identify variation in the examples and how they created opportunities for learning. To strengthen the analysis of how the examples created opportunities for learning, the goals and objectives of the lessons were studied to understand how the examples assisted or hindered the process of achieving them. The PSTs interviews responses to their choices of examples and ability to facilitate the learning process are also presented to substantiate the observations.

6.2 The intended object of learning and its alignment with the CAPS document

The intended object of learning was analyzed in the lesson objectives of the PSTs' lesson plans and supplemented by the responses they provided to some of the interview questions that were about their lesson planning. The descriptors in Table 6.1 were used to analyze the PSTs' lesson objectives and alignment of the lesson plans content to the CAPS document. In doing this, the Annual teaching plan was consulted as an aid in analyzing the content coverage and time spent on each concept of the topic. When the analysis moves to the choice of examples and their structures, the CAPS recommended cognitive demands in Table 6.1 will be used to describe the demand of the PSTs' selected examples.

Table 6.1 Descriptors for the alignment of content to the CAPS document

<table>
<tr><th colspan="2">The intended object of learning</th></tr>
<tr><td>Teacher’s space</td><td>Descriptors</td></tr>
<tr><td>
<table>
<tr><th>WEEK 5</th><th>WEEK 6</th></tr>
<tr><td colspan="2">ANALYTICAL GEOMETRY</td></tr>
<tr><td colspan="2">1. Revise,
• Distance between the two points
• Gradient of the line segment connecting the two points (and from that identify parallel and perpendicular lines)
• Coordinates of the mid-point of the line segment joining the two points
2. Derive and apply:
• The equation of a line through two given points
• The equation of a line through one point and parallel or perpendicular to a given line
• The inclination (θ) of a line, where $m = \tan\theta$ is the gradient of the line $(0^\circ \leq \theta \leq 180^\circ)$</td></tr>
</table>

2023/24 Annual teaching plan: Mathematics Grade 11</td>
<td>• The CAPS document outlines the main objectives of teaching analytical geometry in Grade 11 as:
“Use a Cartesian co-ordinate system to derive and apply:
• The equation of a line through two given points.
• The equation of a line through one point and parallel or perpendicular to a given line; and
• The inclination of a line.”
• The CAPS document sets the time needed to complete the topic.
• The CAPS document also suggests the cognitive demands for the examples chosen. For instance, an example to enhance knowledge (K), routine procedure (R), complex procedure (C) or problem solving (P).</td></tr>
</table>

The 2023/24 Annual teaching plan for mathematics Grade 11 assigns two weeks to teaching analytical geometry in Grade 11 where some of this time must be used to revise Grade 10 work. Two weeks equates to ±10 school days, and many South African schools allow every grade to have mathematics periods almost every day. Therefore, one would expect six to ten lesson plans on this topic.

During WIL the PSTs were in schools for 4 weeks, and the two weeks of teaching analytical geometry was within this period. However, most of them could only provide three to four lesson plans on this topic with some of them only providing one or two. None of them had more than four lesson plans. Moreover, these lesson plans had the same examples because the PSTs were using the same textbooks or teaching material and resources. For this reason, lesson plans of 21 PSTs had to be analyzed, and the process was not as tedious as one would imagine because it was mostly scanning through and looking for those that had different examples to the others. A total of 16 lesson plans out of all provided by the 21 PSTs had different selected examples, and these were further analysed.

In the analysis of the examples in the lesson plans, three categories were established, and they are discussed in Table 6.2. These were based on the amount of time spent in each subtopic, the sequence in which the subtopics were followed, and the way language is used in the lesson objective. In each of the categories, one PST's lesson objectives are outlined in Table 6.2 to provide a glimpse on how the PSTs write their objectives.

The presented objectives are also from lesson plans of the PSTs who were more elaborate in their responses to the interview questions. This decision came as a result of most of the PSTs responding with short answers (mostly just a yes) when asked if they met their lesson objectives. Even though the contents of their lesson plans did not suggest the same. This will be captured in the next section where the examples in the lesson plans are analyzed together with whether they are at the level of the stipulated cognitive demands as per the CAPS document. The lesson objectives in Table 6.2 are written as they are on the lesson plans from which they were adapted.

Table 6.2 Categories of the PSTs' lesson objectives

Category	Lesson plan	Lesson objective	Observations
A	1	Calculate the distance of a line segment, gradient, and midpoint coordinates of a line.	This is the category of PSTs who only had 1 or 2 lesson plans. All of these lesson plans were just about revising Grade 10 work. These objectives are presented here because of the way language was used on lesson plan 2. The lesson objective did not necessarily specify what would be calculated, and how parallel and perpendicular lines are going to be checked.
	2	They should also be able to draw a graph. Calculate and check if lines are parallel or perpendicular.	
B	1	Distance between two points. Gradient of a line segment.	The lesson objectives of this particular PST were mainly driven by the selected examples. The objective of lesson plan 2 may also indicate that the PST was planning on teaching to the test. The issue of the use of mathematical language also arises in this category, especially in this particular PST's lesson objectives. To say they are calculating the gradient of a line
	2	Finding the equation of a line passing through two points. Finding the equation of a line if one point and information regarding the gradient (parallel or perpendicular) is given.	

	3	Midpoint of a line segment. Introduction to the inclination of a line. Apply formulae to typical examination type of questions. Highlight the importance of the properties of quadrilaterals.	segment might lead to learners thinking that the gradient of a straight line is not the same when it is calculated using points on the same line but beyond this segment. The objective of lesson 2 makes it sound like gradient can be parallel or perpendicular, but this is the case for the lines whose gradients are calculated.
C	1	Find the midpoint between two points.	This category spent a lot of time working on revision work. The most important concepts for Grade 11 are covered superficially. The angle of inclination was introduced at a very basic level. The use of mathematical language is also observed in the objective of lesson plan 2 where the PST referred to colinear lines. This may indicate that the word colinear is not understood well.
	2	Calculate the gradient and differentiate between perpendicular, parallel, and colinear lines.	
	3	Calculate the distance between two points.	
	4	Calculate the angle of inclination using the gradient. Calculate the gradient using the angle of inclination.	

The explored lesson objectives that the PSTs wrote for teaching analytical geometry highlighted four key issues: The PSTs' understanding of lesson objectives, Time management, the depth of the application of concepts, and their use of the LoLT and mathematical language which is discussed in the next chapters.

6.2.1 PSTs' understanding of lesson objectives.

This is the initial and most important part of lesson planning, knowing what you want to achieve in the lesson you are planning. Lim et al. (2018) attest that PSTs have difficulties in sequencing and creating a lesson plan from scratch. This means that they would be able to fit in examples if they were given the lesson objectives but they find it difficult to start from scratch and think about the intended lesson and its goals.

The main interview question that was asked and guides this section was:

Do you think you met your lesson objectives? Explain.

There were follow-up questions depending on the PSTs' explanations. The PSTs responses to the main question were very short and required a lot of probing and in doing so, it seemed that they had different ideas of what lesson objectives really are or are about. An example of this was captured in the responses of PST_3 and PST_{15}. Their responses required that the question be simplified or clarified, and this was done by asking the question as shown in Excerpt 6.1.

Interviewer*: Did you achieve what you had prepared to deliver and wanted learners to learn from the lesson? Explain.*

PST_3*: Yes, I did because I finished everything I wanted to do on time.*

PST_{15}*: Yes, because learners were getting correct answers to the classwork questions.*

Excerpt 6.1 Engagements with PSTs 3 and 15

Based on these responses, and even after the question had been simplified, one can say that these PSTs did not have a good idea of what lesson objectives are. These responses, like most of the other responses did not necessary mention the content or the examples they used, and the way learners responded to them. Learners responding correctly to a class activity can be due to many reasons such as them helping each other and referring to examples on the textbooks. Also, finishing on time does not measure how successful the lesson was because the teacher may have finished with content coverage while learners are left confused.

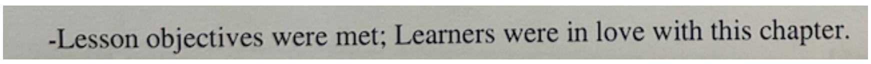
The PSTs showed that they had little knowledge or understanding of what lesson objectives are about. This was also validated by PST_6 during the interview. The interviewer asked a follow-up question about the lesson plan and lesson objectives. Excerpt 6.2 shows what the PST had to say.

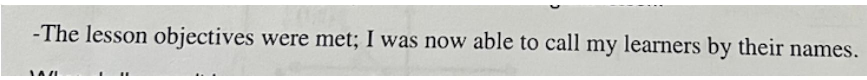
> *Interviewer: The midpoint formulae was supposed to be part of Grade 10 work revision. Why did you have it as a lesson objective for lesson 3?*
>
> *PST_6: I have it on lesson 3 because one of the questions in the activity is… needs them to use the midpoint.*

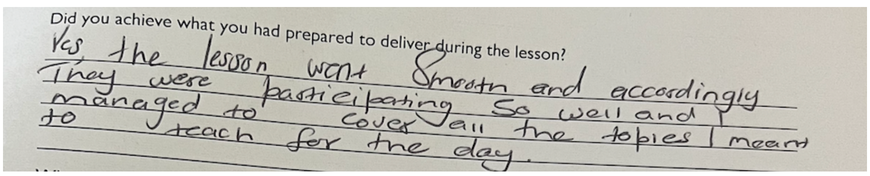
Excerpt 6.2 Engagement with PST_6

This response supports the observations made in category C's lesson plans that the PSTs were planning for teaching to test. Mostly relying on frequently asked questions in examinations. This was also observed in their selected examples, which will be presented later in this chapter.

Some PSTs preferred to give written responses to the interview questions rather than a verbal one. This was very limiting in terms of not giving the researcher the opportunity to ask some follow-up questions. One would think that this gave them more time to think their responses through, but they gave more of shorter or one-word responses here. Snippets 6.1 shows some of the short responses and those that showed how some PSTs seemed to not understand what lesson objectives are and what they are for.


-Lesson objectives were met; Learners were in love with this chapter.


-The lesson objectives were met; I was now able to call my learners by their names.


Did you achieve what you had prepared to deliver during the lesson?
Yes, the lesson went smooth and accordingly
They were participating so well and I
managed to cover all the topics I meant
to teach for the day.

-The lesson objectives were met; I was now able to call my learners by their names.

Did you achieve what you had prepared to deliver during the lesson?

Yes I achieve what I prepored but Ididn't cover Everything I wanted to do.

Did you achieve what you had prepared to deliver during the lesson?

Yes

Snippet 6.1 Interviews written responses.

Zaragoza, Seidel, and Santagata (2023) explain that planning a lesson is a process that involves pedagogical reasoning, in which PSTs need to relate theories and concepts to practical classroom situations. The responses of the above PSTs exposed a lack of pedagogical reasoning, and this was more exposed by their inability to consider the depth of the application of concepts. At face value, these PSTs show some level of self-efficacy, which seems to be based on 'mastery experience' as the PSTs believe they succeeded in the tasks they were executing. This is in line with a claim made by Bjerke and Solomon (2020) that PSTs develop and improve their self-efficacy in their work integrated learning experiences. However, in the case of the PSTs in question, their self-efficacy seems to be based on experiences that were not entirely of successful task execution.

6.2.2 Time management

According to Zaragoza et al. (2023) one of the important aspects of lesson planning is the *timing*, which includes allocating an appropriate amount of time, sequencing, and pacing of the set content. However, the PSTs did not seem to be cognizant of the time factor when planning lessons, nor were they considering the structure and amount of the content that was to be delivered. This was observed in all the analyzed lesson plans, and Excerpt 6.3 shows what some of the PSTs had to say when they were asked follow-up questions about time management and content coverage.

> Interviewer: *There was so much content to cover and two weeks to do so, but you only have three lesson plans, why?*

PST$_{11}$: *When you follow the ATP and textbooks for content to cover, it looks like it will not be a lot but when you teach the content takes longer to cover. Many of my Grade 11 learners needed more time when we were revising Grade 10 work. They can calculate the midpoint but it's a problem when you ask them to find the coordinated of a point on a line when you give them midpoint, and this is important from now to Grade 12.*

Excerpt 6.3 Engagements with PST$_{11}$

PST$_{11}$'s comment suggested that they could not envision the enactment of content during the lesson planning. This is in line with an argument made by Lim et al. (2018) that PSTs and entry level teachers have difficulty in using the lens of enacted content to see mathematical ideas within concepts and tasks.

Some of the PSTs admitted that they put many things on one lesson plan knowing that they would cover the content in more than one day because they had printing problems in schools. This is problematic because, clearly, the lesson plans are done for the sake of doing them and not to guide the lessons. It may also be the reason some of the PSTs showed lack of pedagogical reasoning which develops further when one engages with lesson planning. This is as Grossman et al. (2009) emphasized that lesson planning provides the opportunity to authentically engage in the teaching practice.

6.2.3 Depth of the application of concepts

According to Marton (2015), the object of learning is structured, which means that there is a relationship between the small concepts within one big concept (Olteanu 2017). The response provided by PST$_{11}$ above revealed that the PST did not have knowledge of the different problems that may be solved using a concept and connecting it to others. For instance, this is similar or talks to the observation made in the lesson objectives in Table 6.2 that some PSTs had the notion that teaching about the angle of inclination is merely showing that it is calculated using the gradient. This was the observed trend in most of the lesson plans on teaching the angle of inclination. Some of the PSTs did not go on to show that the angle of inclination can also be used to calculate the angle between two intersecting lines.

One lesson plan that set a good example of the above-mentioned scenario is shown in Snippet 6.2. Here, PST$_5$ revised concepts that were covered in Grade 10 but only superficially, which does not help the purpose of the revision, which is to apply these concepts in more complex Grade 11 problems.

PRESENTATION: In grade 10 the formulae of the **distance** between the two points, the **midpoint** of the line segment and the **gradient** of a line were established. We will now revise these formulae and their use. Please note that these formulae can also be used to determine the coordinates of the points. **Distance between the two points** The distance formulae can be used to determine the length of the line segment between two points or the coordinate of a point when the length is known. The formulae to calculate the length of a line segment between two points. $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formula: $AB^2 = (x_B ; x_A)^2 + (y_B ; y_A)^2$ $AB = \sqrt{(x_B ; x_A)^2 + (y_B ; y_A)^2}$ **Midpoint of a line segment.** The formulae for point M, the midpoint the line segment AB joining the points $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formula. $M(x_M ; y_M) = M\left(\frac{(x_B ; y_A)}{2} ; \frac{(y_A ; y_B)}{2}\right)$ **Gradient of a line** The gradient of the line between any two points on the line is the ratio: $m = \frac{change\ in\ x\ values}{change\ in\ y\ values}$ A formulae to calculate the gradient of a line joining two points $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formulae: The gradient of line AB: $m_{AB} = \frac{y_B - y_A}{x_B - x_A}$	• Listening carefully • Listening carefully	20 min.
CONCLUSON: We use these formula to calculate the distance between the two points $AB = \sqrt{(x_B ; x_A)^2 + (y_B ; y_A)^2}$ We use these formula to calculate the midpoint of a line segment $M\left(\frac{(x_B ; y_A)}{2} ; \frac{(y_A ; y_B)}{2}\right)$ We use these formula to calculate the gradient of line $m_{AB} = \frac{y_B - y_A}{x_B - x_A}$		5 min.

APPLICATION (ASSESSMENT QUESTIONS AND MEMORANDUM):

ENRICHMENT ACTIVITY:

1. **Determine the length of the line segment joining each pair of points**
   a) **A(1;-4) and B(-2;-7)**
   b) **A(3;0) and B(-6;3)**
   c) **A(5;-3) and B(-1;-3)**
2. **Calculate the coordinate of the midpoint of the line joining the two points**
   a) **A(1;-4) and B(-2;-7)**
   d) **A(3;0) and B(-6;3)**
   b) **A(5;-3) and B(-1;-3)**
3. **Calculate gradient of the line joining the following pairs.**
   a) **A(1;-4) and B(-2;-7)**
   e) **A(3;0) and B(-6;3)**
   f) **A(5;-3) and B(-1;-3)**

Snippet 6.2 Sample lesson plan content coverage

This Grade 11 lesson plan showed that the lesson would have started and concluded with going through and substituting values into the gradient, midpoint, and distance formulae. There was no manipulation of the formulae, and this is also evident from the classwork activities given to learners. This is not the sort of revision that should be helpful for the topic because these concepts are applied to solve more complex problems than just plugging into the formulae. The details of this lesson plan also revealed that the PST spent 20 minutes just introducing the formulae. This takes us back to time management. The PST seemed to not know where to spend most of their time, and it could have been on applying the concepts in more complex problems.

6.3 The selected examples for teaching analytical geometry.

In this section, the PSTs' choices of examples for teaching analytical geometry are analyzed by looking at the cognitive demands of the chosen examples, and the structure of the examples by analyzing the observable patterns of variation in them. The suggested cognitive demands in the CAPS document are namely knowledge (K), routine procedure (R), complex procedure (C), and problem solving (P).

The patterns of variation and their descriptors that are used to analyse the examples are outlined below in accordance with the analytical framework.

- Contrast: An example in which some features of the object of learning vary while some are kept the same.
- Generalisation: Examples in which similar instances of the object of learning can be compared.
- Separation: Examples in which the feature of interest in the object of learning varies while all other features are kept the same.
- Fusion: Examples that simultaneously expose learners to the variation of a number of features of the object of learning.

The main focus of this study was on the choice of examples that PSTs select for teaching analytical geometry in multilingual classrooms. This focus was also driven by the statement made by Leinhardt, Zaslavsky, and Stein (1990) that the design and/or selection of examples is a potent mechanism towards achieving the objectives of a mathematics lesson. Hence, it is important to seriously consider exploring and analysing the choice of examples because the use of those examples facilitates the success of a lesson.

To analyse the selected examples, the study employs the patterns of variation from the variation theory, which is part of the theoretical framework of this study. Using the variation theory to analyse examples for teaching and learning mathematics is an approach that scholars in mathematics education attest to. Kullberg and Skodras (2018) explain that having variation in the examples that are selected for teaching mathematics is considered a vital component in facilitating the learning process.

The selection/choice and design of examples is an aspect of teaching that involves mainly the teacher. Here, the teacher considers what they want to achieve in the lesson and create a plan on how they will do so by selecting or designing examples. This is what the variation theory refers to as the intended object of learning. Olteanu (2014) proposed that this aspect of teaching be referred to as the 'intended critical aspect', which is the aspect of content that the teacher considers to be critical for learners, and intends to present in a lesson.

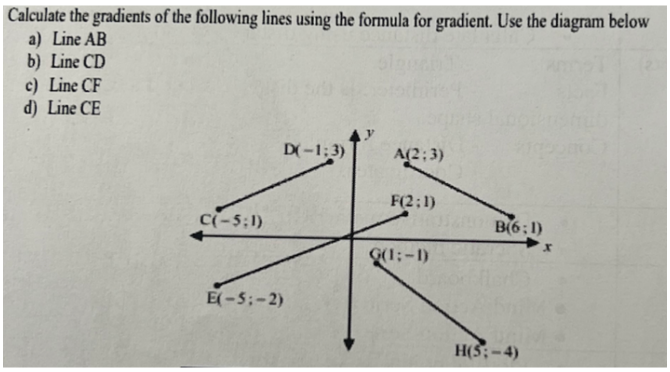
This section of analysis ties to the previous section by exploring whether opportunities for learning the intended critical aspects that were noted in the PSTs lesson plans were created by the examples. Similar to the finding made by Olteanu (2017) that teachers tend to choose or simply select examples from textbooks. This was also the case with the PSTs in this study. All the presented examples were directly extracted from textbooks, sometimes following the order of the examples in the textbooks, or simply choosing the ones they were comfortable enough to teach. Comfortable here, refers to the easy examples that do not entail a lot of problem solving but mostly just showing procedure. Having most or all the examples being extracted from textbooks does not change anything when it comes to the patterns of variation that could be used because scholars in mathematics education (Watson and Mason, 2006; Bartolini, Xuhua, and Ramploud, 2013; Kullber and Skodras, 2018) have demonstrated that examples for mathematics teaching can be described using patterns of variation and invariance.

This study explores the selected examples to understand how the patterns of variation may have been incorporated in the way they are structured in the lesson plans. The examples are presented in terms of the pattern of variation that is observable in them, and the ones that presented the opportunity to be used. The used labelling for the examples follows the format: $PST_{n\text{ - pattern of variation}}$, where PST is for preservice teacher, *n* is the PST number, and pattern of variation is the observed pattern in the examples.

As already stated, almost all the presented examples in this study were extracted from textbooks by the PSTs and presented in lesson plans. This study does not explore or directly use the examples but only focuses on the PSTs' choices and selection of the examples. This is also without criticising or endorsing any of the examples.

6.3.1 Contrast in the selected examples

The presented examples in Snippet 6.3 allowed for contrast to be used as a pattern of variation as they are mostly about showing what a concept or feature of a concept is and what it is not (Al-Murani et al., 2019).



$PST_{2\text{ – contrast}}$

Cognitive demand: **R**

Calculate the gradient (if possible) of a line with the Inclination

a) 135° b) 45° c) 150° d) 90°

PST12- contrast

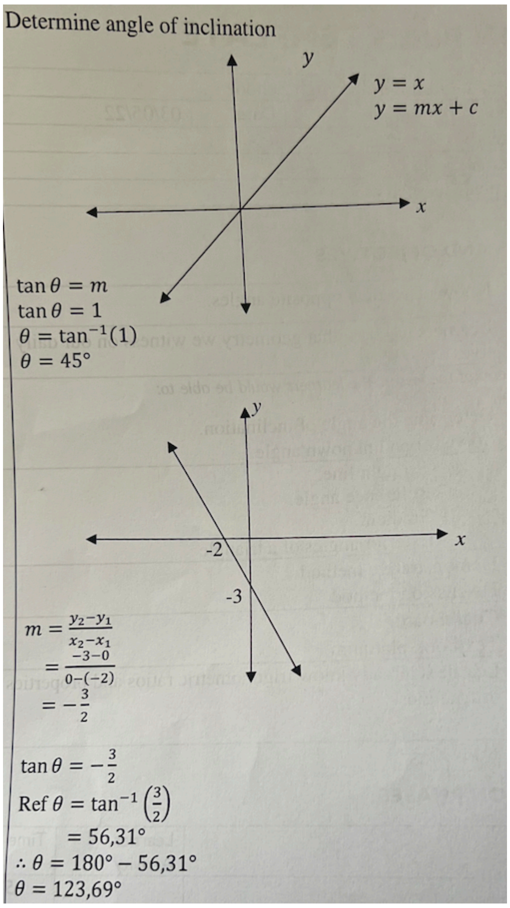
Cognitive demand: **R**

Given: A(−2;1), B(k;−5), C(4;6) and D(−2;7)
Calculate the value of k in each case if:

(a) AB∥CD b) AB ⊥ CD

PST13- contrast

Cognitive demand: **R**



PST14- contrast

Cognitive demand: **R**

Snippet 6.3 Selected examples in which contrast was identified

According to Handy (2021), contrast is about identifying what something is from what it is not, and counter examples in mathematics proofs provide an instance of contrast.

It is the deliberate and systematic introduction of contrasting examples to facilitate the learning process. The goal of using contrast is to help learners discern critical aspects of a concept or phenomenon by highlighting the differences between various instances. Exposing learners to varying examples of a concept or problem helps them begin to notice the essential features and patterns that define it. This also helps learners differentiate between what is relevant and what is not, leading to a deeper understanding.

In the case of PST_2's selected examples, learners were allowed the opportunity to learn about the gradients of lines that are positioned differently. The intention was to have learners calculate and identify that the gradients of lines with their different positions is different in the sense that a line that is; oblique-up is positive, oblique-down is negative, horizontal is zero, and a vertical line has an undefined gradient. For instance, learners will understand what a horizontal line is when they are shown a vertical line, and this also applies when they learn about the gradients of these lines. Therefore, when one explains that a vertical line has an undefined gradient we tend to use the horizontal line as a counter example by showing that its gradient is zero. Although, language tends to interfere with the explanation in multilingual classrooms. I have had learners and PSTs in my classes saying that the vertical line has no gradient (in isiZulu), which actually means zero gradient instead of undefined.

$PST_{2\text{-contrast}}$ had the gradient formulae and how the gradients of lines vary in terms of their difference in positioning written and explained before the example. This allowed for contrast in the examples to be more observable as learners would know from the notes that the gradient of a horizontal line is different to that of a vertical line. There is a great chance that the PST would have completely achieved the goal of the lesson if they had given more examples of this kind.

In another lesson that was also on the concept of gradient, PST_{13} selected an example that required learners to remember the relationship between the gradients of parallel lines, and that of perpendicular lines. The lesson plan had no explanation or notes made about this. The lesson was introduced with this example ($PST_{13\text{- contrast}}$). Clearly, the assumption was that learners would remember and had understood all Grade 10 work. Although, this does not make for a good justification because some learners may not have understood the concept at all in Grade 10. However, the selected examples have contrast because they are in line with Olteanu's (2017) explanation of contrast which is when the teacher lets the learners discern one quality of a concept, and a different quality simultaneously. In this case, the selected examples allowed learners to discern the relationship between gradients of parallel lines whilst also letting them learn about gradients of perpendicular lines. This was also the case with the examples in Snippet 6.4. The examples were also chosen by PST_{13} for a short activity.

Calculate the value of x in each case if A, B, C and D are the points A$(3;4)$, B$(-1;7)$, C$(x;-1)$ and D$(1;8)$ and:

(a) AB∥CD

(b) AB ⊥ CD

(c) B, C and D are collinear

($PST_{13\text{- contrast2}}$: *Cognitive demand*: **R**)

Snippet 6.4 Second example with contrast by PST_{13}

These examples allow contrast in a similar manner as the $PST_{13\text{- contrast}}$ example.

PST_{12} and PST_{14} were working on the concept of the angle of inclination. $PST_{12\text{- contrast}}$ had contrast because it showed the difference between the gradients of lines that have the angles of inclination which are acute, obtuse, or right angles. $PST_{14\text{- contrast}}$ used graphs to show that the angles of inclination of an increasing and a decreasing line are acute and obtuse, respectively. The two PSTs allowed learners to discern one quality of the angle of inclination, and a different quality simultaneously (Olteanu 2017).

6.3.2 Separation in the selected examples

Separation is about varying the dimensions of variation at different rates while keeping some aspects invariant (Olteanu, 2017). Essentially, separation isolates specific dimensions for focused learning. It is closely related to contrast, which is why Marton (2015) argues that it is a result of contrast.

In line with the above argument by Marton (2015), Handy (2021) argues that separation is not necessarily a condition for learning because learners discern certain aspects when the other aspect are kept invariant. This is seen in contrast, for instance when we refer to the example presented in $PST_{2\text{ – contrast,}}$ all the lines are on the same cartesian plane and the main thing is to calculate the gradients of the lines. This is the invariant aspect, and the positions of the lines are different, which is what varies. The goal here was to draw learners' attention to a particular feature (gradients for differently positioned lines) and help them understand its significance and relationship within the broader context. Similarly, separation was observed in the examples that were selected by PST_{13} in the contrast section in Snippet 6.3.

Handy (2021) makes an example that a learner will know the colour black by experiencing the other colours, like red or white. In this experience, colour is separated from the invariant aspects such as keeping whatever is being used to show the colours (a block or paper), the same in size and shape (Handy, 2021). In the context of the selected examples in PST_{2} – contrast, learners may separate the gradients of the lines from their different positions when the lines are on the same cartesian plane and of the same length. Separation would be observed here because the targeted dimension, the gradients of different lines, is the only varying factor while the other aspects are kept constant. This would guide learners to focus on the isolated dimension.

As separation is declared a result of contrast by Marton (2015), one can argue that the selected examples that show contrast in the previous section could lead to separation in one or more ways. Although, it was quite difficult to find examples that had separation as clear as the colours example made by Handy (2021) from the examples that the participating PSTs selected.

6.3.3 Fusion in the selected examples

Fusion is observed in instances where the examples that are selected by the PSTs created opportunities for learners to simultaneously discern various dimensions where they are integrated to show how they interact and contribute to the overall understanding of a concept. These are instances where the examples highlight the interconnectedness and relationships between different dimensions. The fusion of dimensions allows learners to discern how these dimensions are interdependent and influence each other. Snippet 6.5 shows the examples that were selected by the PSTs and in which fusion was observed.

1. Determine the midpoint M of the line segment EF with end point $E(-5;-2)$ and $F(2;1)$
2. If $K(-1;-5)$ is the midpoint of the line segment joining the points $A(x;y)$ and $B(-6;-3)$. Calculate the values of x and y.

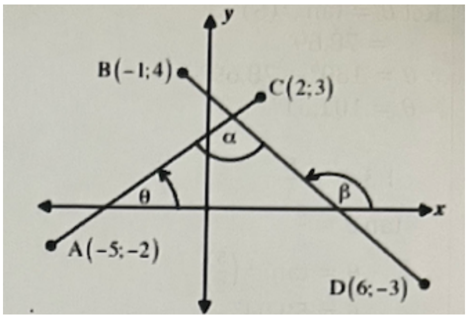
Determine the values of x and y if M (5; 2) is the midpoint of the line segment joining the points A (x ;1) and B (8; y).

PST4-fusion

Cognitive demand: **R**

Given: $A(-5;-2)$, $B(-1;4)$, $C(2;3)$ and $D(6;-3)$.

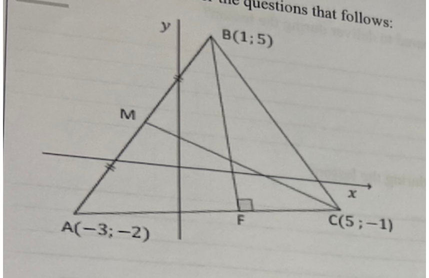
Determine the α Formed between lines AC and BD



PST5- fusion

Cognitive demand: **C**

Use the diagram to answer the questions that follows:

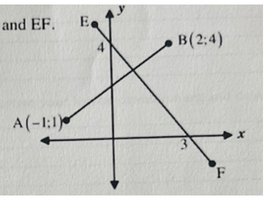


1. Determine coordinates of M the midpoint of AB.
Solution: $M(-1;\ 3\ 2)$
2. Determine the equation of the median CM.
Solution: $y = -5\ 12\ x + 13\ 12$

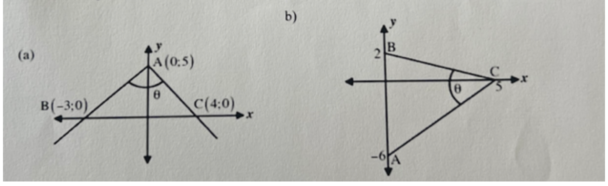
PST11- fusion

Cognitive demand: **R**

Calculate the angle of inclination of a line AB and EF.



Calculate the angle θ which is formed by the two lines of the following sketches. Round off your answer to two decimal places



PST12- fusion

Cognitive demand: **C**

PRESENTATION:

CLASSWORK

1. Determine the equation of the line passing through the points $F(-4;2)$ and $G(-1;-2)$.
2. Determine the equation of the line with a gradient of -2 passing through the point $(3;-4)$.
3. Determine the equation of the line that is parallel to $3y-2x=6$ and passes through the point $(9;-1)$
4. Determine the equation of the line that is perpendicular to $y=\frac{1}{2}x+1$ and passes through the point $(-6;2)$

PST14- fusion.

Cognitive demand: **R**

Snippet 6.5 Selected examples in which fusion was identified

$PST_{4\text{-fusion}}$ had two examples where learners must find the midpoint followed by an example where they must use the midpoint to find the coordinate of a point. Learners get to use the midpoint formulae to find the midpoint of a line segment, the coordinate of a point on a line segment given another point and the midpoint between the two points, and to determine the value of x or y where coordinates are given such that one of the values is missing. By doing this, the teacher is allowing learners to experience many dimensions of variation simultaneously (Olteanu, 2017) whereby learners need to distinguish between manipulating the midpoint formulae to find a coordinate, and manipulating it to find missing values of a coordinate.

Fusion was also observable in $PST_{11\text{- fusion}}$ because learners were to learn to distinguish between a median and a midpoint of one of the sides of a triangle. They will discern that a median has an equation while a midpoint has coordinates. $PST_{12\text{- fusion}}$ and $PST_{5\text{- fusion}}$ presented opportunities for fusion in a similar way. In the examples that were selected by PST_5 learners must use the angles of inclination of two lines to calculate the angle formed where the two lines intersect. Learners would learn that the angle of inclination for an oblique-up line is not the same and requires different calculations to one that is oblique-down. They would also learn that the angle formed where two lines intersect without one of the lines being the x-axis or parallel to the x-axis in not the angle of inclination. Therefore, one cannot use the gradient to find this angle.

In $PST_{12\text{- fusion}}$, learners would simultaneously discern that the angle of inclination is formed between the x-axis and an oblique line and that when two oblique lines intersect the angle formed is not the angle of inclination. Also, the methods used to calculate the two are different. There is also fusion in this example because in addition to this, learners are to understand the difference between the angle of inclination for oblique up and oblique down lines.

In the case of $PST_{14\text{- fusion}}$ the examples exposed learners to the different cases where they had to use the gradient to find the equation of a line. In these examples, learners may understand how to calculate the gradient using the formulae, use the relationship between parallel, and perpendicular lines to find the gradient, and find equations of lines using gradients and given points. These examples combine multiple dimensions of a gradient, allowing learners to understand how these dimensions work together and influence one another.

It is crucial to note that the PSTs hardly selected examples that combined multiple dimensions of a concept. They mostly selected repetitive examples where, for example, they had three or more different examples that simply required learners to find the distance, or the angle of inclination without necessarily having different aspects of the concept involved in the examples.

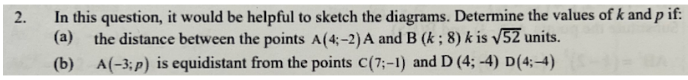
6.3.4 Generalisation in the selected examples

According to Olteanu (2017), generalisation is when a teacher provides different examples of a concept by keeping one aspect invariant while varying the other aspects. Generalisation help learners in identifying the core features that apply across different cases, leading to a broader and more abstract understanding of the concept.

This pattern of variation is important in examples, especially for teaching analytical geometry because they serve as concrete instances that elucidate abstract concepts. Generalisation was by far the most observed pattern of variation in the PSTs' selected examples.

Referring back to Snippet 6.3 $PST_{2 - contrast}$ examples which focused on the same aspect of a concept, that is calculating the gradients, one can argue that generalisation was observable because the instruction is to calculate the gradient for all the lines, which are different in size and direction. Even though a follow-up question such as, *"what do the calculated gradients tell you about the lines?"* could have assisted the PST in achieving or emphasizing it more, especially because lines CE and CF are not drawn. A question of this sort and a few more examples on calculating the gradients of lines in different positions would have created more opportunity for generalisation. This is because learners would discern the critical feature for themselves which would allow them to be part of their learning process.

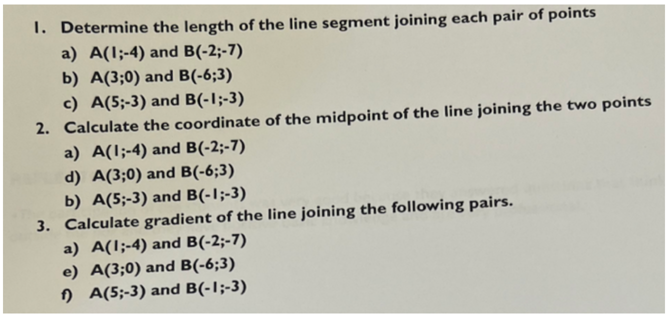
PST$_{10}$ also selected examples which were similar to PST$_{2 – contrast}$ examples, and they provided the same opportunity to have generalisation in them. However, PST$_{10}$ does not ask learners to find the length of all the lines and doing so would have allowed generalisation to be enhanced, and learners to learn. See PST$_{10- generalisation1}$ in Snippet 6.6, and some of the other examples that the PSTs selected, and in which generalisation could be observed.



2. In this question, it would be helpful to sketch the diagrams. Determine the values of k and p if:
(a) the distance between the points $A(4;-2)$ A and B $(k ; 8)$ k is $\sqrt{52}$ units.
(b) $A(-3;p)$ is equidistant from the points $C(7;-1)$ and D (4; -4) $D(4;-4)$

PST$_{4- generalisation}$

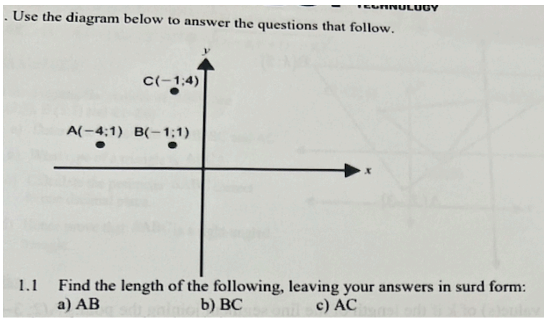
Cognitive demand: **R**



1. Determine the length of the line segment joining each pair of points
   a) A(1;-4) and B(-2;-7)
   b) A(3;0) and B(-6;3)
   c) A(5;-3) and B(-1;-3)
2. Calculate the coordinate of the midpoint of the line joining the two points
   a) A(1;-4) and B(-2;-7)
   d) A(3;0) and B(-6;3)
   b) A(5;-3) and B(-1;-3)
3. Calculate gradient of the line joining the following pairs.
   a) A(1;-4) and B(-2;-7)
   e) A(3;0) and B(-6;3)
   f) A(5;-3) and B(-1;-3)

PST$_{5- generalisation}$

Cognitive demand: **R**

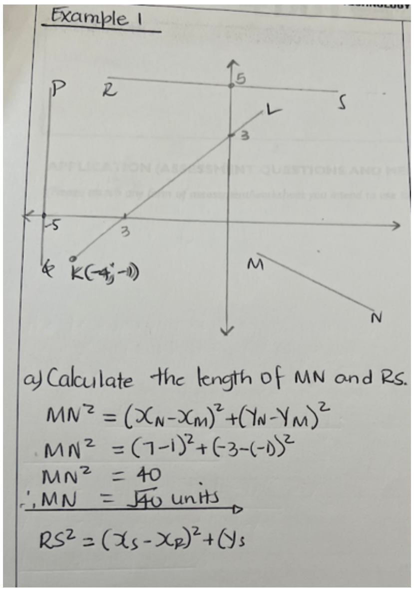


. Use the diagram below to answer the questions that follow.

1.1 Find the length of the following, leaving your answers in surd form:
a) AB b) BC c) AC

PST$_{8- generalisation}$

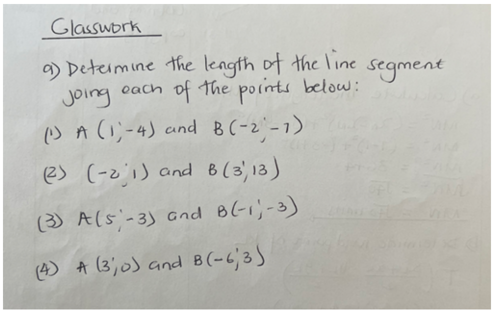
Cognitive demand: ***R***



a) Calculate the length of MN and RS.
$MN^2 = (x_N - x_M)^2 + (y_N - y_M)^2$
$MN^2 = (7-1)^2 + (-3-(-1))^2$
$MN^2 = 40$
$\therefore MN = \sqrt{40}$ units
$RS^2 = (x_S - x_R)^2 + (y_S$

PST$_{10- generalisation1}$

Cognitive demand: R



Classwork

a) Determine the length of the line segment joing each of the points below:
(1) A (1;-4) and B (-2;-7)
(2) (-2;1) and B (3;13)
(3) A(5;-3) and B(-1;-3)
(4) A (3;0) and B(-6;3)

PST$_{10- generalisation2}$

Cognitive demand: **R**

Snippet 6.6 Selected examples in which generalisation was identified

The PST$_{4- generalisation}$ examples lived up to the notion that generalisation is about exposing learners to a variety of examples that have a common underlying concept. The examples presented different instances in which learners must use the distance formulae to solve the problems. In the first instance, learners were given two points and the distance between them, but one of the points is missing the x value of its coordinate. In the second example, three points, A, C and D, and their coordinates are given. A is equidistant to the other two points, and learners must calculate the value of p, the y value of the coordinate of A. These examples are some of the diverse examples that shared the common underlying concept, the distance between two points. The constant aspect here was the use of the distance formulae, while the varying aspects were, (1) the first example had two points and the second one has three, and (2) in the first example, the distance was given, while the second example required learners to find the distance first.

$PST_{5\text{- generalisation}}$, $PST_{8\text{- generalisation}}$, and $PST_{10\text{- generalisation2}}$ examples allowed generalisation in a similar manner, with $PST_{8\text{- generalisation}}$ using points that are plotted on the cartesian plane, which also helps learners visualise the difference in the two -line segments that are formed by AB and BC. Leung (2012), who refers to the patterns of variation as the types of variation interactions explains generalisation as an activity that is about making verifications and conjectures. The way in which PSTs 5, 8, and 10 selected the examples above looks like they were mainly focusing on having learners verify their procedures.

The selected examples allowed generalisation in the way that Cheng (2016) explains generalisation, that is, it is about allowing learners to experience varying appearances of the same concept. The selected examples in $PST_{5\text{- generalisation}}$ $PST_{8\text{- generalisation}}$, and $PST_{10\text{- generalisation2}}$ did this by varying the coordinates of the points that learners must use to calculate the length, midpoint, and gradient of the lines that form when connecting the two points (which are varied). The use of different point to form different lines allowed learners to adapt what they are learning to different situations (Cheng, 2016).

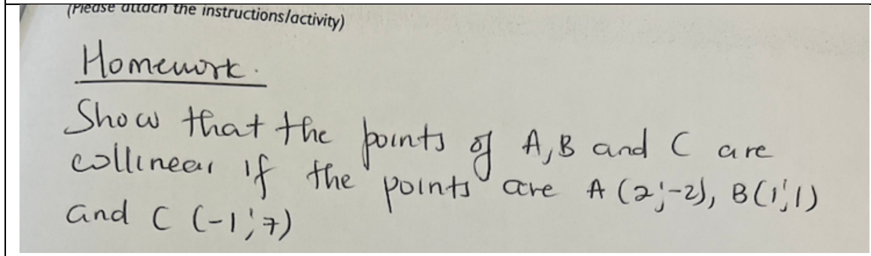
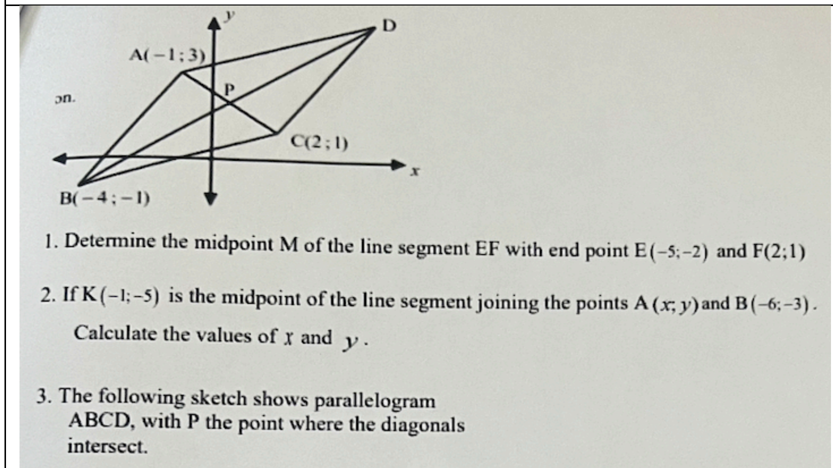
It is important to note that the study is also cognisant of the fact that the selected examples that were presented by the PSTs may not have been used in the order they were presented, and some may not have been presented at all during the lessons. This emanates from the researcher's teaching experience and experience in observing and evaluation PSTs during their teaching practical. In some cases, PSTs will use examples that are completely different to the ones they selected and have in the lesson plans. However, this study focuses mainly in the choices of examples that the PSTs selected for teaching analytical geometry, and most often than not, the used examples will be similar to the originally selected.

6.3.5 Cognitive demands in the PSTs' selected examples

The most observed cognitive demands in the selected examples was the routine procedure (R), which was showed that the PSTs were teaching for procedural understanding rather than conceptual understanding. They did not ask many questions for knowledge or problem solving, actually there was hardly any. This confirmed the claim that mathematics PSTs rely more on procedural or instrumental understanding instead of conceptual or relational understanding (Özpınar and Arslan, 2022).

Table 6.3 shows more of some of the other examples that were selected by the PSTs and which have varying cognitive demands. The cognitive demands are indicated for each question in the examples as it is done in the CAPS document.

Table 6.3 Cognitive demands in some of the selected examples

Selected examples.	**Cognitive demands**
 (Please attach the instructions/activity) Homework. Show that the points of A, B and C are collinear if the points are A (2;-2), B(1;1) and C (-1;7) PST_{10}	**K** Learners must know what collinear points are to know what method to use. This instruction is problematic because it says " …the points of A, B, and C…" this makes it sound like there are points within the points and it is simply a language issue, which might hinder the learners' understanding.
 1. Determine the midpoint M of the line segment EF with end point E(−5;−2) and F(2;1) 2. If K(−1;−5) is the midpoint of the line segment joining the points A(x;y) and B(−6;−3). Calculate the values of x and y. 3. The following sketch shows parallelogram ABCD, with P the point where the diagonals intersect. PST_8	1. **R** 2. **R** 3. Question three was incomplete on the lesson plan but upon asking the PST, they said learners were asked to find the coordinates of P and D. Therefore, this would be **C**

<table>
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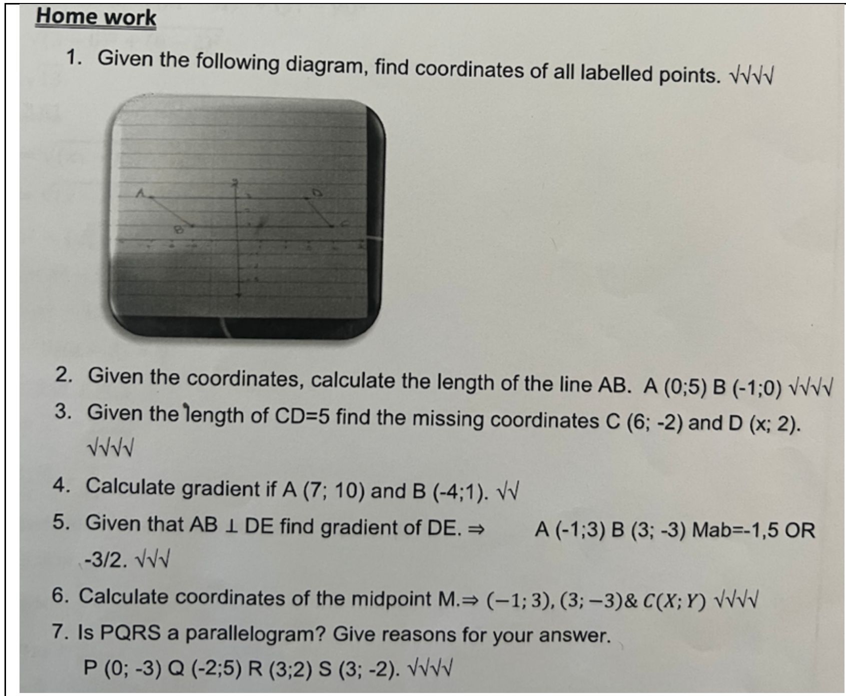

Home work

1. Given the following diagram, find coordinates of all labelled points. √√√√
2. Given the coordinates, calculate the length of the line AB. A (0;5) B (-1;0) √√√√
3. Given the length of CD=5 find the missing coordinates C (6; -2) and D (x; 2). √√√√
4. Calculate gradient if A (7; 10) and B (-4;1). √√
5. Given that AB ⊥ DE find gradient of DE. ⇒ A (-1;3) B (3; -3) Mab=-1,5 OR -3/2. √√√
6. Calculate coordinates of the midpoint M.⇒ $(-1;3), (3;-3) \& C(X;Y)$ √√√√
7. Is PQRS a parallelogram? Give reasons for your answer. P (0; -3) Q (-2;5) R (3;2) S (3; -2). √√√√

PST_{14}
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<td>
1. The instruction on this question seems to be incomplete. So, the PST was asked to explain what the instruction was, and the explanation was that the drawing had clear numbering on the board and learners had to use the points on the cartesian plane to find the coordinates. This would then be K because learners must know how to use the cartesian system.
2. R
3. R – One must note the issue of language in this question. There is a missing value, not the whole coordinate, which the PST wrote coordinates.
4. R
5. R – The use of notation is incorrect here. The PST use a capital letter M for gradient and small letters for line segment AB.
6. R – The way in which the instruction in this question is expressed is quite confusing. There is no M on the diagram, and the C(X; Y) on the instruction did not seem to mean anything. The PST said she did not write it on the board. Although, it’s difficult to not notice that PST_{14} uses shorthand instead of a full and clear instruction. This may lead to learners also not understanding the mathematics notation and language.
7. C – Learners must know the properties of a parallelogram and connect concepts to respond to the question.
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As already noted, the most observed cognitive demand in the PSTs' selected examples was the routine procedure. This did not align with the education provided by Stein et al. (2009) that tasks will not always be created such that they are equal or of the same levels, and it is important to note that tasks of different levels will stimulate different levels and kinds of learners' thinking.

When a task requires learners to perform a memorized procedure, a small opportunity is provided to develop the learners' thinking, while tasks that require conceptual thinking incite learners to make connections between concepts and create opportunities for learners' thinking to develop (Stein and Smith 1998). According to Stein and Lane (1996), when one intends to develop the capacity to think, reason, and solve problems, it is important to consider high level and cognitively complex tasks. Most of the PSTs avoided high level tasks or examples and the reason seemed to be the lack of confidence in their content knowledge. However, some PSTs like PST_1 made excuses for covering content at basic levels. Excerpt 6.4 shows what PST_1 had to say.

> Interviewer*: It looks like you only calculated the angle of inclination and did not use it to find the angle between two intersecting lines, why?*
>
> PST_1*: There was no time, and the teacher said she will do revision with the learners.*

Excerpt 6.4 Engagement with PST_1

This may indicate that the PSTs are also learning to teach to the test from their work integrated learning. For this reason, it was important to have interview sessions with the PSTs to learn and understand their experiences and process of developing PCK.

The next chapter presents the identified themes in the PSTs interviews about their analytical geometry content knowledge, selection of examples for teaching the topic, and issues of language in learning and having to teach the topic in multilingual classrooms.

CHAPTER SEVEN: PRESENTATION AND ANALYTIS OF DATA - THEMES FROM THE INTERVIEWS

7.1 Introduction

The focus of this chapter is on the interview engagements about the PSTs choices and selection of examples. The chapter reports on the investigation of the role of PCK in the PSTs choice of examples for teaching analytical geometry. It intends to provide further insight into the PSTs experiences of analytical geometry concepts, and their thinking about the examples to teach the concepts. The chapter does this by discussing themes which were identified from the recurring patterns of meaning from the PSTs responses.

The themes were guided by the research questions as per the definition of a theme by Braun and Clarke (2006) who state that a theme captures aspects of data that represent some level of patterned responses which are also related to the research questions. Most of the interview questions are aligned with the main research questions. Therefore, the PSTs' responses produced data that spoke to the purpose of this study and assisted in responding to the research questions. However, very few PSTs from the participating group were vocal and made time to elaborate on their responses, the rest gave short responses and required a lot of probing. Identifying themes from their responses was mostly guided by single words and very short phrases, which mostly revealed the extent to which the PSTs engaged or made considerations when they chose the examples.

The identified themes from the interview responses are, the source and choice of examples, the purpose of selected examples, and use of language in the selected examples. The development of these themes was guided by the illustration that Lochmiller (2021) makes on shaping themes. Figures 7.1, 7.2, and 7.3 are used to illustrate how each theme was shaped by the sub-themes which were a result of the presented recurring terms and patterns in phrases. The themes are interdependent and have overlapping concepts, therefore the discussion draws from their similarities and differences, and connects them.

In each theme, there will be an analysis of the observed or identified aspect of PCK. This is possible because the interviews provided a platform for the PSTs to express and justify their reasoning and ideas when selecting examples, and their responses provided more information about their PCK and its role in their examples' choice. To analyse the role of PCK in their examples, this chapter will refer to Table 7.1 which is a portion of the analytical framework. Table 7.1 shows components of the MTSK model, which are explained in the theoretical framework, and employed in the analytical framework with descriptors that make it applicable to this study.

Table 7.1: MTSK model and descriptors for analysis

MTSK components		Descriptors
Mathematical knowledge	KoT	Knowledge of the analytical geometry topic. "Knowing the what and in what way" (Carrillo-Yanez et al., 2018). There is evidence of explaining the 'what', 'why', and 'how' aspects of concepts.
	KSM	Showing the knowledge of the inter and intra connections in the concepts. Able to show and clearly indicated the relationships between concepts within the topic/subject and to other topics/subjects.
	KPM	Evidence of being able to demonstrate clear understanding of how analytical geometry concepts are used in practical applications. That is, how they are used to perform mathematical operations and solve real world problems.
	KMT	The PST is aware of the issues around the topic. Knows the theories, teaching strategies and resources, and misconceptions that learners may have in this topic. There is evidence of being

Pedagogical Content Knowledge		able to address the misconceptions and use relevant examples for the Grade and age group.
	KFLM	There is evidence of adopting theories of learning and uses different strategies/ aids/ activities to create opportunities for learning. The examples are also set and structured in a way that provides enough opportunities for learners to grasp the critical features of the concept.
	KMLS	Learning outcomes of the lesson are in line with the CAPS document outcomes. The level at which content is pitched matches the national standard as prescribed in the CAPS document.

The components of the MTSK framework will be used to examine the PSTs mathematical knowledge and PCK by analyzing all the provided responses and matching them with the PSTs' performance on the test, and their selection of examples. This will be done in all the themes.

7.2 Theme 1: The source and choice of examples

One of the crucial aspects of teaching mathematics is selecting examples to illustrate concepts. This makes the source of examples a vital resource for teaching and learning mathematical concepts. Selecting appropriate and relevant examples is significant, and serious considerations must be made to ensure that the chosen examples create opportunities for learning (Olteanu, 2017). This theme focuses on how the PSTs chose their examples and what guided their choice of examples. Figure 7.1 provides a schematic view of this theme together with the three sub-themes for which the terms and phrases that created them are indicated.

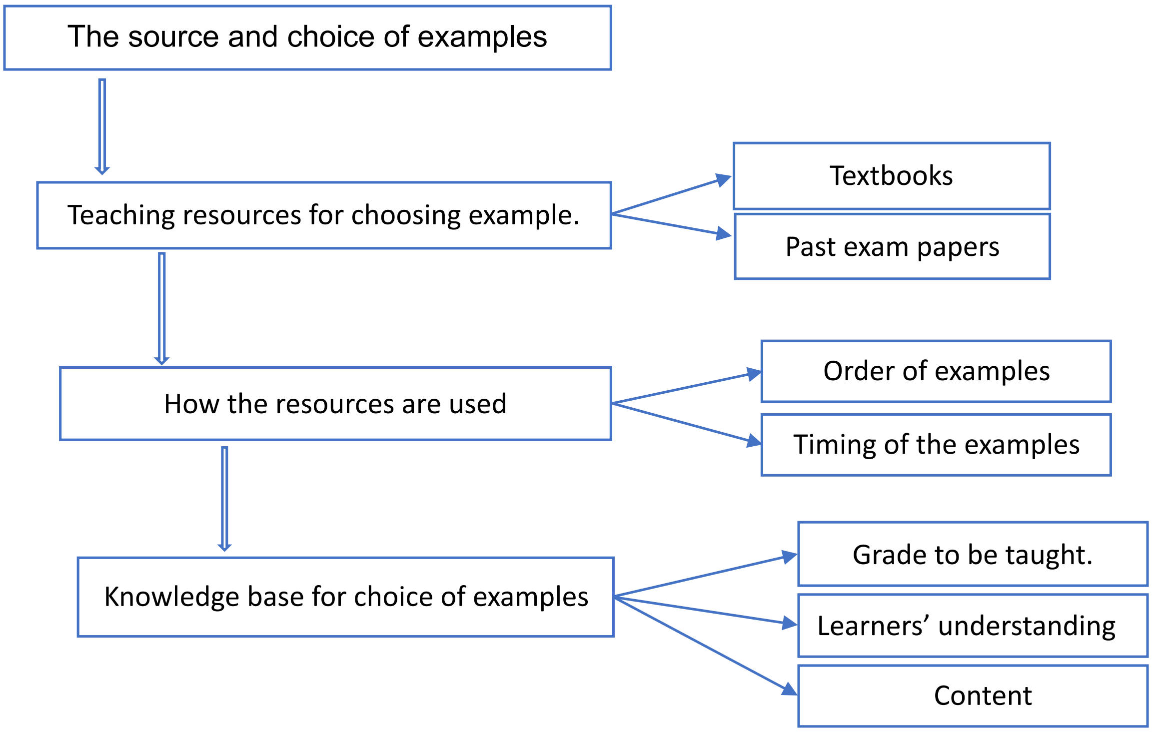


Figure 7.1 Theme 1- The source and choice of examples

To understand the PSTs process of examples selection for teaching analytical geometry concepts, this section delves into the teaching resources that the PSTs used for choosing their examples, how the resources were used, and the knowledge base that guided their choice of examples.

7.2.1 Teaching resources for choosing examples.

None of the PSTs was asked about teaching resources, but most of them mentioned textbooks and past examination papers in response to the first interview question, which was:

> *How did you choose the examples you used in the lesson, and what did you have to consider?*

The responses mostly started with the source of examples as the PSTs explained where they had adapted the examples from and how they selected them. One of the informative responses to this question was given by PST_8 who also started by referring to the source of examples. The other PSTs showed signs of not engaging in cognitive pedagogical processes and a lack of KFLM, and this is seen in the way PST_5 responded in Excerpt 7.1.

PST_8: I checked the examples on the textbooks and took the ones that match the goal of my lesson. I was also thinking about what learners know.

PST_5 said: I was using past papers because learners were going to write tests.

PST_{11}: I start with choosing easy examples and move to difficult examples on textbooks.

Excerpt 7.1 Engagements about the choice of examples

KMLS is also a lacking component as the PSTs did not seem to consider the learning outcomes of the topic. This is seen in the response by PST_{11} which also revealed that their focus was on preparing learners for the test, which means that some important concepts might have been left out because the PST focused on what was to be tested. The use of examples from past exam papers is ideal in situations where content is covered extensively, and these examples are used for formative assessment.

There is a wide array of teaching resources at the disposal of the PSTs. These include hard and soft copy textbooks, online educational websites and worksheets, curriculum guides such as the CAPS document and annual teaching plans, educational software, and many more. However, the participating PSTs revealed that their most relied on teaching resource from which they chose examples were textbooks and past examination question papers, and most of them relied on textbooks. This suggests that these were the most available resource in the schools they would have been practicing, which is not far-fetched because most of the PSTs in this institution do their practical in schools that are in the rural KwaZulu Natal. While this resonates with the argument made by Ronda and Adler (2017) that textbooks are the most available teaching and learning resource, especially in developing countries, including South Africa, it should be noted that the PSTs have access to online platforms on their phones and university facilities which they could use to prepare for their teaching.

7.2.2 How resources are used.

As already indicated, the participating PSTs seemed to only rely on textbooks for example selection. They did not harness the variety of teaching resources they had at their disposal. This could mean that they were not empowered and equipped enough to tailor their teaching to the learners' specific learning needs and interests, which is in line with having KFLM. In these PSTs process of lesson planning, learners did not seem to be the main concern, but content coverage was. This is evident in the responses to the two questions that shaped this particular sub-theme. PST_{11} in Excerpt 7.1 also explained how they use textbooks when selecting examples. So, the two questions below are not the only questions that highlighted how the PSTs used resources for selecting examples.

> Interviewer*: How did you decide on the number of examples to use in each lesson?*
>
> PST_1*: I try to use short examples to save time and I can do more.*
>
> PST_{10}*: I look at how long are the explanations on the textbook and then do more examples if there is time.*
>
> Interviewer*: How did you decide on the appropriate time to use an example?*
>
> PST_5*: I follow the examples on the textbooks, and then sometimes I take questions from past papers for classwork and homework.*
>
> PST_9*: The textbooks show examples you can use to introduce the topic.*

Excerpt 7.2 Engagements about the time and number of examples the PST choose.

The response from PST_1 was related to the trend that was observed in chapter 6 which showed that the PSTs tend to choose "comfortable" examples for teaching. These are easy examples of low cognitive demand, and normally require knowledge and use of procedures without connections and application. This, and the PSTs performance in the test may suggest that the PSTs did not have KSM and KPM as they hardly chose examples that show inter and intra connectedness of concepts.

The PSTs did not seem to invest time in thinking about how they should order their examples, and what the appropriate time for a particular example was. At this point it looks like some of the PSTs simply presented textbook lessons, which are not always problematic. However, some create limited opportunities for learners to engage in formal mathematical discourse (Ronda and Adler 2017). Therefore, following examples as they are presented in textbooks may not always be ideal.

7.2.3 Knowledge base for choice of examples.

In teaching mathematics, the selection of examples is an ongoing process that requires constant adaptation to meet the learners' ever evolving learning needs. This is entirely dependent on content knowledge and knowledge for teaching, which encompasses the selection and sequencing of examples and tasks, and understanding pedagogical choices and decisions that have an effect on learning (Wilson, Sztajn, Edgington, and Confrey, 2014).

Many of the interview questions revealed the PSTs knowledge base in their selection of the examples. This is also seen in Excerpt 7.1 where PST_8 indicated that they thought about their lesson objectives when they selected examples. Knowing and understanding the lesson objectives is the first step of lesson planning and it guides the process of choosing examples to promote learning. It also initiates thoughts about the mathematical concepts and skills to convey to learners, alignment of content to curriculum, learners' prior knowledge, and more, which shows KMT and KMLS. However, lesson objectives, learner diversity, and other aspects of pedagogical knowledge were very rare in the PSTs' responses. In regard to this finding, Sahin-Taskin (2017) points out that PSTs have difficulty in selecting appropriate examples and activities for their learners because they do their teaching practice in different schools and have limited time with each group of learners.

Considering that the PSTs did not have enough time to engage with learners to know them sufficiently, the main question that drove the development of this sub-theme was intended to find out what guided the PSTs when they were choosing examples. The question was:

> *What knowledge mainly informs your choice of examples, is it the knowledge of content, learners, or any other knowledge?*

The question was poorly answered because of the attempt made to expand it for clarity. As a result, the PSTs simply chose one of the options in the question. Content knowledge seemed to be the main aspect of teaching that the PSTs thought was most needed for example selection. Although this is not entirely incorrect, but content knowledge alone does not assist in choosing and structuring examples to create opportunities for all learners to learn. These PSTs' responses confirm a finding made by Güle and Çelik (2018) in a study that explored the relationship between PSTs content knowledge and PCK. In this study, Güle and Çelik (2018) found that there is a deficiency in the PSTs knowledge of learners, and presentation of content.

Excerpt 7.3 shows what some of the PSTs had to say in response to the question.

PST_1: I use the right content for the grade.

PST_3: Content and what learners already know.

PST_4: Content and learners.

PST_{11}: I know the content for Grade 11

Excerpt 7.3 Engagements about the knowledge base for the choice of examples

There is a relationship between the way the PSTs were responding to the question, and the way they structured their lesson objectives. At this point the impression is that they equate teaching to content delivery. In these responses, the lack of KMT is exposed. This sub-theme is also related to the next theme, which is the purpose of the selected examples because some of the responses revealed the PSTs' thinking process and knowledge base.

7.3 Theme 2: The purpose of selected examples.

Referring back to literature and the purpose of examples in mathematics teaching, especially analytical geometry, terms, and phrases such as conceptual understanding, connecting concepts to the real world, visualizing abstract concepts, and problem solving are expected to be the basis of this theme. However, this theme was shaped by the PSTs focus on content coverage and assessments as shown in Figure 7.2.

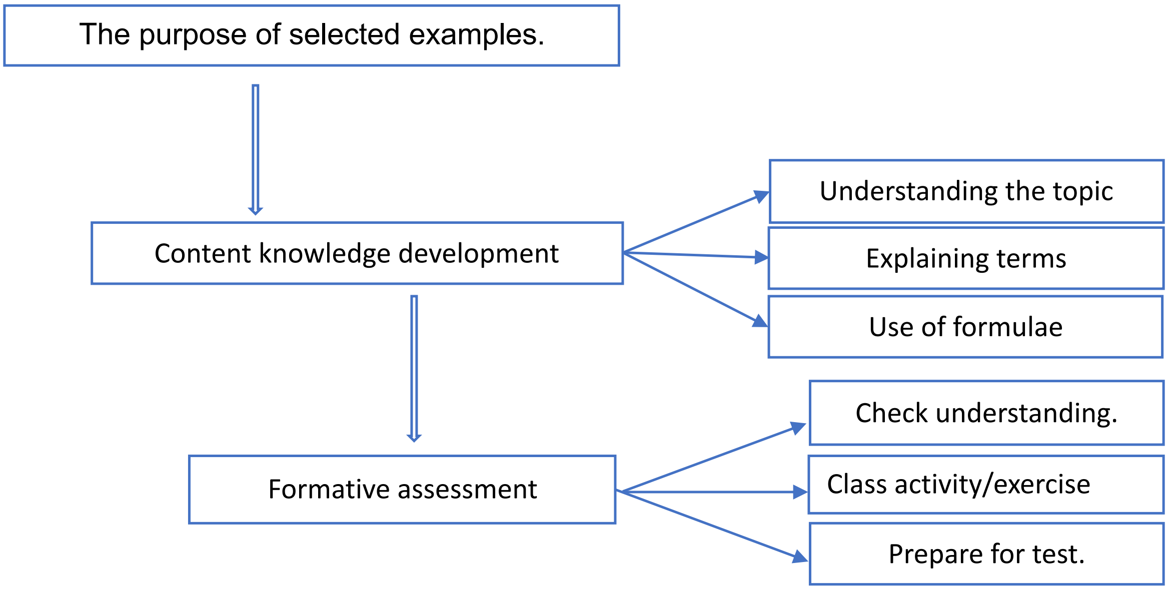


Figure 7.2 Theme 2 – The purpose of selected examples

The sub-themes of this theme were built from responses to the following questions.

1. *What was the purpose of the selected example (name specific example)?*
2. *How did the selected examples enable understanding in your opinion?*

Once again, the responses showed that the PSTs were consumed with teaching to complete the syllabus in preparation for assessments.

7.3.1 Content knowledge development

Responses to the first question were mainly about content knowledge, which the PSTs expressed in different words including use of formulae and revising and explaining concepts. PST_{13} was asked about a specific example which is shown in Snippet 7.1 and their response is captured in Excerpt 7.4. The first question was asked for this example because the lesson objective on the lesson plan was:

"*By the end of the lesson, the learners would be able to calculate gradient of a line.*"

Calculate the value of x in each case if A, B, C and D are the points $A(3;4)$, $B(-1;7)$, $C(x;-1)$ and $D(1;8)$ and:

(a) $AB \parallel CD$

(b) $AB \perp CD$

(c) B, C and D are collinear

Snippet 7.1 Specific example that was enquired on with PST_{13}.

The response to the first question was as follows:

> *PST_{13}: So, I thought it would help me remind them about gradients of perpendicular lines, parallel lines, and collinear points. I did this example with them in class.*
>
> **Excerpt 7.4 Response of PST_{13} on the purpose of a specific example**

This response supports the observation that was made in chapter 6, that most of the PSTs did not know how to structure or set lesson objectives. PST_{13} assumed that learners already knew how to calculate the gradient when this example was selected to be a worked example, but the lesson objective of the lesson is to calculate the gradient. The calculations of the gradient is prior knowledge in this case because it was covered in Grade 10. It seems that lesson objectives were confused with lesson concepts and content, which shows that the PST lacked KMT and KMLS.

Having observed a similar misunderstanding in the lesson objectives and examples that were selected by PST_5, the same question was asked about the example in Snippet 7.2.

The formula to calculate the length of a line segment between points A and B, with $A(x_1;y_1)$ and $B(x_2;y_2)$,

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ or $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

EXAMPLE 1

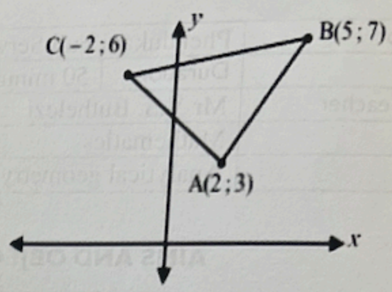
In the diagram, the vertices of ΔABC are A (2;3), B (5;7) and C(−2;6)

a) Determine the lengths of AB, BC and AC

b) What type of a triangle is ABC?

c) Calculate the perimeter ΔABC correct to one decimal place.

d) Hence prove that ΔABC is a right-angled triangle.



Snippet 7.2 Specific example that was enquired on with PST_5

The example was the first worked-out example on a lesson plan that had the lesson objective that is shown is Snippet 7.3.

Lesson Objective *(The purpose of the lesson)*	*By the end of the lesson, the learners would be able to:* • Calculate distance using distance formula.

Snippet 7.3 Lesson objective of PST_5

PST_5 provided the responses in **Excerpt 7.5** when asked the two questions.

Interviewer*: What was the purpose of the selected example (name specific example)?*

PST_5: *The purpose of this example is to revise the distance formulae and use it to solve different problems.*

Interviewer*: How did the selected examples enable understanding in your opinion?*

PST_5: *Learners had to calculate the distance for all the sides of the triangle. Many examples help learners to understand.*

Excerpt 7.5 Response of PST_5 on the purpose of a specific example

Again, prior knowledge was used as a lesson objective. There is a lot to be learned from this example, but the PST made it all about the first question in the example. The PST was also of the view that more examples would help learners understand the concepts.

As much as the PSTs made examples about content knowledge development, they did not seem to understand exactly what they had intended to convey to their learners. This is concerning because according to Süral (2019, p. 2), not having clear intentions about a lesson may lead to an "ineffective and purposeless instruction".

7.3.2 Formative assessment

A similar problem as that in section 7.2.1 is visible in this section, where the PSTs did not elaborate much on the content in the example besides stating that they were assessing learners' understanding. Formative assessment plays a vital role in mathematics teaching and learning. It provides teachers an idea of the progress of concept understanding and encourages learners to take responsibility for their own learning (Shepard, 2017). Examples play a crucial role in formative assessment by highlighting problematic aspects of a concept.

PST_{12} was very aware of what they were assessing and seemed to be using the example strategically to help learners work with the angle of inclination both visually (on a cartesian plane), and numerically. This shows that the PST had the KFLM and KMLS.

Calculate the gradient (if possible) of a line with the Inclination

a) 135° b) 45° c) 150° d) 90°

Snippet 7.4 Specific example that was enquired on with PST_{12}.

The PST was asked about the example in Snippet 7.4, and their responses to the two questions are captured in Excerpt 7.6

> *Question 1- PST_{12}: Learners must be able to calculate the angle of inclination even when it is not on the cartesian plane.*

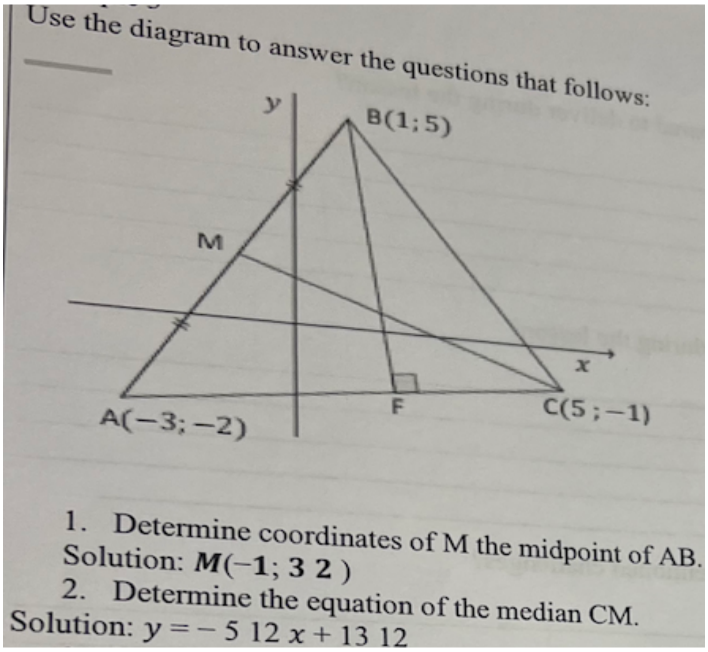
Question 2 - PST_{12}: It helped them remember acute and obtuse angles because they are important when you calculate the angle of inclination.

Excerpt 7.6 Responses of PST_{12} on the purpose of a specific example

PST_{12} was intentional when they selected these examples. It was intended to help learners know when the angle of inclination is acute or not. Most of the other PSTs expressed that the examples were to check if learners understand. This was without picking or pointing to specific aspects of a concept.

When asked about the example in Snippet 7.5, PST_{11} said:

"The example is to prepare learners for tests because they get questions like this."



Snippet 7.5 Specific example that was enquired on with PST_{11}.

The response from PST_{11} seemed to present an idea of formative assessment that agrees to the argument that is presented by Shepard (2017) that some people's or educators' view of formative assessment is better described as warning for summative assessment instead of authentic formative assessment.

Based on the analysis of the way the PSTs responded to the interview questions that were meant to highlight the purpose of the examples, it is evident that these PSTs could not think about connecting concepts. They mostly referred to concepts individually without linking them to others, which is crucial for analytical geometry.

7.4 Theme 3 Use of language in the selected examples

Considering that the study involves participants who are training to teach mathematics in multilingual classrooms, and practice and will teach in multilingual classrooms, the issue of language cannot be overlooked. Many responses to the different questions hint on the issue of language and the way the PSTs tried to accommodate learners.

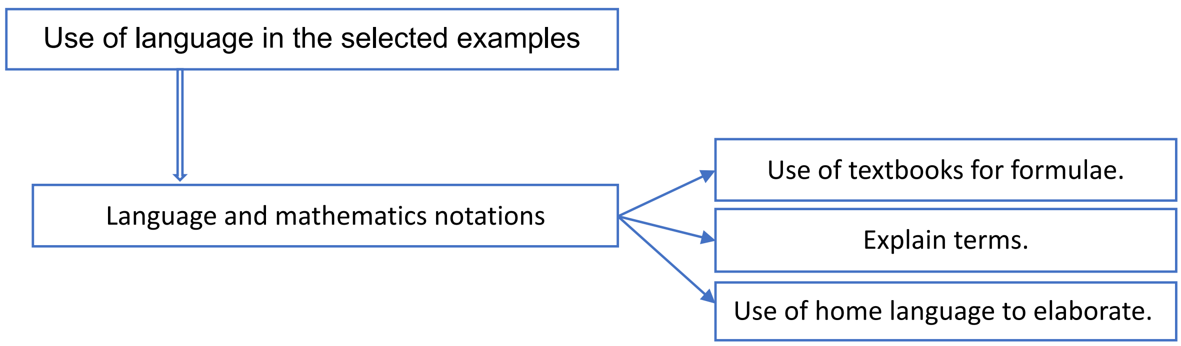
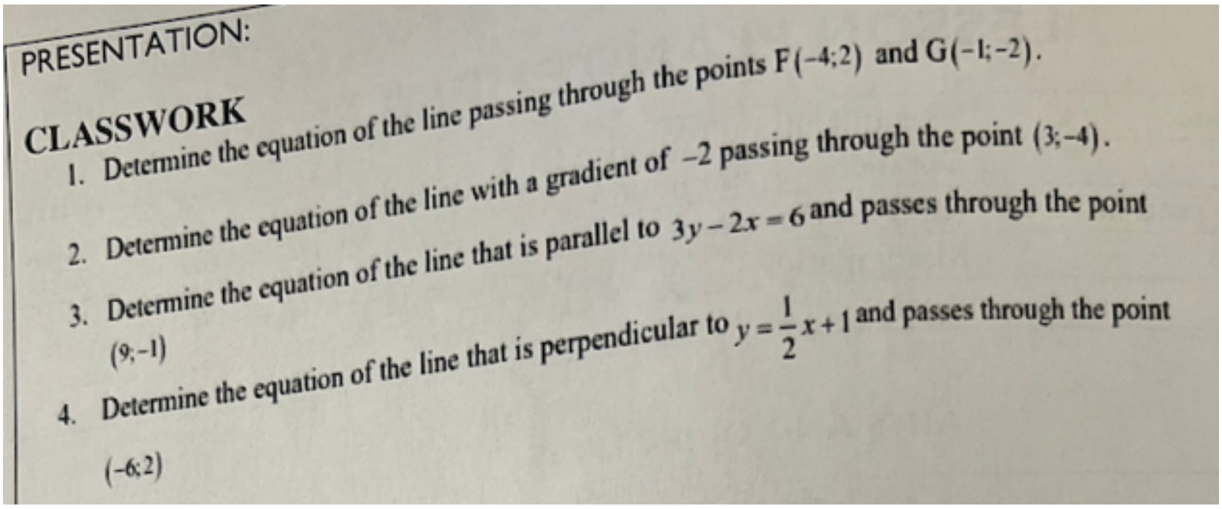


Figure 7.3 Theme 3 - Use of language in the selected examples

Language plays a pivotal role in connecting words and numbers to facilitate meaning making for problem solving in mathematics. This theme looks at the use of language and mathematics notations in the selected examples. The main question that was about language in the examples was:

> *How did you choose the words in your examples so that learners understand the language used?*

It is quite evident from the lesson plans and analysis in Chapter 6 that the PSTs did not necessarily think about the words in the examples because they adapted all their examples from textbooks and selected them as they are. The example in Snippet 7.6 and the PST's intentions attest to this.



PRESENTATION:

CLASSWORK

1. Determine the equation of the line passing through the points $F(-4;2)$ and $G(-1;-2)$.
2. Determine the equation of the line with a gradient of -2 passing through the point $(3;-4)$.
3. Determine the equation of the line that is parallel to $3y-2x=6$ and passes through the point $(9;-1)$
4. Determine the equation of the line that is perpendicular to $y=\frac{1}{2}x+1$ and passes through the point $(-6;2)$

Snippet 7.6 Specific example that was enquired on with PST$_{14}$.

The following statement was provided by the PST as the purpose of this example.

> *For learners to see that they can calculate the equation of a line by using one point and gradient, or two points, or another equation and a point*

When it came to choosing words in their selected examples, the PST said:

> *I chose this example because ibhalwe ngamagama (it is written in words) to explain the problem, not numbers kuphela (only)*

This is just one example of how poorly the question was answered. Other PSTs were clear in saying they did not choose the words. This prompted a follow-up question for which the responses are shown in Excerpt 7.7. The question was:

> *How do you help learners understand the mathematics language notations?*

> *PST$_1$: I start by explaining and revising the big words.*

PST_7: Show how to substitute and use formulae.

PST_{11}: Sometimes I explain and show them what a word means by using their home language, and they understand.

Excerpt 7.7 Responses of PSTs 1, 7, and 11 on the choice of words in their example

The PSTs touched on explaining terminology, formulae, and codeswitching to facilitate the understanding of mathematics language and notations. However, it is quite alarming that most of them admitted to not thinking about the words in the examples because they get them from textbooks. This is so because according to Ronda and Adler (2017), some textbooks provide little access to formal mathematical discourse. For this reason, they add that teachers should be aware of what is afforded and limited in textbooks, be it is examples, use of words, or tasks, so that they could supplement the textbooks accordingly. The explored responses in this chapter indicated that the PSTs had limited knowledge when looking at it in terms of the PCK components of the MTSK model. This is further discussed in the next chapter where all the findings made in the three data presentation and analysis chapters are discussed.

CHAPTER EIGHT: DISCUSSION OF FINDINGS

8.1 Introduction

The purpose of this study was to investigate the role of PCK in the PSTs' choices of examples, and whether it informs the structure of those examples. It was also to investigate how the selected examples created opportunities for learning in multilingual analytical geometry classrooms. To fulfil this purpose, data analysis focused on the key aspects of PCK in the PSTs, and their process of selecting examples. The examples they selected for teaching the topic were analysed to check for patterns of variation and to understand their purpose in the lessons. In this chapter, the findings and observations made in the three preceding chapters are explored and consolidated to make conclusions about the knowledge that guides the PSTs in their selection of examples. The steering theoretical framework and the main research questions are employed to make guided conclusions.

The theoretical framework of this study was inspired by an argument made by Al-Murani et al. (2019) and other scholars in mathematics education who argued that the variation theory is a suitable framework for analyzing examples for teaching mathematics. Therefore, the framework of this study is an integration of the patterns of variation to analyze the structure of the selected examples, and the PCK born MTSK model to investigate the PSTs' PCK and its role in the selection of the examples. The merger PCK – PV framework is presented again in Figure 3.4 from Chapter Three, which is now labeled Figure 8.1 for ease of reference

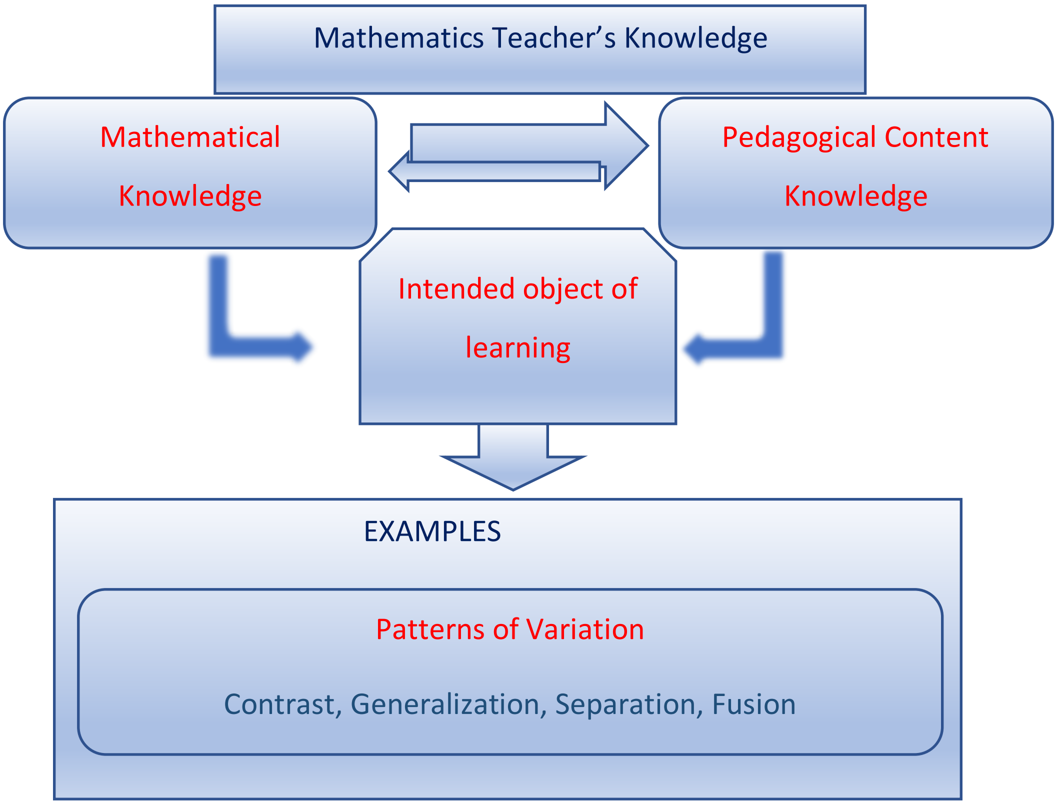


Figure 8.1 PCK – PV Integration of PCK and the Patterns of Variation framework

The notion of this merger framework is in line with the argument that mathematical knowledge alone is not sufficient for teaching, PCK is also necessary for one to teach effectively (Güle and Çelik, 2021). Therefore, the view is that mathematical knowledge and PCK work together in informing one's ability to formulate lesson objectives, which are a major part of the intended object of learning. In turn, the lesson objectives determine the types of examples to be used, which essentially guides the choice of examples for teaching mathematics. These examples can then be described using the patterns of variation, which make a fitting device for structuring and analysing examples.

The PCK – PV framework was also the basis of the analytical framework of this study, whose aspects have been used in the preceding data analyses chapters. In this chapter, all the aspects and phases of the analytical framework are drawn from, and the main research questions guide the discussion of the findings. The main research questions, are:

1. What pedagogical content knowledge do preservice teachers have for teaching analytical geometry in multilingual classroom?
2. What informs the types of examples preservice teachers choose for teaching analytical geometry in multilingual classrooms?
3. How does the preservice teachers' choice of examples in analytical geometry create opportunities for learning?

The discussion is in three parts or sections, each derived from the research questions. It first explores the PSTs' PCK by discussing their content knowledge together with their ability to think about creating opportunities for learning. The second section looks at their rationale for selecting examples, and lastly, the discussion explores how the selected examples facilitated opportunities for learning analytical geometry in multilingual classrooms.

8.2 PCK for teaching analytical geometry in multilingual classroom.

Teaching mathematics in multilingual classrooms requires deep understanding of content and efficacious pedagogical strategies. These are skills that are crucial for teachers to effectively enact the knowledge of specific content. This section presents the findings on the content and pedagogical knowledge that the PSTs had and used for preparing to teach analytical geometry.

8.2.1 Analytical geometry concepts understanding and knowledge.

The content knowledge of the PSTs was assessed in an analytical geometry test that covered most of the aspects of the South African curriculum as stipulated in the CAPS document. The diagnostic report in Chapter Five highlighted the content areas that need further development in the PSTs. These key problematic areas seem to match the reported misconceptions that the 2019 and 2020 matriculants had as indicated in the NSC Diagnostic Report (DBE, 2019/20). This period coincides with the years in which most of these PSTs would have been in the Grades 11 and 12. Essentially, this is the same cohort.

The years 2019 and 2020 were also the years in which the world was grappling with the Coronavirus pandemic, and "emergency remote education" (Valverde-Berrocoso et al., 2021: 1) was introduced. This was different from distance learning. Teaching and learning in South Africa had to take place online, which was a challenge for both basic and higher education because many schools are in rural areas where there is no access to technological tools for teaching and learning. Although this challenge seemed to be better in higher education because some students were provided with laptops and learning cellular data, there were still some challenges with network coverage for students in rural areas.

This led to many classes being missed, and mathematics being taught to test because at the end of all the interrupted years, examinations were still administered. Evidence of this group of PSTs having been taught to make it through examinations in their secondary schooling is in the fact that they got admitted into a higher education institution to specialize in mathematics teaching. Clearly, they passed mathematics well enough for the program, but their responses to the test and interview questions revealed that they did not have conceptual understanding, specifically in analytical geometry. This was seen in the way they performed well in questions that simply required routine procedure, and the performance drop in questions that required explanation and connection of concepts (see Table 5.3 and Figure 5.1).

The sub-question 1.1 required the PSTs to determine the gradient of straight-line AC given the coordinates of the two endpoints. They had to merely substitute into the gradient formula and attain the gradient. In sub- question 1.2 the equation of the straight-line was sought. Again, the standard form of a straight-line was used and so the performance in these questions was 100% and 95%, respectively. Almost all the PSTs had correct answers to these types of questions. This could imply that school learners will benefit from the PSTs in these type of calculations in Analytical Geometry. This claim is made in accordance with a statement made by Bjerke and Solomon (2020) that there is a strong correlation between the mathematics teachers' quality of mathematics knowledge and their ability to create opportunities for learners to learn the subject. For this reason, Gule and Celik (2018) highlighted and emphasised that content knowledge is an essential component for mathematics teaching.

As already indicated, this particular group of PSTs had their learning interrupted from high school and into university. The missed content could not be covered extensively in their teacher training because of time and curriculum constraints (Alex 2019). Bosica et al. (2021) also pointed out that the amount of knowledge that PSTs should acquire for mathematics teaching is a lot compared to the amount of time they have in the teacher training programs. This is in agreement with Kaiser and König (2019) who explained that the knowledge that is needed by mathematics teachers is inclusive of deep content knowledge and knowledge for teaching. Therefore, teacher training cannot only focus on content knowledge development and closing the gaps, even though Lowrie and Jorgensen (2016) and Yet et al. (2021) expressed their concerns about the focus on mathematics PSTs' content knowledge being suppressed by pedagogical content knowledge in research. This concern is justified, and more research on how PSTs could be further developed in content is essential, but this needs serious consideration of time and curriculum coverage to ensure that the PSTs have all aspects of MTSK.

The PSTs in this study had major challenges in working out questions of higher-level demand where calculations required procedures with connections between different mathematical concepts. Sub-questions 2.2.3 and 3.4 were the most poorly answered questions with 15% and 17% average performance per sub-question, respectively, because they required deep understanding of concepts. Sub-question 2.2.3 shown in Figure 8.2 simply needed one to have clear understanding and a visual idea of properties of a circle. This question did not need much use of procedures as the response relied on previously worked out calculations of the radii and manipulations of the equations of the two circles in question.

2.2 Two other circles are given:

- One has centre K, and equation $x^2 - 6x + y^2 + 4y - 12 = 0$.
- The other has centre T, and equation $(x-12)^2 + (y-10)^2 = 100$

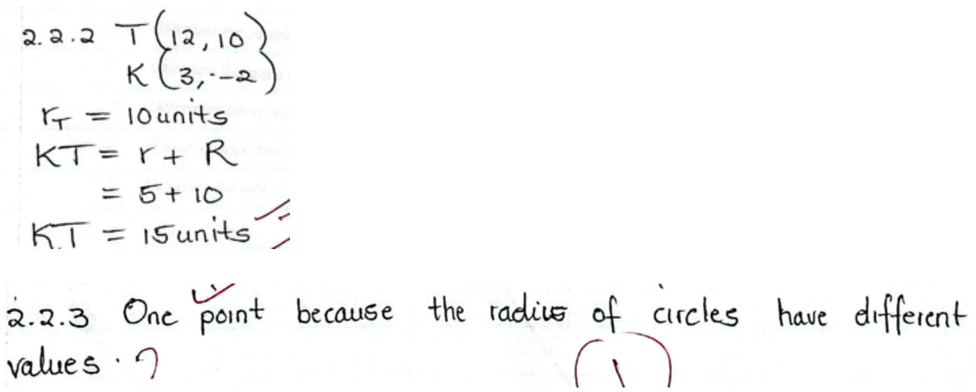
2.2.1 Determine the centre and radius of the circle with centre K. (4)

2.2.2 Calculate the distance between the centres, K and T. (2)

2.2.3 At how many points do the two circles intersect? Motivate your answer. (2)

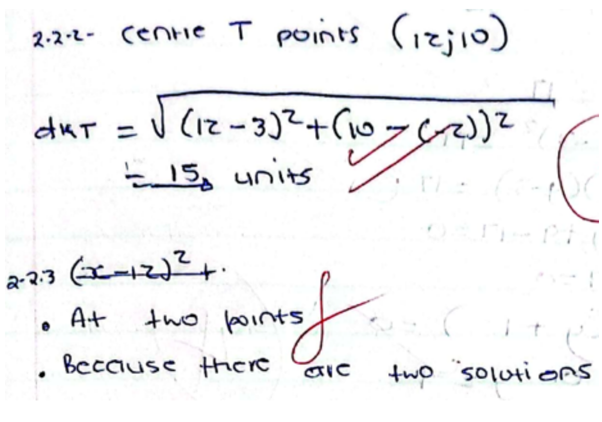
Figure 8.2 Sub-questions of Question 2.2.

Snippets 8.1 and 8.2 show some of the responses of the PSTs who got correct answers for sub-questions 2.2.1 and 2.2.2 but could not make a connection between the distance between the centres and the sum of the radii lengths. This mishap is related to their inability to visually imagine the two circles as Makamure and Jojo (2021) explained that PSTs have a deficiency in visual-spatial reasoning.



Snippet 8.1 Sample 1 response to question 2.2.3

Snippet 8.1 indicates a misconception that circles with different radii lengths will always touch at one point. Again, this PST could not visually imagine this scenario in different situations.

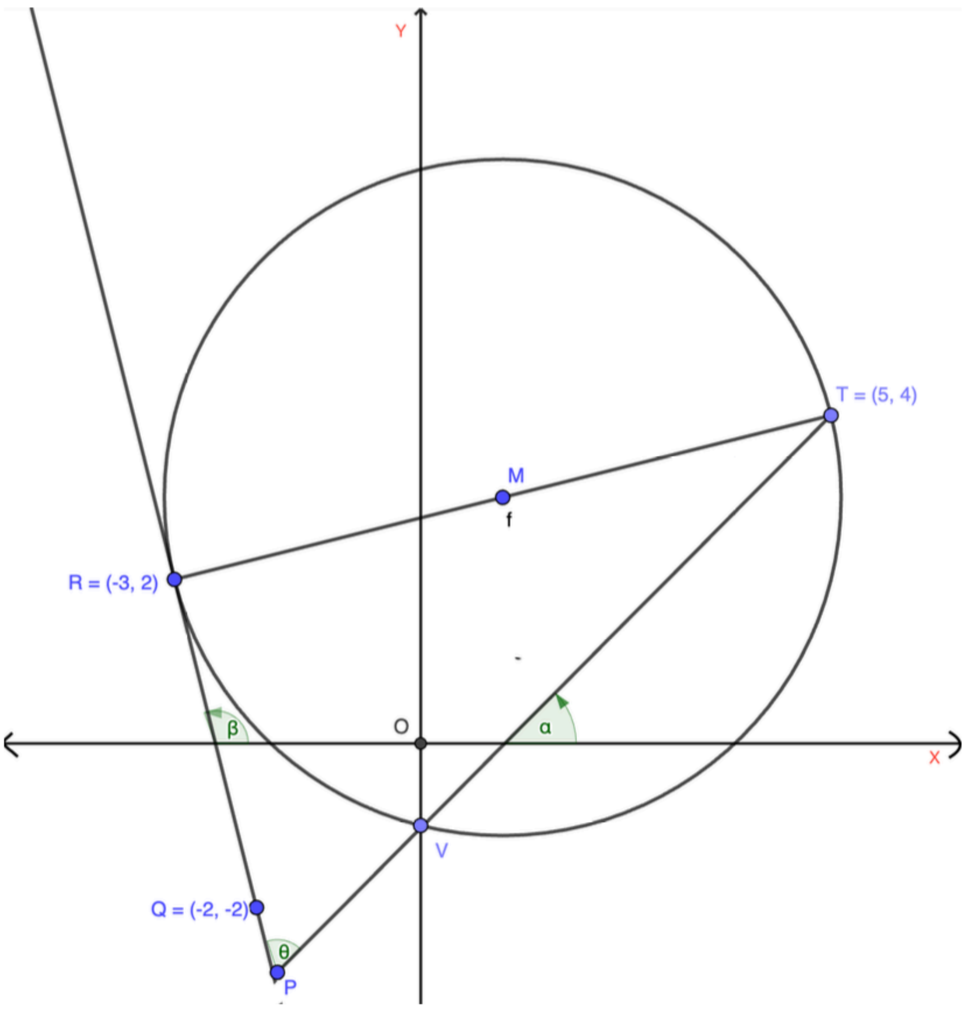


Snippet 8.2 Sample 2 response to question 2.2.3

In Snippet 8.2, the PST got the distance between the centers correct and could not link this to find the solution to sub-question 2.2.3. The PST was not available for interviews in which I needed to understand what they meant by the "*two solutions*" in their response.

These responses concur with claims made by researchers in the literature review where Murtafiah et al. (2018) argued that most PSTs can only rely on descriptive explanations when explaining or dealing with mathematics problems that require some sort of explanation. They are able to explain the "how" more than the "why" and "what", but unable to give reasons and provide interpretative explanations. According to Hargie (2006) reason giving and interpretative explanations mean responding to the "why" and "what" respectively.

Sub-question 3.4 in Figure 8.3 was about calculating the angle between intersecting lines using the angle of inclination. This sub-question was the least attempted with 11 of the 20 PSTs leaving it blank, and 5 of the remaining 9 got zero on the attempts. Only three out of the 20 PSTs got the full seven marks, and one PST got three out of seven in this sub-question. This is directly related to the choice of the PSTs' examples for teaching about the angle of inclination, which was observed in Chapter Six. Most of them chose basic examples and avoided dealing with examples with complex calculations, see Snippets 6.2 and 6.3. This observed relationship reveals how the PSTs content knowledge affected their ability to choose examples to create opportunities for learning.

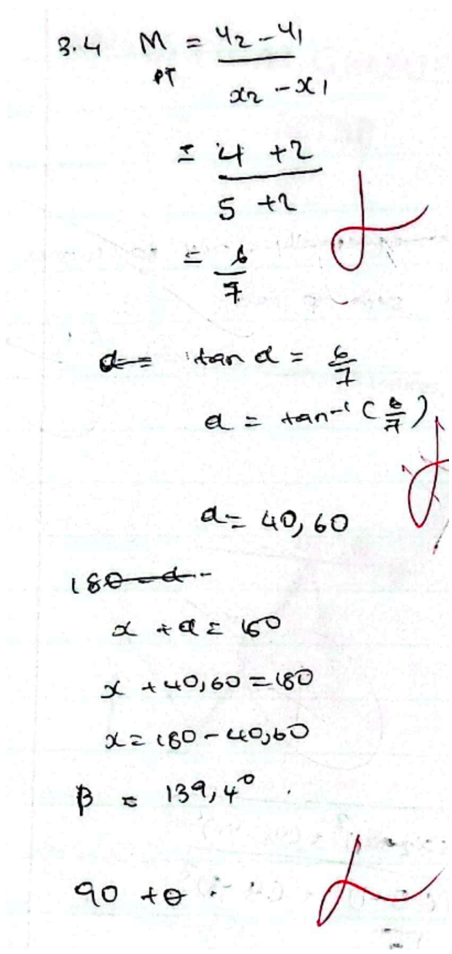


3.4 If $R\hat{P}T = \theta$, calculate θ to ONE decimal place. (7)

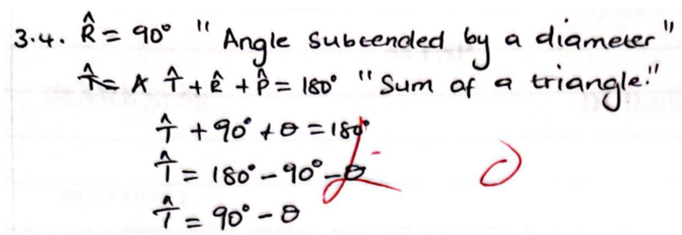
Figure 8.3 Sub-question 3.4 and the diagram for calculation

Different approaches could have been used to answer this question, but the PSTs failed to do so because of misconceptions and the lack of understanding of basic concepts. Snippet 8.3 shows some of the incorrect attempts and misconceptions.

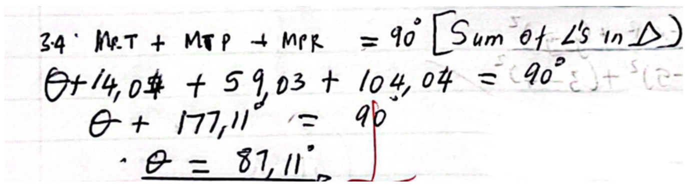
A.



B.



C.



Snippet 8.3 Sample of incorrect responses to sub-question 3.4.

Most of the PSTs who had incorrect attempts used different approaches. In Snippet 8.3A, the PST tried using the angle of inclination, but did not use the gradient of TV instead of PT as the coordinates of V were already calculated in one of the questions before sub-question 3.4 and those of P are unknown. This PST could not link this problem to the concept of collinear points and the rest of the solution shows that they did not know what to do next. In an attempt to use circle geometry, the PST in Snippet 8.3B incorrectly applied the 'angle subtended by the diameter', which does not only expose a misconception, but also the lack of content knowledge. This is because $\hat{R}$ is indeed 90^0 because of the tangent radius theorem, and in sub-question 3.2, the PSTs had to prove that PR is a tangent at R by using analytical methods showing that RT is perpendicular to RQ. Snippet 8.3C shows another case of a lack of content knowledge because the PST uses incorrect notation and adds gradients instead of angles. The PST does not seem to have adaptive reasoning (Kilpatrick, Swafford, and Findell, 2001) as they calculated and wrote that $\theta + 177{,}11° = 90°$ where $\theta = 87{,}11°$.

As already noted, the lack of analytical geometry content knowledge affected the PSTs' ability to choose and select examples for teaching the topic. According to the observed issues in content, the PSTs did not have KoT and KSM (Carrillo-Yañez et al., 2018). The lack of KoT was seen in the way they could not use analytical geometry concepts to solve problems like using the angle of inclination to find the angle between two intersecting lines. According to Carrillo-Yañez et al. (2018) a mathematics teacher should have knowledge of inter and intra conceptual connections in mathematics teaching, and as much as the PSTs try to make connections, this is still done incorrectly. For instance, in Snippet 8.3B the PST tried to use circle geometry, but they failed to do so and this indicates a lack of KSM.

8.2.2 Knowledge for teaching analytical geometry in multilingual classrooms.

According to Jones (2003) analytical geometry can help learners in linking different areas of mathematics, and for this reason, teaching the topic in multilingual classrooms requires a teacher to consider approaches that will cater for the diverse learners in the classroom. This includes choosing and using examples that encourage learners to engage with concepts by drawing diagrams or making extensions to provided diagrams and explaining their solutions. It also entails using language in examples to facilitate meaning making. This study explores the role of PCK in the PSTs' choice of examples for teaching multilingual analytical geometry classrooms, which requires an exploration of the use of language in examples.

As Planas et al. (2018) explained that research on language in mathematics classrooms must be about 'the language/s of the learners', 'the language of the teacher and the classroom', and 'the language of mathematics'. In this study, the focus was on the language of the PSTs and classroom, and the language of mathematics. The language of the teacher and classroom is a facilitator of the learners' understanding of language of mathematics. Hence, the main question that guided the exploration of the use of language by the PSTs in this study was:

How did you choose the words in your examples so that learners understand the language use?

The responses exposed the discomfort that the PSTs have with the language of the classroom (LoLT), English in this case. The response from PST_{14} in Chapter Seven, theme 3 showed this in the way they quickly switched to IsiZulu their home language for words they know in English. The response was "I chose this example because ibhalwe ngamagama (It is written in words) to explain the problem, not numbers kuphela (only)". This switch was a trend in most of the PSTs' responses, especially when they had to elaborate. In this regard, it is crucial to retell the context and background of the PSTs in this study as they are mostly from rural areas that predominantly speak IsiZulu.

Beyond the PSTs' issue of the LoLT command, the response also indicates that they do not think about the language and structure of examples as they see it already done on textbooks. Most of the responses triggered a follow-up question, which was:

How do you help learners understand the mathematics language notations

Chapter seven shows how the PSTs resorted to codeswitching and translanguaging without being cautious of it because none of them mentioned the two terms in their responses. Also, the PSTs were not cognizant of the mathematics register, as their responses show how they treat terms, formulae, and notations instrumentally without conceptual explanations. They did not mention how they encouraged learners to use mathematics terms and notations, especially in writing. Again, this could also be because they did not have deep knowledge of the register as they also use some of the terms and notations incorrectly. One instance of this can be seen in Snippet 8.3B, where the PST wrote $\hat{R} = 90°$ "angle subtended by a diameter". In this case, RT is the diameter (see Figure 8.3), and so the PST seems to have thought that an angle subtended by a diameter is the angle formed when a diameter intersects with another line.

This takes me back to my trigger questions in the literature review.

- *Are the mathematics PSTs cognisant of the grammatical systems involved in the mathematics register as they prepare to teach in multilingual classrooms?*

- *Are PSTs taught to learn and teach the mathematics language?*

These questions were triggered by an argument made by Sibanda (2021) that STEM subjects are unique languages on their own and learning to read them is not overtly and systematically taught, and teaching to read the language is not supported by research. In response to the first question, the data in this study shows that the PSTs were aware of the mathematics register, but they did not understand it to think about it carefully when preparing to teach. This can also be seen in Snip 8.4B where the PST wrote 'sum of a triangle', which is clear from the calculation that they were referring to angles, but unfortunately, the statement does not capture this.

At this point, the response to the second question would be superficial because it is difficult to tell exactly what PSTs are taught in all of their training. Concurring with this argument Althauser (2018) explained that what takes place within the "Black box", that is what PSTs are actually taught is a minimal discussion. This discussion needs to be broadened, especially for PSTs in multilingual teacher training classrooms. Karisan et al. (2019) also enlightened that lasting change in mathematics classrooms relies on carefully guiding PSTs when it comes to thoughtfully considering the outcomes of their pedagogical choices. The PSTs in this study did not seem to be able to connect their example choices or lessons structure to what they intended for their learners to learn, and this is because they did not have deep KoT and KSM as already established.

The quality of mathematics knowledge is strongly correlated to pedagogical choices, and according to Bjerke and Solomon (2020), it is related to the teachers' ability to create opportunities for learning. This is seen in the analysis of the lesson plans. The most common observation was that the PSTs selected examples that required basic calculations and hardly any explanation, which also aligns with their performance in questions that required some sort of explanation in the test. Furthermore, the PSTs did not use examples that encouraged learners to make diagrams for visual explanations, and this is also in line with their inability to visualize the solution to sub-question 2.2.3 as shown in Snippets 8.1 and 8.2. Here, the PSTs showed insufficient KMT.

There is also evidence of the paucity of KFLM and KMLS in the PSTs as they mostly relied on formative assessments as a technique to gauge learners' understanding. This was not done to inform or guide their teaching methods as some of the PSTs indicated in the interviews that they selected their examples to assess understanding in preparation for tests. The PSTs did not consider learning outcomes of the topic as stipulated in the CAPS document or even adopt teaching strategies from their training to facilitate learning. When they were asked about their knowledge for selecting some of the examples. The responses revealed that the PSTs made little considerations of the types of learners they are working with besides knowing their Grade. Therefore, one could conclude that this group of PSTs had little knowledge for teaching, especially analytical geometry in multilingual classrooms.

8.3 The rationale for choosing examples to teach analytical geometry in multilingual Classrooms.

Examples are the main pedagogical tool for teaching mathematics (Codes and Contreras, 2023). They are very useful in the teaching and learning of analytical geometry because they help in visualising abstract concepts, and showing their application in real life. Learners in multilingual mathematics classrooms rely mostly on examples for the understanding of concepts and mathematical discourse. It is crucial for PSTs to be purposeful in their selection of examples because just as much as they assist in facilitating learning, they can also hinder the process if not selected and used carefully.

In the interviews, most of the PSTs indicated that they did not necessarily think about the structure and sequence of the examples and the reasons for choosing them because examples are readily available in textbooks. This suggests that the PSTs did not have the ability to introduce spontaneous examples (Rowland, 2008) in class, which are essential to further elaborate or illustrate a concept in mathematics. It also endorses an argument made by Zodik and Zaslavsky (2008) that the ability to generate effective examples is dependent on the knowledge of mathematics, the knowledge of the student's learning abilities, and PCK. The study has established from the previous sections that some of the PSTs in question lack these forms of knowledge.

Depending on textbooks for examples made the PSTs think less about the purpose of their examples and how language is used in the examples to help learners understand. The PSTs confirmed this in responses that indicated that they thought all examples are ready to use, which again shows no consideration of the type of learners they were working with. Here, 'noticing' as proposed by Wilson et al. (2017) is one of the professional learning processes that could enhance the PSTs' expertise on how to be cognisant of the learners' different cultures and learning abilities. In line with this claim Guler and Celik (2021) argue that professional development approaches that focus on real classroom reflections can improve PCK and the knowledge of learners in PSTs.

The PSTs' rationale for choosing their examples also exposed their deficit KFLM which is essential in creating opportunities for learning. One other instance of this is seen in the way PST_{13} used examples that did not match the lesson objective on the lesson plan (see Snippet 8.5), which was:

By the end of the lesson, the learners would be able to calculate gradient of a line

It was expected that these PSTs would choose the ideal examples to provide to their learners for effective learning and achievement of this objective. This is assumed as all PSTs demonstrated in sub-question 1.1 that they could find the gradient of a straight-line (see Figure 5.1). But this assumption is proved incorrect when one observes the choice of example in Snippet 8.5. The PST here actually chose an example that tests the application of the gradient of a straight line. One would think that PST_{13} would have scaffolded an example by asking firstly to calculate the gradient of the straight-lines AB and CD and then asked if these lines were parallel. Furthermore, according to Norton (2019), mathematical content knowledge is a basis for mathematical pedagogical content knowledge and so just knowing how to find a solution does not guarantee that a teacher would have necessary skills to use analogies, representations, illustrations, and explanations that effectively create opportunities for learning (Shulman 1986). This justifies the need for teacher development at tertiary institutions as teaching is therefore not an "easy" profession.

Calculate the value of x in each case if A, B, C and D are the points $A(3;4)$, $B(-1;7)$, $C(x;-1)$ and $D(1;8)$ and:

(a) $AB \parallel CD$

(b) $AB \perp CD$

(c) B, C and D are collinear

Snippet 8.4 Revisiting Snippet 7.1.

The PST used the examples in Snippet 8.5, yet the lesson objective suggests that the lesson would be about learning to calculate the gradient. When asked what the purpose of this example was, the response was, "I *thought it would help me remind them about gradients of perpendicular lines, parallel lines, and collinear points. I did this example with them in class"*. In the last part of the response, the PST reveals that they think learning took place because they did the example in class. However, they do not realise that the examples suggest that calculating the gradient is prior knowledge in this lesson, which is indeed true because it was covered in Grade 10. This also shows lack of KMLS because the level at which the content is pitched does not match the national standard as per the CAPS document.

8.4 How the choice of examples facilitated opportunities for learning analytical geometry.

As Ronda and Adler (2017) explain it, textbook examples do promote learning, but this also depends on how the teacher uses them as there are many considerations to make depending on the context of the lesson. It is crucial to reiterate that this study was not intended to discuss and analyse the textbook lesson examples, but the PSTs choice of examples and how they are ordered to promote learning. However, in the study the explored examples were mostly adapted from textbooks.

Owing to Kullberg and Skodras (2018), and other mathematics education scholars, patterns of variation were used to understand and describe the PSTs' selected examples, and how they could have promoted learning. In this analysis, it was observed that the PSTs selected examples which mostly had one pattern of variation, and hardly two or more in one set of examples.

As much as they did not necessarily learn about the variation theory to think about the patterns of variation when selecting and structuring examples, the PSTs had many other theories to draw from when preparing lessons. Thinking about the background and learning context of the learners would also have helped the PSTs in having features that could promote learning in their examples. At this point, the selected examples and responses to interviews questions showed that the PSTs lacked the KFLM as an aspect of PCK in the MTSK model. They did not seem to consider or link the theories they learned in their training to prepare lessons that cater for diverse learners in multilingual mathematics classrooms.

The examples were mostly examples that facilitated routine procedure (R) as shown in Table 6.3 which shows the cognitive demands of the examples that were selected by the PSTs. This shows that the PSTs were teaching for procedural understanding rather than conceptual understanding. According to Stein et al. (2009), tasks must be created and presented at different levels in order to stimulate and facilitate different levels and kinds of learners' thinking. The PSTs did not consider this because they shy away from using examples that require complex procedures and problem solving. This corresponds with their performance in the test in which they did well in calculations that did not require procedures with connections. The PSTs in this study lack KoT and KSM, which are two of three components of mathematics content knowledge in the MTSK model. They mostly know procedures without connections, which does not help in teaching because it does not develop or improve teaching skills (Norton, 2019).

This chapter referred to the research questions, literature review, and the theoretical framework to make informed discussions of the findings that were made in the three data presentation and analysis chapters. The next chapter draws from this discussion and other aspects of the study to respond to the research questions and make conclusions, and present the limitations of this study and recommendations that emanate from this study.

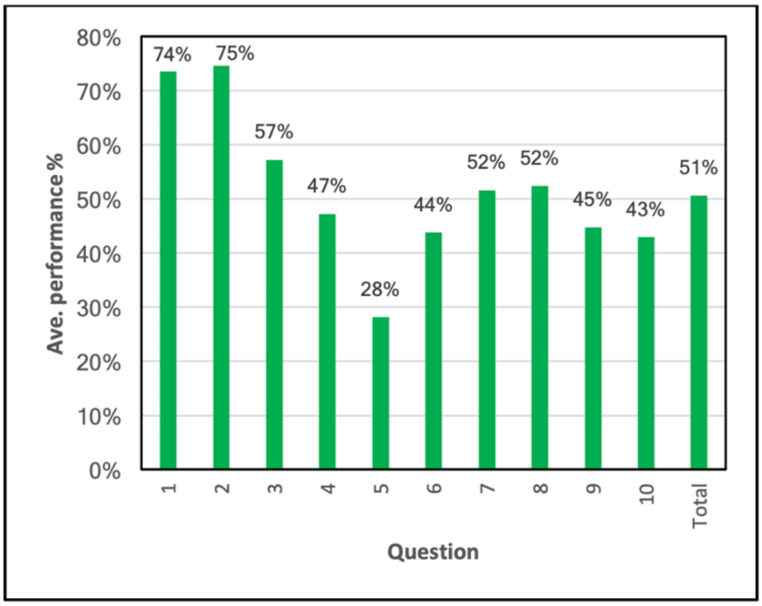
CHAPTER NINE: CONCLUSION, LIMITATIONS OF THE STUDY, AND RECOMMENDATIONS

9.1 Introduction

Chapters Five, Six, and Seven focused on the analysis of the PSTs' content knowledge, choice of examples, and the pedagogical choices they made when selecting examples. In Chapter Eight, the discussion of the findings that were made in the three analysis chapters drew from the framework and leads to this chapter. This chapter refers to all the findings and other chapters to make conclusions, respond to the main research questions, and explore the limitations of the study, and recommendations from the study. The response to the research questions will focus on each question individually, thus making each question a section. To do this, the chapter will also revisit the context and purpose of the study.

9.2 Revisiting the purpose of the study.

The purpose of this study was motivated by calls for more research in multilingual PSTs' classrooms (Thompson et al. 2016; Rangnes and Meane, 2021), which directed the study to explore the PSTs' knowledge for teaching in multilingual classrooms. The study was also motivated by the performance and understanding of analytical geometry by the South African learners. Figure 9.1. shows the average performance in the two analytical geometry questions written in the year 2020, which was 52%.



Q	Topic/s
1	Data Handling
2	Data Handling
3	Analytical Geometry
4	Analytical Geometry
5	Trigonometry
6	Trigonometry
7	Trigonometry
8	Euclidean Geometry
9	Euclidean Geometry
10	Euclidean Geometry

Figure 9.1 Average performance of the 2020 matriculants in paper 2

This average performance also exposes the learners' understanding of other topics in mathematics because analytical geometry connects measurements, space, shape, trigonometry (Bansilal and Naidoo 2012), and calculus (Campuzano and Crisanto, 2022). The learners' understanding of the topic is directly related to the teacher's content knowledge and understanding because this is an essential component of teaching (Gule and Celik, 2018). On that note, the average performance in analytical geometry suggests a need for a probe into the teachers' knowledge of the topic. However, in-service teachers are constantly developed, and their PCK improves because of professional development (Sakaria, Maat, and Matore, 2023).

At this point, professional development seems to bridge the existing gap in teachers' knowledge. This gap exists because PSTs did not get enough time and experience in actual classrooms for them to learn to implement instruction that caters for all learners (Lee, Han, Kim, and Herner-Patnode, 2021), and create opportunities for learning. They should gain more of this knowledge through experience, research on their teaching, and participating in professional learning communities (Brijlall 2014; Evens et al., 2017). Hence, this study explored the PCK that PSTs have, and how it guides their choices of examples for teaching analytical geometry in multilingual classrooms. This may inform relevant stakeholders on appropriate measures to further develop this group of PSTs, future PSTs, and the same cohort of teachers. Backing this idea, Rocha (2020) reasoned that knowing what needs to be done to facilitate professional development from the beginning of the teacher training program in PSTs, is crucial.

9.3 Addressing the context and research questions.

Focusing on the role of PCK on the choice of examples stems from examples being the main instructional tool that facilitates learning in mathematics (Cayo et al., 2023). The multilingual aspect is based on the context of the study, and the intention to add more informed knowledge to research on factors that contribute to the challenges faced in multilingual mathematics classrooms. The context of this study was in an institution that is located in KZN, and according to the EMIS data in Table 9.1, of all South African provinces, KZN has the greatest number of schools, and in the year 2022, the province had the greatest number of learners and teachers. KZN is also one of the most rural provinces with a predominantly African population (See Figure 4.1).

Table 9.1 Number of schools, teachers, and learners per sector and province in South Africa in 2022 (Source: www.education.gov.za)

Province	Public			Independent			Public and Independent					
	Learners	Educators	Schools	Learners	Educators	Schools	Learners	As % of National Total	Educators	As % of National Total	Schools	As % of National Total
Eastern Cape	1 751 496	57 509	5 046	74 654	3 895	265	1 826 150	13.6	61 404	13.6	5 311	21.4
Free State	707 664	23 114	946	21 180	1 229	83	728 844	5.4	24 343	5.4	1 029	4.1
Gauteng	2 254 391	71 966	2 056	357 250	21 487	935	2 611 641	19.5	93 453	20.7	2 991	12.0
KwaZulu-Natal	2 822 526	91 298	5 801	60 828	4 352	220	2 883 354	21.5	95 650	21.2	6 021	24.2
Limpopo	1 715 130	50 599	3 646	83 516	3 993	201	1 798 646	13.4	54 592	12.1	3 847	15.5
Mpumalanga	1 109 466	35 119	1 649	35 821	2 460	135	1 145 287	8.5	37 579	8.3	1 784	7.2
Northern Cape	299 014	10 127	545	7 042	540	41	306 056	2.3	10 667	2.4	586	2.4
North West	847 044	27 232	1 448	28 051	1 691	100	875 095	6.5	28 923	6.4	1 548	6.2
Western Cape	1 178 155	38 662	1 452	66 743	5 720	302	1 244 898	9.3	44 382	9.8	1 754	7.1
South Africa	**12 684 886**	**405 626**	**22 589**	**735 085**	**45 367**	**2 282**	**13 419 971**	**100.0**	**450 993**	**100.0**	**24 871**	**100.0**

The institution in which this research was conducted receives PSTs from the many rural parts of KZN. Most of these PSTs leave the university area and go back to their homes for work integrated learning, which would mean that most of them get their teaching experience in multilingual mathematics classrooms. Therefore, the conclusions made in this chapter will inform a big part of the South African education. As already mentioned, the conclusion will be based on the responses to the research questions which are addressed in the forthcoming sections.

9.3 1 What PCK do PSTs have for teaching analytical geometry in multilingual classroom?

The MTSK model was used to examine the PCK in the PSTs because Carrillo-Yañez et al. (2018) advise that the framework does not only explore the knowledge that is *used* for teaching, but also examines the knowledge that is *needed*. Thus, with the help of this model, the study was able to establish that the PSTs lacked in many of the components of MTSK, which implies a lack in PCK. The two domains whose subdomains were the basis of analysis are Mathematical Knowledge, and Pedagogical Content Knowledge.

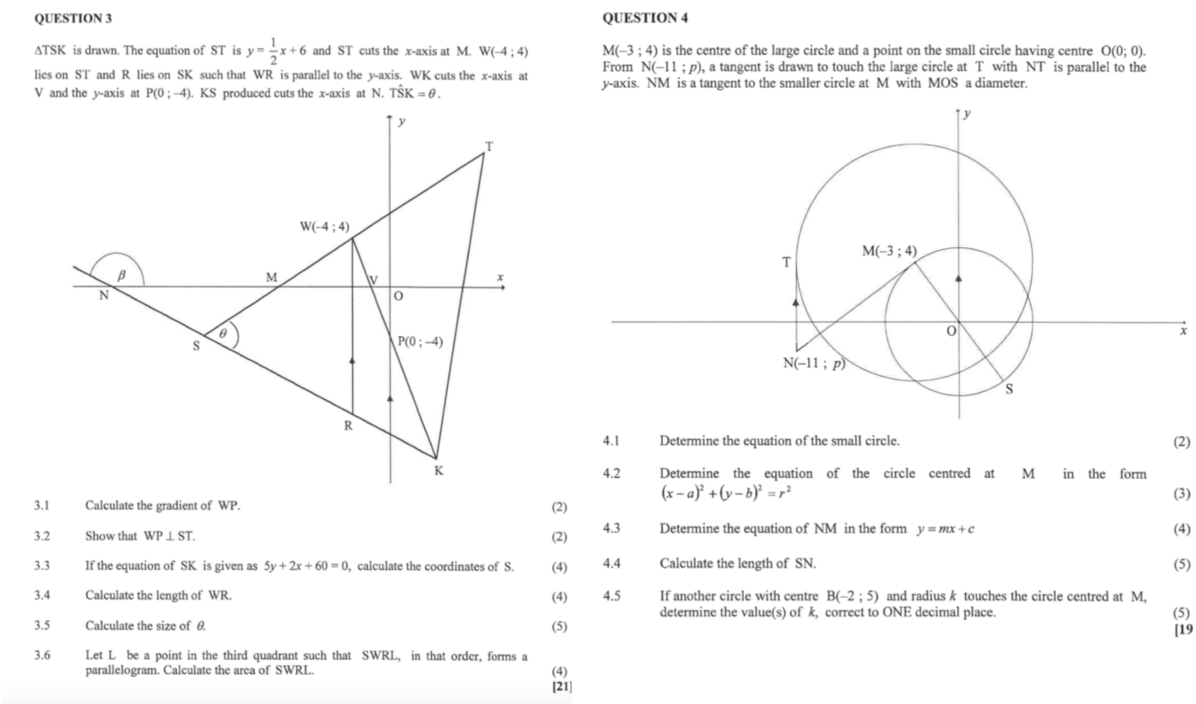
When it comes to the mathematical knowledge, the study established that the PSTs had limited KoT as their test responses confirmed that they had procedural understanding more than conceptual understanding. Further analysis of their responses also revealed that the PSTs did not have KSM and KPM. The lack of KSM was seen in the way the PSTs could not show their knowledge of inter and intra connections in mathematics concepts. In the test they performed well in sub-questions that required calculations with procedures without connections. The sub-questions that required the application of concepts from other topics did not receive the same performance. Evidence of not having KPM was seen in the PSTs lesson plans and examples as none of them had demonstrated how analytical geometry concepts are used in practical applications and to solve real world problems.

It is apparent that most of these PSTs were in the critical grades (11 or 12) or starting teacher training when the Coronavirus pandemic struck. During this time, teaching and learning was online, which Sevimli (2023) argues that the amount of time that is spent on examples in the online learning platforms is shorter than that spent in contact classes. Therefore, these PSTs may have had very limited exposure to mathematics examples for them to develop their mathematical and pedagogical knowledge. The performance of these PSTs in the test corresponds with the performance of the 2020 matriculants in analytical geometry. Looking at the problematic questions in the test, Questions 2 and 3 had a combined average of 45% as shown in Table 9.2.

Table 9.2 Average performance of the PSTs in the overall questions

Question 1	Question 2	Question 3
66	47	43

This average performance corresponds with that of the 2020 Grade 12s in Question 4 shown in Figure 9.2, which was 47% (see Figure 9.1). They also correspond because in both cases, the questions involved the equation of the circle. This may indicate that this gap is still not closed, and the PSTs need to further engage with the content to develop and improve their KoT.


QUESTION 3

ΔTSK is drawn. The equation of ST is $y = \frac{1}{2}x + 6$ and ST cuts the x-axis at M. W(–4 ; 4) lies on ST and R lies on SK such that WR is parallel to the y-axis. WK cuts the x-axis at V and the y-axis at P(0 ; –4). KS produced cuts the x-axis at N. $T\hat{S}K = \theta$.

3.1	Calculate the gradient of WP.	(2)
3.2	Show that WP ⊥ ST.	(2)
3.3	If the equation of SK is given as $5y + 2x + 60 = 0$, calculate the coordinates of S.	(4)
3.4	Calculate the length of WR.	(4)
3.5	Calculate the size of θ.	(5)
3.6	Let L be a point in the third quadrant such that SWRL, in that order, forms a parallelogram. Calculate the area of SWRL.	(4) [21]

QUESTION 4

M(–3 ; 4) is the centre of the large circle and a point on the small circle having centre O(0; 0). From N(–11 ; p), a tangent is drawn to touch the large circle at T with NT is parallel to the y-axis. NM is a tangent to the smaller circle at M with MOS a diameter.

4.1	Determine the equation of the small circle.	(2)
4.2	Determine the equation of the circle centred at M in the form $(x-a)^2 + (y-b)^2 = r^2$	(3)
4.3	Determine the equation of NM in the form $y = mx + c$	(4)
4.4	Calculate the length of SN.	(5)
4.5	If another circle with centre B(–2 ; 5) and radius k touches the circle centred at M, determine the value(s) of k, correct to ONE decimal place.	(5) [19

Figure 9.2 Grade 12 paper 2 Questions 3 and 4 of the year 2020

(Source: www.education.gov.za)

The performance of the 2020 matriculants in Question 3 was also not good, and one of the sub-questions owing to this was sub-question 3.5. The Diagnostic Report (DBE 2020) shows that many of the learners had difficulty using the formula $tan\theta = m$ correctly, and either used incorrect angles or gradients to substitute into the formula. This was also seen in the PSTs' responses to sub-question 3.4. The data and findings from the test also exposed that the PSTs had little understanding of the basic geometric and trigonometry concepts they needed to solve analytical geometry problems. This agrees to Spangenberg's (2021) finding that teachers have low levels of PCK for teaching trigonometry, which is also evident in the 2020 matriculants' performance in Figure 9.1. Due to the insufficient knowledge of trigonometry, the PSTs did not select enough examples that needed the application of trigonometric concepts, which shows that they also lack PCK for teaching the topic.

Not to focus on comparisons, but this gives a clear perspective of the factors that contributed the PSTs mathematical knowledge. Based on their test responses and the examples they selected, which were pitched at introductory level for the Grade, one may conclude that the PSTs did not have enough content knowledge for teaching analytical geometry.

Literature shows that mathematical content knowledge is crucial in the development of PCK (Norton, 2019). Therefore, one can argue that the need for further development of mathematical knowledge in these PSTs suggests that their PCK also needs developing. This was observed in the way the PSTs structure their lesson objectives and select examples. The PSTs did not know how to align the lesson objectives and the examples they intended to use in their lessons. This is in line with an observation made by Lim et al. (2018) that PSTs have difficulties in lesson planning, which exposes their inability to make pedagogical choices.

Pedagogical choices are informed by many things, including the teaching and learning context, and the learners' background. This seemed to be misunderstood as PSTs indicated that finishing a lesson on time and having learners get correct answers to a class activity meant the lesson was effective. In terms of the MTSK model, this indicates a deficiency in KMT and KFLM in these PSTs. They had little knowledge of mathematics teaching, and were not able to identify features of learning mathematics, or apply teaching and learning theories into their lesson planning. As already discussed when looking at the PSTs' mathematical knowledge, the selected examples pitched the content at a low and basic level. Here, the PSTs did not consider the learning outcomes of the topic in Grade 11 signalling that the PSTs also did not have KMLS.

Consolidating the PSTs mathematical knowledge and PCK, the paucity of most of the components of the MTSK model in both domains show that this group of PSTs did not have sufficient knowledge for teaching analytical geometry in multilingual classrooms.

9.3.2 What informs the types of examples PSTs choose for teaching analytical geometry in multilingual classrooms?

The thematic analysis of the interviews with the PSTs uncovered that the PSTs were mostly guided by teaching resources in their selection of examples. Some of the other factors that contributed to their choices of examples were the amount of time they had to complete the topic, and learners' assessments.

In the interviews, the PSTs mentioned the use of textbooks and past examination question papers as a director of the types of examples they should use for the lessons. Textbooks are curriculum resources that are meant to improve the teaching and learning experience, and introduce different pedagogical approaches (Rezat, Fan, and Pepin, 2021). Even so, Rezart et al. (2021) went on to explain that the enactment of the content in curriculum resources is dependent on the way teachers and learners interact with the resource. The responses of the PSTs indicated that they did not interact with the resource much or consider the learners or learning environment when choosing those examples. They took the examples, and their sequence as they were and plugged them into their lesson plans.

The PSTs further indicated that the number of examples they selected was determined by the amount of time they had for the lesson. As much as timing and pacing content is crucial in lesson planning (Zaragoza et al., 2023), sequencing of the content is also vital. Relying on textbooks for sequencing does not take the learners context and different teaching and learning theories into consideration as the PSTs did. The sequencing of their examples was mostly guided by time, and not their knowledge for teaching.

In some cases, the PSTs showed that they did not apply their knowledge for teaching analytical geometry when choosing examples. They claimed that they chose certain examples because assessment dates were approaching, and they had to use examples from past question papers as formative assessment. This set a good example for an argument made by Shepard (2017) that in some cases, formative is not authentic and would be best described as warning for summative assessment. These PSTs seem to be repeating their experiences of the way they were taught the topic or subject, which is teaching to test.

All of this goes back to these PSTs not having PCK for teaching analytical geometry in multilingual classrooms. They did not use PCK in the factors that inform the types of examples they chose. This could have hindered learning because the selection of examples that cater for all learners and their evolving learning needs depends on PCK (Wilson et al. 2014).

9.3.3 How does the PSTs' choice of examples in analytical geometry create opportunities for learning?

As alluded to earlier in the responses to the two questions above, the PSTs selected examples that created limited opportunities for learning. This is owing to their mathematical content knowledge and pedagogical knowledge.

The study also discovered that language played a role in the PSTs content knowledge, which in turn affected their choice of examples. This was observed in the explanation responses the PSTs provided to the test, and how their selected examples in the lesson plans hardly had questions that required explaining. According to del Rosario Zavala (2017), research in mathematics education has documented that PSTs find it difficult to learn the necessary skills to have mathematical discussions, and they must learn to create opportunities for learners of different backgrounds to engage in mathematical discussions. Having learned mathematics in rural multilingual classrooms where codeswitching was often necessary because of the low proficiency in English (Thompson et al., 2016), the PSTs seemed to also find it difficult to use the LoLT and mathematical language in their problem solving in the test.

The PSTs' lived experience of the topic corresponds with the experiences they are creating for learners. This observation aligns with the finding made by Karisan et al. (2019) that PSTs have difficulty in reflecting and adapting practices that are different from what they may have experienced in their schooling. The recurrence of the PSTs experiences is also seen in the levels of cognitive demand of the examples they select. They hardly use examples that require complex procedure or problem solving.

In regard to using the patterns of variation as a tool to describe and analyse the chosen examples by the PSTs, the results show that the chosen examples facilitated learning at the level at which the curriculum resource from which they were adapted would. Hence, the opportunities for learning were limited because, the PSTs took the examples as they were without restructuring or sequencing them to suite their learners. In some cases, the PSTs selected similar examples without considering variation for learners to be exposed to different features of the object of learning. For these PSTs, using different values in examples is all they know when it comes to variance, even if it means the question remains invariant. Such examples can be seen in Chapter Six, but you may refer to Snippet 9.1 for another one of them.

ENRICHMENT ACTIVITY:

1) **Calculate the gradient of AB and DC in each of the following and state whether AB and DC are Parallel, Perpendicular or Neither**
   - **a)** A(2;-1), B(5;-3), C(-1;1), D(-;3)
   - **b)** A(4;2), B(-1;-2), C(2;0), D(10;-10)
   - **c)** A(5;4), B(2;7), C(-7;1), D(-5;3)
   - **d)** A(-5;-1), B(-1;4), C(-3;2), D(5;2)

Snippet 9.1 Selected examples for enrichment in Grade 11

As much as this counts for variation, there are still many other varying examples the PST in Snippet 9.1 could have used to help learners remember the difference between the gradients of parallel lines and perpendicular lines. The examples are also selected for an enrichment activity, which for Grade 11 is too basic because the concept of gradient should have been under revision. The PSTs did not find it problematic to select the examples as they are from the textbook because they used Grade 11 textbooks, and therefore, the content was seen to be relevant. However, for content enactment, selecting and sequencing examples to suit different learners is crucial, and may entail using different textbooks and creating one's own examples.

9.4 Impact of the theoretical framework and conclusion

Having revisited the purpose of the study, considered its context and addressed the research questions, the conclusion starts by revisiting the employed framework on this study. According to the framework, mathematical knowledge and pedagogical content knowledge are interdependent. They both play a major role in lesson planning, and especially in the selection of examples. Hence the variation theory was employed as part of the framework as scholars in mathematics education have dubbed it suitable for the analysis of examples. The framework led to findings that spoke to the purpose of the study, and exposed some aspects that needed further developing in the PSTs, who had very limited PCK. The background of the PSTs affected their development of PCK as they also struggled with the mathematical content, and had difficulty in using and understanding the mathematical language. The PSTs misconceptions in the mathematical content also seemed to stem from their command of the LoLT, which plays a major role in the mathematics register (Thompson et al., 2016). This had a negative impact on the PST's ability to select examples that promote learning in multilingual analytical geometry classrooms.

9.5 Limitations of the study

This was a qualitative study intended to provide insights into the experiences and perceptions of the PSTs' selection of examples for teaching multilingual analytical geometry classrooms, and the role of PCK in this process. Like all qualitative studies, the study also had its limitations. These are discussed in this section in the following order: Context based findings, limited quantification and generalizability, dependence on participants, and time constraints.

One of the limitations that came with this study being a qualitative research study is that the findings are context based (Queirós, Faria, and Almeida, 2017). The study was conducted in a university that is based in the KZN province of South Africa where the population is mostly African and IsiZulu speaking. Most of the PSTs in this particular institution are from different parts of the province, and this is unlike other institution who receive PSTs from different parts of the country. It is therefore reasonable to foresee that the findings are highly contextual, and applicable to specific institutions and PSTs.

Besides the location of the institution, one may also need to consider that the findings may also not be applicable to PSTs in other institutions because of the different educational settings and admission requirements in those institutions. Most importantly, it was noted that the group of PSTs who participated in this study had their education disturbed by the emergency online teaching and learning that was introduced in 2020 and carried on for more than a year. Some studies, such as one conducted by Batmang et al. (2021) show how this group of PSTs education was affected. In their study, Batmange et al. (2021) found that some PSTs had challenges in understanding the learning technologies that were used for teaching and learning during the lockdown (remote teaching and learning), and this affected their learning. This finding is applicable to the PSTs in this study because of their learning context in high school and exposure to technological tools. Therefore, the findings of this study, may also be limited to this cohort of PSTs.

The findings of the study being specific to its context made it difficult to generalize the findings to a larger population (Queirós et al., 2017). This is also owing to the study being a small-scale study that worked with one group of third year mathematics PSTs from only one university out of four in KZN, and twenty-six in the country excluding all private institutions that offer teacher training programs. For this reason, the findings of the study can only be applicable to a small population of PSTs.

As if working with one group of PSTs was not limiting enough, the study had challenges in having informative responses from the participating PSTs because most of them would provide very brief answers to the interview questions. Some PSTs even expressed discomforts with having one on one interviews and requested to have the interview questions in writing for them to provide written responses, this made it even worse, because these were limited responses one could not follow-up on. This was a limitation that came with the study being qualitative and relying on participants. It was not possible to avoid this because it was crucial to hear and learn about the PSTs' reasons for selecting particular examples to understand the role of their PCK, and this needed to come directly from them.

Relying on the PSTs for data collection was also a challenge because data collected depended on their availability, and when available, the process was time consuming. Understandably, a qualitative study is time-consuming in its nature because it requires extensive data collection and analysis. Interviewing the PSTs took longer than I had anticipated. Queirós et al. (2017) explained that the pitfall of interviews in a qualitative study is that they are time intensive. In some cases, I would have to explain a question further, or do a lot of probing to get a clear response. For this reason, a limited number of PSTs were interviewed.

9.6 Recommendations

The findings and limitations of this study exposed a need for more research in certain areas. More studies with bigger sample sizes in the different South African universities are necessary to understand the different factors that contribute to the lack of PCK in PSTs, especially for teaching analytical geometry. This mathematics topic has received a lot of attention when it comes to using technological tools to facilitate its teaching and learning, but unfortunately, its not possible to use these tools to teach it if one doesn't understand it. This is another case of PCK suppressing content knowledge (Yet et al., 2021). Hence, future research could explore the PSTs knowledge of analytical geometry and how it could be further developed. This would also inform involved stakeholders about the different measures that could be implemented in each different context.

In an attempt to bridge the gap in mathematical content knowledge, teacher training must also consult the national diagnostic reports to make relevant adjustments to the first-year content modules in teacher training. This would help in focusing on key areas and addressing all misconceptions to ease the process of developing the PSTs' PCK. It would also facilitate the consolidation of the high school mathematics, which Bansilal, Brijlal, and Trigueros (2017) argued would present the opportunity to move from pitching the standard topics at high school level to an advanced level. This would also help in developing specialized mathematics content knowledge in the PSTs.

The field of education has explored the variation theory with some scholars exploring mathematics pedagogy through variation, where Leung (2012) posited that mathematical concepts can be developed through interaction with variation. Employing the theory in mathematics teaching could lead to an improvement in the way the subject is taught and learned. This is especially because the main driving tool in its teaching and learning is the use of examples which are best described using the patterns of variation from the variation theory (Al-Murani et al., 2019).

Future research could also explore ways in which teacher training institutions could be more specific in what exactly happens in their classrooms by focusing on teaching PSTs how to select examples and use the patterns of variation to evaluate their examples. This is possible because some scholars have explored the application of the variation theory in teacher training. In a study that looked at the lived experiences of the PSTs when learning to plan, evaluate, and revise lessons using the variation theory, Martensson and Ekdahl (2021) found that it is possible for mathematics PSTs to actually learn and apply the principles of the variation theory in their teaching practices.

This study explored the PSTs PCK and its role in their selection of examples for teaching analytical geometry in multilingual classrooms. In doing so, the data revealed another factor that could not be focused on in this study because the data collection tools were not designed for it. The language aspect kept on coming up in the data, and this was a constant reminder that the PSTs are also learning mathematics in multilingual classrooms where codeswitching is mostly impossible to use because their teachers also have different languages. Thompson et al. (2016) conducted a study that explored ways in which multi-language diversity could be addressed in mathematics PSTs education. One of their contributions is that mathematics PSTs educators must help PSTs learn 'how' to facilitate mathematical language and the understanding of mathematical content in their instruction. However, this is possible when the gap in the PSTs knowledge is addressed. A clear framework on how to address multi-language diversity issues and develop PCK in multilingual PSTs' classrooms could lead to a positive impact in the South African mathematics education.

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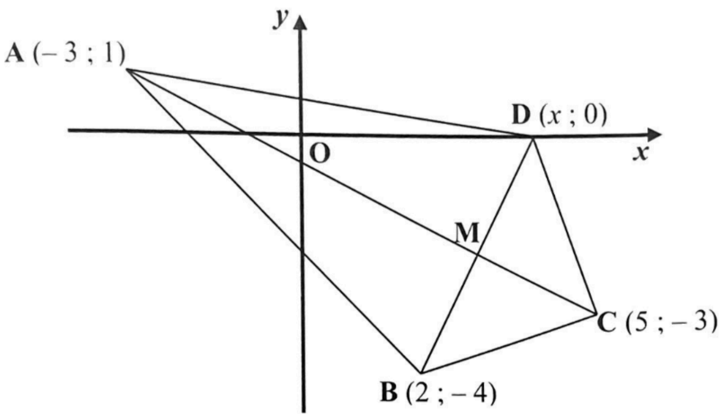
APPENDIX A: ANALYTICAL GEOMETRY TEST/ACTIVITY

MARKS: 60 **DURATION: 1 HOUR**

QUESTION 1

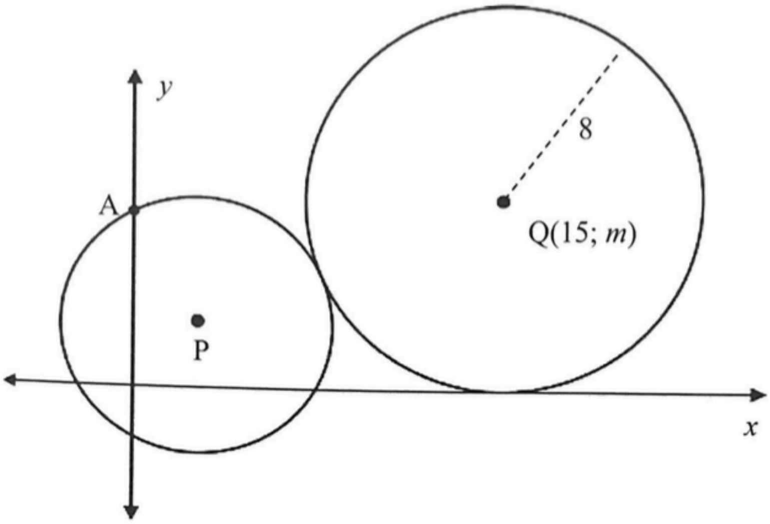
ABCD is a quadrilateral on the Cartesian plane with vertices, A(–3; 1), B(2 ; –4), C(5 ; –3) and D(x ; 0).



1.1	Determine the gradient of AC.	(2)
1.2	Calculate the equation of AC.	(3)
1.3	Calculate the angle of inclination of AC.	(3)
1.4	Determine the coordinates of D if AC is the perpendicular bisector of DB.	(5)
1.5	Hence, deduce that ABCD is a kite.	(3)
1.6	Calculate the area of ΔABD.	(5)
		[21]

QUESTION 2

2.1 In the diagram below, Q(15; m) is the centre of the larger circle which touches both the x-axis and the circle centred at P. The circle with centre P, has the y-intercept at A and has equation $(x-3)^2 + (y-3)^2 = 25$. The radius of the circle, centre Q, is 8 units.



2.1.1 Determine the equation of the circle with centre Q in terms of x; y and m. (1)

2.1.2 Determine the value of m. (2)

2.1.3 Determine the length of PQ. (2)

2.1.4 Calculate the coordinates of A. (3)

2.1.5 Determine the equation of the tangent to the circle with centre P, at A. (4)

2.2 Two other circles are given:

- One has centre K, and equation $x^2 - 6x + y^2 + 4y - 12 = 0$.
- The other has centre T, and equation $(x-12)^2 + (y-10)^2 = 100$

2.2.1 Determine the centre and radius of the circle with centre K. (4)

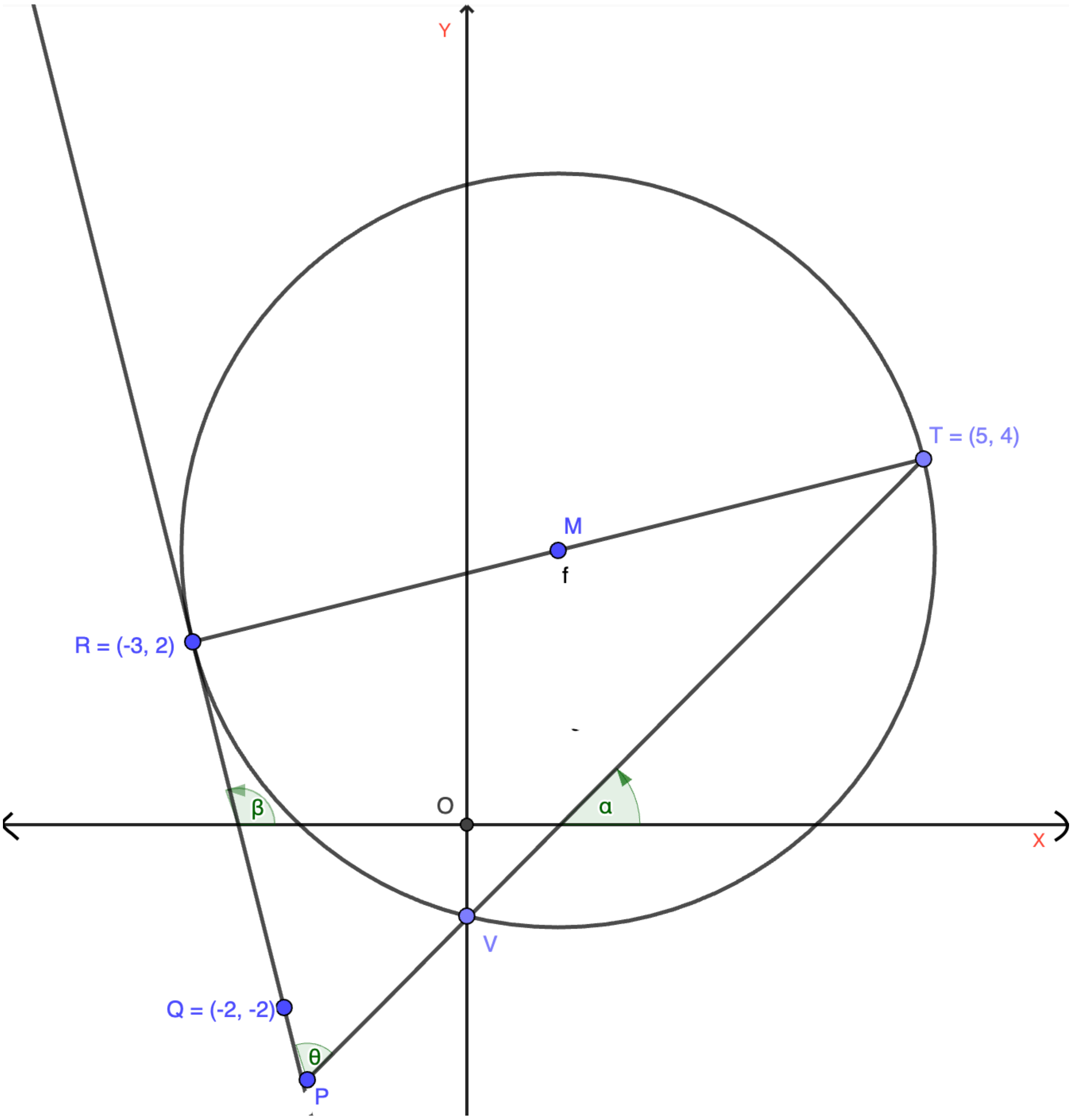
2.2.2 Calculate the distance between the centres, K and T. (2)

2.2.3 At how many points do the two circles intersect? Motivate your answer. (2)

[20]

QUESTION 3

In the diagram below the circle with centre M passes through V, R (-3;2) and T (5; -4). Q is the point (-2; -2) and the lines through RQ and TV meet at P. The inclination angle of PT is α and the angle of inclination of PR is β. Vis the y-intercept of both the circle and line TP.



3.1 Determine the equation of the circle with centre M. (4)

3.2 Show, using analytical methods, that PR is a tangent to the circle at R. (4)

3.3 Determine the coordinates of V. (4)

3.4 If $R\hat{P}T = \theta$, calculate θ to ONE decimal place. (7)

[19]

APPENDIX B: LETTER OF INFORMATION

LETTER OF INFORMATION

Title of the Research Study: The role of Pedagogical Content Knowledge in the preservice teachers' choice of examples to create opportunities for learning analytical geometry in Kwa Zulu Natal multilingual classrooms.

Principal Investigator/s/researcher: Sibongile Zulu, Master of Education

Co-Investigator/s/supervisor/s: Prof Deonarain Brijlall, PhD and Dr Mamothibe Thamae, PhD

Good day,

My name is Sibongile Zulu. I am a Doctor of Education (mathematics education) student in the School of Education at the Durban University of Technology.

I am doing research on the role of pedagogical content knowledge in the preservice teachers' choice and use of examples to promote learning in multilingual analytical geometry classrooms. In this research, preservice teachers are students who are currently in their training to become teachers, in this case, mathematics teachers. Multilingual classrooms are classrooms in which more than one language is used, or present the opportunity to be used. For example, a classroom in which the teacher's home language is IsiZulu, and teaching and learning takes place in English, while the learners in the classrooms also have different home languages.

The research involves working with third year preservice teachers who will be teaching Grade 11 mathematics classes. The preservice teachers will be given a short analytical geometry test which is basically an activity, and then I will request their lesson plans on analytical geometry to analyze the examples in the lesson plans. This will be followed by an interview with each preservice teacher to enquire more about what I would have observed in the lesson plans. This will take place over a period of 2-3 weeks, and there will be no lesson disruptions caused. The interviews would take place when the preservice teachers are available and would be for a maximum of 20 minutes per interview.

I would like to invite you as a mathematics preservice teacher to participate in this research which seeks to investigate the role of pedagogical content knowledge in the preservice teachers' choice of examples, and whether they would create opportunities for learning in multilingual analytical geometry classroom. This research will provide opportunities for preservice teachers and teacher training institutions to reflect on the knowledge necessary to promote learning in multilingual analytical geometry classrooms, and skills that are necessary in mathematics teacher training. The research will also add to the body of knowledge on how to teach analytical geometry in multilingual classrooms, and lead to an improvement in teacher training for multilingual classrooms, which will ultimately lead to an improvement in the performance of mathematics in South Africa.

As the research participant, you will not be advantaged or disadvantaged in any way. You may also withdraw your permission to participate in this research at any time during this project without any penalty. The research may be terminated early in particular circumstances, and as the researcher, I may, under certain circumstances, decide to withdraw the participant from the study. This will be done professionally through a written letter and will not disadvantage you as a participant in anyway. There are no foreseeable risks in participating in this study, and you will not need to pay or be paid for participating in this research.

Your identity (name and surname), the name of the university in which you are training to become a teacher will be kept confidential and anonymous at all times, and in all academic writing about the research. To conceal your identity, all participants will be referred to as 'preservice teacher A, B or C' and so on. Alphabets will be used to conceal the participant's identity. Your individual privacy will be maintained in all published and written data resulting from the research. All research data will be securely stored with the Durban University of Technology, and will be destroyed between 3-5 years after completion of the project.

Please let me know if you require any further information. You are entitled to discuss this with your family and friends, and you are under no obligation to commit at this stage.

For queries, please contact me, the researcher on 0764874508, and my supervisor at deonarianb@dut.ac.za or the Institutional Research Ethics Administrator on 031 373 2375. Complaints can be reported to the Director: Research and Postgraduate Support Dr L Linganiso on 031 373 2577 or researchdirector@dut.ac.za.

I look forward to your response as soon as is convenient.

Yours sincerely,

Sibongile Zulu

APPENDIX C: INFORMED CONSENT

CONSENT

Full Title of the Study: The role of Pedagogical Content Knowledge in the preservice teachers' choice of examples to create opportunities for learning analytical geometry in Kwa Zulu Natal multilingual classrooms.

Names of Researcher/s: Sibongile Zulu

Statement of Agreement to Participate in the Research Study:

- I hereby confirm that I have been informed by the researcher, (name of researcher), about the nature, conduct, benefits and risks of this study - Research Ethics

Clearance Number: ___________________________,

- I have also received, read and understood the above written information (Participant Letter Of Information) regarding the study.
- I am aware that the results of the study, including personal details regarding my sex, age, date of birth, initials and diagnosis will be anonymously processed into a study report.
- In view of the requirements of research, I agree that the data collected during this study can be processed in a computerised system by the researcher.
- I may, at any stage, without prejudice, withdraw my consent and participation in the study.
- I have had sufficient opportunity to ask questions and (of my own free will) declare myself prepared to participate in the study.
- I understand that significant new findings developed during the course of this research which May relate to my participation will be made available to me.

______________________	___________	______	________________
Full Name of Participant	**Date**	**Time**	**Signature/Right**

Thumbprint

I, ________________ ____
(name of researcher) herewith confirm that the above participant has been fully informed about the nature, conduct and risks of the above study.

___________________	___________	_____________________
Full Name of Researcher	**Date**	**Signature**
___________________	___________	_____________________
Full Name of Witness (If applicable)	**Date**	**Signature**
_________________	__________	___________________
Full Name of Legal Guardian (If applicable)	**Date**	**Signature**

APPENDIX D: SEMI-STRUCTURED INTERVIEW TOOL

SEMI-STRUCTURED INTERVIEW TOOL

FOLLOW – UP SEMI-STRUCTURED QUESTIONS FOR PRESERVICE TEACHERS AFTER ANALYSING THEIR TEST RESPONSES.

Preservice teacher: _________________________

1. Why did you do this (explain based on response)?
2. What was the purpose of this step (explain specific step)?
3. What does this explanation mean (specific written response)?

FOLLOW – UP INTERVIEW WITH PRESERVICE TEACHERS AFTER ANALYSING THE LESSON PLANS.

Preservice teacher: ___________________________

Your lesson objective

1. Do you think you met your lesson objectives? Explain.
   (Did you achieve what you had prepared to deliver and wanted learners to learn from the lesson? Explain.)

Your choice of example

1. How did you choose the examples you used in the lesson? What did you have to consider?
2. How do you decide on the number of examples to use in each lesson?

3. What knowledge mainly informs your choice of examples? Is it the knowledge of content, your learners, or any other knowledge?
4. How do you decide on the appropriate time to use an example (whether when introducing or on the body of the lesson)?
5. What was the purpose of the selected example (name specific example) used in the lesson plan?
6. What response did you expect from learners for the specific examples used? (if necessary)
7. In what ways are the examples connected?
8. How did the selected examples enable mathematical understanding in your opinion?
9. How did you choose words in your examples so that learners understand the language used?
10. Do the ways in which learners respond to the examples used in class influence your next choice of examples?
    a. (If question is not answered adequately, ask: When do you come up with unplanned examples in class, and what leads to this?

APPENDIX E: GATEKEEPER LETTER

Directorate for Research and Postgraduate Support
Durban University of Technology
Open House
P.O. Box 1334, Durban 4000
Tel.: 031-3732576/7
Fax: 031-3732946

20 September 2022

Ms Sibongile Zulu
c/o School of Education
Faculty of Arts and Design
Durban University of Technology

Dear Ms Zulu

PERMISSION TO CONDUCT RESEARCH AT THE DUT

Your email correspondence in respect of the above refers. I am pleased to inform you that the Institutional Research and Innovation Committee (IRIC) has granted **Gatekeeper Permission** for you to conduct your research "The role of Pedagogical Content Knowledge in preservice teachers' choice of examples to create opportunities for learning analytical geometry in KwaZulu-Natal multilingual classrooms." at the Durban University of Technology. **Kindly note that this letter must be issued to the IREC for approval before you commence data collection**.

The DUT may impose any other condition it deems appropriate in the circumstances having regard to nature and extent of access to and use of information requested.

We would be grateful if a summary of your key research findings would be submitted to the IRIC on completion of your studies.

Kind regards.
Yours sincerely

MS V GOVENDER
ACTING-DIRECTOR: RESEARCH AND POSTGRADUATE SUPPORT DIRECTORATE

APPENDIX F: ETHICS TRAINING CERTIFICATE

TRREE

Zertifikat

Certificat

Certificado

Certificate

Promouvoir les plus hauts standards éthiques dans la protection des participants à la recherche biomédicale
Promoting the highest ethical standards in the protection of biomedical research participants

Clinical Trials Centre
The University of Hong Kong

Certificat de formation - Training Certificate

Ce document atteste que - this document certifies that

Sibongile Zulu

a complété avec succès - has successfully completed

Research Ethics Evaluation

du programme de formation TRREE en évaluation éthique de la recherche
of the TRREE training programme in research ethics evaluation

Release Date: 2021/11/09
CID : s5dUMLS7fx

Professeur Dominique Sprumont
Coordinateur TRREE Coordinator

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[REV : 20170310]

APPENDIX G: ETHICAL CLEARANCE CERTIFICATE

DUT
DURBAN UNIVERSITY OF TECHNOLOGY
INYUVESI YASETHEKWINI YEZOBUCHWEPHESHE

INSTITUTIONAL RESEARCH ETHICS COMMITTEE

Institutional Research Ethics Committee
Research and Postgraduate Support Directorate
2nd Floor, Berwyn Court
Gate 1, Steve Biko Campus
Durban University of Technology

P O Box 1334, Durban, South Africa, 4001

Tel: 031 373 2375
Email: lavishad@dut.ac.za
http://www.dut.ac.za/research/institutional_research_ethics

www.dut.ac.za

28 November 2022

Ms S Zulu
79 Collard Road
Durban North

Dear Ms Zulu

The role of Pedagogical Content Knowledge in preservice teachers' choice of examples to create opportunities for learning analytical geometry in KwaZulu-Natal multilingual classrooms
Ethical Clearance number IREC 118/22

The DUT-Institutional Research Ethics Committee acknowledges receipt of your final data collection tool for review.

We are pleased to inform you that the data collection tool has been approved. Kindly ensure that participants used for the pilot study are not part of the main study.

In addition, the DUT-IREC acknowledges receipt of your gatekeeper permission letter.

Please note that FULL APPROVAL is granted to your research proposal. You may proceed with data collection.

Any adverse events [serious or minor] which occur in connection with this study and/or which may alter its ethical consideration must be reported to the DUT-IREC according to the DUT-IREC Standard Operating Procedures (SOP's).

Please note that any deviations from the approved proposal require the approval of the DUT-IREC as outlined in the DUT-IREC SOP's.

Yours Sincerely

Prof J K Adam
Chairperson: DUT-IREC

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APPENDIX H: TURNITIN REPORT

PhD Thesis

by Sibongile Zulu

Submission date: 28-Nov-2023 10:44PM (UTC+0200)
Submission ID: 2241187155
File name: Sibongile_Edited_thesis.docx (19.92M)
Word count: 51216
Character count: 276013

PhD Thesis

ORIGINALITY REPORT

6%	%	6%	%
SIMILARITY INDEX	INTERNET SOURCES	PUBLICATIONS	STUDENT PAPERS

PRIMARY SOURCES

1	Happy Maybe Maambiwa Khangale, Ndidzulafhi Selina Raliphaswa, Azwidihwi Rose Tshililo. "HIV Status Disclosure to Adolescents Who Are Perinatally Infected in Rustenburg Sub District Northwest Province", Children, 2022 Publication	<1%
2	Hetrick, Cara. "Novice Secondary Mathematics Teacher Support from Mentoring Relationships in the First Five Years of Teaching", San Diego State University, 2022 Publication	<1%
3	"Multilingual Education Yearbook 2021", Springer Science and Business Media LLC, 2021 Publication	<1%
4	Magabvu, Bright. "Elements of Pedagogical Content Knowledge Displayed by Mathematics Teachers in the Teaching of Trigonometry", University of Johannesburg (South Africa), 2023 Publication	<1%

APPENDIX I: EDITING CERTIFICATE

AH

EDITING CERTIFICATE

CONTACT

Dr Anita Hiralaal

BA, HDE, B ED HONS, B COMM HONS, M ED, PH D, CERTIFICATE IN COPY-EDITING AND PROOFREADING (UCT), CERTIFICATE IN COPY-EDITING AND PROOFREADING: SA WRITER'S COLLEGE

17 Fairfield Avenue
Scottsville
Pietermaritzburg
Telephone: 0333864913
0825352777
anitah0106@gmail.com

GUILD OF COPY EDITORS

TO: MS SIBONGILE ZULU

28 November 2023

Editing of thesis submitted in fulfilment of the degree of Doctor of Education

THE ROLE OF PEDAGOGICAL CONTENT KNOWLEDGE IN THE PRESERVICE TEACHERS' CHOICE OF EXAMPLES TO CREATE OPPORTUNITIES FOR LEARNING ANALYTICAL GEOMETRY IN KWA ZULU NATAL MULTILINGUAL CLASSROOMS

DR ANITA HIRALAAL

This thesis has been academically edited to ensure technically accurate and contextually appropriate use of language, grammar, logical coherency and appropriate presentation