Asia Pac. J. Math. 2024 **11**:99

# ON A CLASS OF PIECEWISE CONTINUOUS LYAPUNOV FUNCTIONS AND UNIFORM EVENTUAL STABILITY OF NONLINEAR IMPULSIVE CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS VIA NEW GENERALIZED DINI DERIVATIVE

JACKSON EFIONG ANTE[1,*], JEREMIAH UGEH ATSU[2], ADHIR MAHARAJ[3], ETIMBUK EMMANUEL ABRAHAM[4], OJEN KUMAR NARAIN[5]

[1]Department of Mathematics, Topfaith University, Mkpatak, Nigeria

[2]Department of Mathematics, University of Cross River State, Calabar, Nigeria

[3]Department of Mathematics, Durban University of Technology, Durban, South Africa

[4]Department of Electrical/Electronics, Topfaith University, Mkpatak, Nigeria

[5]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa

[*]Corresponding author: jackson.ante@topfaith.edu.ng

Received Jul. 31, 2024

Abstract. In this paper, the uniform eventual stability of nonlinear impulsive Caputo fractional differential equations with fixed moments of impulse is examined using the vector Lyapunov functions which is generalized by a class of piecewise continuous Lyapunov functions. Together with comparison results, sufficient conditions for the uniform eventual stability of impulsive Caputo fractional differential equations are presented. An illustrative example is given to confirm the suitability of the obtained results.

2020 Mathematics Subject Classification. 34A08, 34A34, 34A37, 34D20.

Key words and phrases. uniform eventual stability; Caputo derivative; impulsive fractional differential equations; Lyapunov functions.

## 1. Introduction

The stability of solutions of differential equations via Lyapunov method has been intensively investigated in the past [15,40], and in many real cases, it is obligatory to study the stability of certain sets, which are not invariant with respect to a given system of differential equations and thereby excludes the stability in the sense of Lyapunov. As stated in [40], examples of such sets can be found when self-controlled systems of management are being studied (see [13]). Thus, to allay the problem that will arise subsequently, [28] introduced a new concept called eventual stability, maintaining that, the set under consideration, despite not being invariant in the usual sense, is invariant in the asymptotic sense (see also [42]). Accordingly, the eventual stability of solutions of impulsive differential systems have been extensively studied (see [22,40] and the references therein).

DOI: 10.28924/APJM/11-99

As observed in [37], there are many perturbations and adaptive control problems where the point in question may not be an equilibrium (invariant) point but eventually stable sets that are asymptotically invariant, which enables us to consider Lyapunov stability as a special case of eventual stabilities. For decades, a large number of researchers have shown explosive interests in the study of the qualitative properties of impulsive differential equations (See [6, 9, 17, 22, 25, 32, 39–41]).

Now, concerning the stability of perturbed differential equations, [38] obtained results on the eventual stability of impulsive differential systems but with the perturbations considered as bounded functions. In [37], sufficient conditions for the retention of uniform eventual stability of impulsive differential system was established with non fixed moments of impulses under varnishing perturbations by employing piecewise continuous auxilliary functions which is assumed to be a generalization of the Lyapunov functions. Results on the uniform eventual stability for impulsive differential equations with non fixed moment of impulses having bounded perturbations were established in [19], while [40] obtained results on the eventual stability and eventual boundedness of impulsive differential systems with supremum using a class of piecewise continuous functions (which are analogues of classical Lyapunov functions) together with the Razumikhin technique.

Unarguably, the theory of impulsive differential equation is richer than the corresponding theory of differential equations [22], as they constitutes very important models in the description of the true state of several real life processes and phenomena, as many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations acts instantaneously, that is, in the form of impulses. It is also known for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects [22].

Now, the efficient applications of impulsive differential system require the finding of criteria for stability of their solutions [36], and one of the most versatile methods in the study of the stability properties of impulsive systems is the method of Lyapunov function (Lyapunov's second method). The method was originally developed for studying the stability of a fixed point of an autonomous or nonautonomous differential equations. However, as was argued in [33], the method was then extended from fixed points to sets, from differential equation to dynamical systems and to stochastic equations.

There are several approaches in the literature in the study of the stability of solutions, one of which is the Lyapunov's second method. However, the novelty of the Lyapunov's second method over other methods of examining stability properties like the Razumikhin technique, the use of matrix inequality, etc. stems from the fact that the method allows us to examine the stability of solutions without first solving the given differential equation. Again, the method involves seeking an appropriate continuously differentiable function that is positive definite and whose time derivative along the trajectory curve or solution path is negative semidefinite.

The stability of the zero solution of impulsive differential equations have been extensively studied in [6,11,34].

Furthermore, the study of stability for fractional order systems is quite recent and one of the main difficulties in the application of a Lyapunov function to fractional order differential equations is the appropriate definition of its derivative among the fractional differential equations (see [5]). The stability of fractional order systems is examined in [1–5,7,10,15,20]. Using the generalized Caputo fractional Dini derivative and scalar impulsive fractional differential equations, [5] established the comparison results together with sufficient conditions for the stability properties of impulsive fractional differential equations using the scalar Lyapunov function.

In this paper, the uniform eventual stability of impulsive Caputo fractional differential equations using the vector Lyapunov functions is examined. Together with the comparison results, sufficient conditions for the uniform eventual stability of the set $x(t) = 0$ is established with illustrative example.

## 2. Preliminaries, Notations and Definitions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|.\|$, let $\Omega$ be a domain in $\mathbb{R}^n$ containing the origin; $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $t_0 \in \mathbb{R}_+, t > 0$.
Let $J \subset \mathbb{R}_+$. Define the following class of functions $PC^q[J, \Omega] = \alpha : J \to \Omega, \alpha(t)$ is a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t)$ exists.

Fractional calculus generalizes the classical calculus to non integer order and allows for the extension of the classical concepts of derivative and integral to functions with fractional orders. It allows for functions with non integer orders which makes it much more flexible in describing real world systems (see [18,30,31,35]). There are several definitions of fractional derivatives and fractional integrals.

**General case.** Let the number $n - 1 < q < n, q > 0$ be given, where $n$ is a natural number and $\Gamma(.)$ denotes the gamma function.

**Definition 2.1.** The Riemann-Liouville fractional derivative of order $q$ of $x(t)$ is given by (see [35])

$$ {}^{RL}_{t_0}D^q_t x(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-s)^{n-q-1} x(s) ds, t \geq t_0. $$

**Definition 2.2.** The Caputo fractional derivative of order $q$ of $x(t)$ is defined by (see [35])

$$ {}^{C}_{t_0}D^q x(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} x^{(n)}(s) ds,\ t \geq t_0. $$

The Caputo derivatives has many properties that are similar to those of the standard derivatives, which makes them easier to understand and apply. The initial conditions of fractional differential equations using the Caputo derivative are also easier to interpret in physical context, which is another reason why it is often used in applications of fractional calculus.

**Definition 2.3.** The Grunwald-Letnikov fractional derivative of order $q$ of $x(t)$ is given by (see [4])

$$D_0^q x(t) = \lim_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t-rh),\ t \geq t_0,$$

and the Grunwald-Letnikov fractional Dini derivative of order $q$ of $x(t)$ is given by (see [4])

$$D_0^q x(t) = \limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{(t-t_0)}{h}]} (-1)^r ({}^qC_r) x(t-rh),\ t \geq t_0,$$

where ${}^qC_r$ are the binomial coefficients and $[\frac{(t-t_0)}{h}]$ denotes the integer part of $\frac{(t-t_0)}{h}$.

**Particular case.** (when n=1). In most applications, the order of q is often less than 1, so that $q \in (0,1)$. For simplicity of notation, we will use ${}^CD^q$ instead of ${}^C_{t_0}D^q$ and the Caputo fractional derivative of order q of the function $x(t)$ is

$$ {}^CD^q x = \frac{1}{\Gamma(1-q)} \int_{t_0}^t (t-s)^{-q} x' ds,\ t \geq t_0. \tag{1}$$

## 3. Impulses in Fractional Differential Equations

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}^CD^q x = f(t,x), t \geq t_0, x(t_0) = x_0, \tag{2}$$

where $x \in \mathbb{R}^N$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N]$, $f(t,0) \equiv 0$ and $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^N$.

Some sufficient conditions for the existence of the global solutions to (2) are considered in [8,12,26,27, 32,35,43]. The IVP for FrDE (2) is equivalent to the following Volterra integral equation (See [5]),

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, x(s)) ds,\ t \geq t_0. \tag{3}$$

Consider the initial value problem for the system of impulsive fractional differential equations (IFrDE) with a Caputo derivative for $0 < q < 1$,

$$\begin{aligned} {}^CD^q x &= f(t,x), t \geq t_0, t \neq t_k, k = 1,2,... \\ \Delta x &= I_k(x(t_k)), k \in N, t = t_k \\ x(t_0) &= x_0, \end{aligned} \tag{4}$$

where $x, x_0 \in \mathbb{R}^N$, $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, and $t_0 \in \mathbb{R}_+$, $I_k : \mathbb{R}^N \to \mathbb{R}^N$, $k = 1,2,...$

under the following assumptions:

$(A_0)$ (i) $0 < t_1 < t_2 < ... < t_k < ...$, and $t_k \to \infty$ as $k \to \infty$;

(ii) $f : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, is continuous in $(t_{k-1}, t_k]$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$ exists;

(iii) $I_k : \mathbb{R}^N \to \mathbb{R}^N$

In this paper, we assume that $f(t, 0) \equiv 0$, $I_k(0) = 0$ for all $k$, so that we have the trivial solution for (4), and the points $t_k, k = 1, 2, ...$ are fixed such that $t_1 < t_2 < ...$ and $\lim_{k\to\infty} t_k = \infty$. The system (4) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t) = x(t; t_0, x_0) \in PC^q([t_0, \infty), \mathbb{R}^N)$.

*Remark* 3.1. The second equation in (4) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point $t_k$.

**Definition 3.1.** Let $V : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N_+$. Then $V$ is said to belong to class $\mathcal{L}$ if,

(i) $V$ is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^{\mathrm{N}}$ and for each $x \in \mathbb{R}^N$, $k = 1, 2, ...$ and $\lim_{(t,y)\to(t_k^+,x)} V(t, y) = V(t_k^+, x)$ exists;

(ii) $V$ is locally Lipschitz with respect to its second argument $x$ and $V(t, 0) \equiv 0$.

Now, for any function $V(t, x) \in PC([t_0, \infty) \times \xi, \mathbb{R}^N_+)$ we define the Caputo fractional Dini derivative as:

$$^cD^q_+V(t, x) = \limsup_{h\to0^+} \frac{1}{h^q}\{V(t, x) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}(^qC_r)[V(t - rh, x - h^q f(t, x)) - V(t_0, x_0)]\}, \tag{5}$$

$t \geq t_0$, where $t \in [t_0, \infty), x, x_0 \in \xi$, $\xi \in \mathbb{R}^N$ and there exists $h > 0$ such that $t - rh \in [t_0, T]$.

**Definition 3.2.** A function $g \in C[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in $x$, if $x \leq y$ and $x_i = y_i$ for $1 \leq i \leq n$ implies $g_i(x) \leq g_i(y), \forall i$.

**Definition 3.3.** The set $x(t) \equiv 0$ of (2) is said to be:

$(S1)$ eventually stable if for every $\epsilon > 0$ there exists a number $T = T(\epsilon) > 0$ for all $t_0 \in \mathbb{R}_+$ and $\delta = \delta(t_0, \epsilon)$ for all $x_0 \in \mathbb{R}^N$ such that $\|x_0\| < \delta$ implies $\|x(t; t_0, x_0)\| < \epsilon$ for $t \geq t_0$.

$(S2)$ uniformly eventually stable if the $\delta$ in $S1$ is independent of $t$.

**Definition 3.4.** A function $a(r)$ is said to belong to the class $\mathcal{K}$ if $a \in C([0, \psi), \mathbb{R}_+]$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing in $r$.

In this paper, we define the following sets:

$$\bar{S}_\psi = \{x \in \mathbb{R}^N : \|x\| \leq \psi\}$$

$$S_\psi = \{x \in \mathbb{R}^N : \|x\| < \psi\}.$$

Suffice to say that the inequalities between vectors are understood to be component-wise inequalities.

We will use the comparison results for the impulsive Caputo fractional differential equation of the type

$$
\begin{aligned}
{}_{t_0}^{c}D^q u = g(t,u), t \geq t_0, t \neq t_k, k = 1, 2, \ldots \\
\Delta u = \psi_k(u(t_k)), k \in N, t = t_k \\
u(t_0^+) = u_0,
\end{aligned}
\tag{6}
$$

existing for $t \geq t_0$, where $u \in \mathbb{R}^n, \mathbb{R}_+ = [t_0, \infty), g : \mathbb{R}_+ \times \mathbb{R}_+^n \to \mathbb{R}^n, g(t,0) \equiv 0,$
where $g$ is the continuous mapping of $\mathbb{R}_+ \times \mathbb{R}_+^n$ into $\mathbb{R}^n$. The function $g \in C[\mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n]$ is such that for any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the system (6) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in PC^q([t_0, \infty), \mathbb{R}^n)$.

**Lemma 3.2.** *Assume $m \in PC([t_0, T] \times \bar{S}_\psi, \mathbb{R}^N)$ and suppose there exists $t^* \in [t_0, T]$ such that for $\alpha_1 < \alpha_2$, $m(t^*, \alpha_1) = m(t^*, \alpha_2)$ and $m(t, \alpha_1) < m(t, \alpha_2)$ for $t_0 \leq t < t^*$. Then if the Caputo fractional Dini derivative of $m$ exists at $t^*$, then the inequality ${}^C D_+^q m(t^*, \alpha_1) -^C D_+^q m(t^*, \alpha_2) > 0$ holds.*

*Proof.* Let $V(t,x) = m(t,\alpha_1) - m(t,\alpha_2)$.
Applying (5), we have

$$
\begin{aligned}
{}^C D_+^q (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h \to 0^+} \frac{1}{h^q} \{[m(t^*, \alpha_1) - m(t^*, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&\quad - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\}.
\end{aligned}
$$

When $m(t^*, \alpha_1) = m(t^*, \alpha_2)$, we have

$$
\begin{aligned}
{}^C D_+^q (m(t^*, \alpha_1) - m(t^*, \alpha_2)) &= \limsup_{h \to 0^+} \frac{1}{h^q} \{-[m(t_0, \alpha_1) - m(t_0, \alpha_2)] - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qC_r \\
&\quad [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] - [m(t_0, \alpha_1) - m(t_0, \alpha_2)]\} \\
&= -\limsup_{h \to 0^+} \frac{1}{h^q} \{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&+ \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t^* - rh, \alpha_1) - m(t^* - rh, \alpha_2)] \\
&- \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&= -\limsup_{h \to 0^+} \frac{1}{h^q} \{[m(t_0, \alpha_1) - m(t_0, \alpha_2)] \\
&- \limsup_{h \to 0^+} \frac{1}{h^q} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r qC_r [m(t_0, \alpha_1) - m(t_0, \alpha_2)]
\end{aligned}
$$

$$= \quad -\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qC_r[m(t_0,\alpha_1)-m(t_0,\alpha_2)].$$

Applying equation 3.8 in [4], we have

$${}^cD_+^q(m(t^*,\alpha_1)-m(t^*,\alpha_2)) \quad = \quad -\frac{(t-t_0)^{-q}}{\Gamma(1-q)}[m(t_0,\alpha_1)-m(t_0,\alpha_2)].$$

By the lemma, we have that

$$m(t,\alpha_1)-m(t,\alpha_2)<0, \text{for } t_0\le t<t^*.$$

And so it follows that

$${}^cD_+^q(m(t^*,\alpha_1)-m(t^*,\alpha_2))>0.$$

□

*Remark* 3.3. Lemma (3.2) extends Lemma 1 in [4], where the vectors $m(t,\alpha_1)$ and $m(t,\alpha_2)$ are compared component-wise.

## 4. Fractional Differential Inequalities and Comparison Results for Impulsive Vector Fractional Differential Equations

In this section, we assume that $0<q<1$.

**Theorem 4.1.** *Assume that*

(i) $g\in PC[\mathbb{R}_+\times\mathbb{R}_+^n,\mathbb{R}^n]$ *and is continuous in* $(t_{k-1},t_k]$, $k=1,2,\ldots$ *and* $g(t,u)$ *is quasimonotone nondecreasing in* $u$ *for each* $u\in\mathbb{R}^n$ *and* $\lim_{(t,y)\to(t_k^+,u)} g(t,u)=g(t_k^+,u)$ *exists;*

(ii) $V\in PC[\mathbb{R}_+\times\mathbb{R}^N,\mathbb{R}_+^N]$ *and* $V\in\mathcal{L}$ *such that*

$${}^CD_+^qV(t,x)\le g(t,V(t,x)),$$

$t\ne t_k$, $(t,x)\in\mathbb{R}_+\times\mathbb{R}^N$ *and*
$V(t_k^+,x+I_k(x(t_k)))\le\rho_k(V(t,x))$, $t=t_k$, $x\in S_\psi$ *and the function* $\rho_k:\mathbb{R}_+^N\to\mathbb{R}_+^N$ *is nondecreasing for* $k=1,2,\ldots$

(iii) $r(t)=r(t;t_0,u_0)\in PC^q([t_0,T],\mathbb{R}^n)$ *is the maximal solution of the IVP for the IFrDE* (6).

*Then,*

$$V(t,x(t))\le r(t), t\ge t_0, \tag{7}$$

*where* $x(t)=x(t;t_0,x_0)\in PC^q([t_0,T],\mathbb{R}^N$ *is any solution of* (4) *existing on* $[t_0,\infty)$*, provided that*

$$V(t_0^+,x_0)\le u_0. \tag{8}$$

*Proof.* Let $\eta \in \bar{S}_\psi =: \{\eta \in \mathbb{R}^n : \|\eta\| \leq \psi\}$ be a small enough arbitrary vector and consider the initial value problem for the following system of fractional differential equations.

$$\begin{aligned} {}^C D^q u = g(t,u) + \eta, \Delta u = \psi_k(u(t_k)), t = t_k, k = 1,2,... \\ u(t_0^+) = u_0 + \eta, \end{aligned} \tag{9}$$

for $t \in [t_0, \infty)$.

The function $u_\eta(t,\alpha)$ is a solution of (9), where $\alpha > 0$, if and only if it satisfies the Volterra Integral equation

$$u_\eta(t,\alpha) = u_0 + \eta + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1}(g(s,u_\eta(s,\alpha)) + \eta)ds, \; t \in [t_0,\infty). \tag{10}$$

Let the function $m(t,\alpha) \in C([t_0,T] \times \bar{S}_\psi, \mathbb{R}^N)$ be defined as $m(t,\alpha) = V(t,x^*(t))$.

We now prove that

$$m(t,\alpha) < u_\eta(t,\alpha), \quad \text{for} \quad t \in [t_0,\infty). \tag{11}$$

Observe that the inequality (11) holds for $t = t_0$ i.e

$$m(t_0,\alpha) = V(t_0,x_0) \leq u_0 < u_\eta(t_0,\alpha).$$

Assume that the inequality (11) is not true, then there exist a point $t_1 > t_0$ such that

$$m(t_1,\alpha) = u_\eta(t_1,\alpha) \quad \text{and} \quad m(t,\alpha) < u_\eta(t,\alpha), \quad \text{for} \quad t \in [t_0,t_1).$$

It follows from lemma (3.2) that

$${}^C D_+^q m(t_1,\alpha) - {}^C D_+^q u_\eta(t_1,\alpha) > 0,$$

so that

$${}^C D_+^q (V(t_1,x(t_1))) > {}^C D_+^q (u_\eta(t_1,\alpha)),$$

and using (9) we arrive at

$${}^C D_+^q (V(t_1,x(t_1))) > g(t_1,u_\eta(t_1,\alpha) + \eta) > g(t_1,u(t_1,\alpha)).$$

Therefore,

$${}^C D_+^q (m(t_1,\alpha)) > g(t_1,u(t_1,\alpha)). \tag{12}$$

For $t \in [t_0,T]$, we maintain that $x^*(t)$ satisfies (4) and the equality,

$$\limsup_{h \to 0^+} \frac{1}{h^q}[x^*(t) - x_0 - S(x^*(t),h)] = f(t,x^*(t)), \tag{13}$$

holds, where $x^*(t)$ is any other solution of (4).

$$S(x^*(t),h) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[x^*(t-rh) - x_0)], \tag{14}$$

is the Grunwald Letnikov fractional derivative and $[\frac{t-t_0}{h}]$ is the integer part of $\frac{t-t_0}{h}$.
Multiply (13) through by $h^q$ we have,

$$\limsup_{h\to 0^+}[x^*(t) - x_0 - S(x^*(t), h)] = h^q f(t, x^*(t))$$

$$x^*(t) - x_0 - \limsup_{h\to 0^+}[S(x^*(t), h)] = h^q f(t, x^*(t))$$

$$x^*(t) - x_0 - [S(x^*(t), h) + \rho(h^q)] = h^q f(t, x^*(t))$$

$$x^*(t) - h^q f(t, x^*(t)) = [S(x^*(t), h) + x_0 + \rho(h^q)]. \tag{15}$$

For $t \in [t_0, T]$, we have

$$\begin{aligned}
&m(t,\alpha) - m(t_0,\alpha) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[m(t-rh,\alpha) - m(t_0,\alpha)] \\
&= V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \\
&= V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,q}C_r[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \\
&+ \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)\{[V(t-rh, S(x^*(t), h) + x_0 + \rho(h^q) - V(t_0, x_0)] \\
&-[V(t-rh, x^*(t-rh) - V(t_0), x_0)]\}.
\end{aligned} \tag{16}$$

Since $V(t, x)$ is locally Lipschtzian in the second variable, with $L > 0$ as the Lipschitz constant, then from (16) we obtain

$$\begin{aligned}
& V(t, x^*(t)) - V(t_0, x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1\,q}C_r[V(t-rh, x^*(t) - h^q f(t, x^*(t)) - V(t_0, x_0)] \\
& + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)\{[V(t-rh, S(x^*(t), h) + x_0 + \rho(h^q) - V(t_0, x_0)] \\
& -[V(t-rh, x^*(t-rh) - V(t_0), x_0)]\} \\
\leq\ & L|(-1)^{r+1}|||\sum_{r=1}^{[\frac{t-t_0}{h}]} ({}^qC_r)[S(x^*(t), h) + x_0 + \rho(h^q) - x^*(t-rh)]||
\end{aligned}$$

$$\leq L||\sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r[S(x^*(t), h) + \rho(h^q) - (x^*(t-rh) - x_0)]||. \tag{17}$$

Using (14), (17) becomes,

$$
\begin{aligned}
& L\Big\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r( \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}{}^qC_r[x^*(t-rh)-x_0)] + \rho(h^q) - (x^*(t-rh)-x_0)]\Big\| \\
\le\ & L\Big\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(-1)^{r+1}( \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r[x^*(t-rh)-x_0)] \\
& + \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r\rho(h^q) - \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh)-x_0)]\Big\| \\
\le\ & L\Big\|(-1)^{r+1} \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r(x^*(t-rh)-x_0)[ \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^qC_r\rho(h^q)\Big\|. \qquad (18)
\end{aligned}
$$

Substituting (18) into (16) yields,

$$
\begin{aligned}
& V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)[V(t-rh, x^*(t) - h^q f(t,x^*(t)) - V(t_0,x_0)] \\
& + L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r) - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)\rho(h^q)\| \\
& = V(t,x^*(t)) - V(t_0,x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}{}^qC_r[V(t-rh, x^*(t) - h^q f(t,x^*(t)) - V(t_0,x_0)] \\
& + L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)(x^*(t-rh)-x_0)[- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}{}^qC_r - 1] + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}({}^qC_r)\rho(h^q)\|.
\end{aligned}
$$

Dividing through by $h^q > 0$ and taking the $limsup$ as $h \to 0^+$ we have,

$$
\begin{aligned}
{}^C D_+^q m(t,\alpha) \ =\ & \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,x^*(t)) - V(t_0,x_0) \\
& - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}{}^qC_r[V(t-rh, x^*(t) - h^q f(t,x^*(t)) - V(t_0,x_0)]\Big\} \\
& + \limsup_{h\to 0^+} \frac{1}{h^q} L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}{}^qC_r(x^*(t-rh)-x_0)[- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}{}^qC_r - 1] \\
& + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1}{}^qC_r\rho(h^q)\|.
\end{aligned}
$$

Recall,

$$
\lim_{h\to 0^+} \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r}{}^qC_r = -1 \ and \ \limsup_{h\to 0^+} \frac{1}{h^q}\rho(h^q) = 0,
$$

from (3.6) and (3.7) in [4] we have,

$$ {}^{C}D_{+}^{q}m(t,\alpha) = {}^{C}D_{+}^{q}V(t,x^{*}(t)) + L\| \sum_{r=1}^{[\frac{t-t_0}{h}]} {}^{q}C_{r}(x^{*}(t-rh)-x_0)[-(-1)-1]+0\| $$

$$ {}^{C}D_{+}^{q}m(t,\alpha) = {}^{C}D_{+}^{q}V(t,x^{*}(t)) + 0, $$

using condition $(ii)$ of Theorem 4.1 we have

$$ {}^{C}D_{+}^{q}m(t,\alpha) \leq g(t,V(t,x^{*}(t))) = g(t,m(t,\alpha)). \tag{19} $$

Also,

$$ m(t_0^{+},\alpha) \leq u_0 \; and \; m(t_k^{+},\alpha) = V(t_k^{+}, x(t_k) + I_k(x(t_k)) \leq \rho_k(m(t_k)). \tag{20} $$

Now, (19) with $t = t_1$ contradicts (12), hence (11) holds. □

For $t \in [t_0, T]$, we now show that whenever $\eta_1 < \eta_2$, then

$$ u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha). \tag{21} $$

It is obvious that (21) holds for $t = t_0$. Assume the inequality (21) is not true. Then there exist a point $t_1 > t_0$ such that $u_{\eta_1}(t_1,\alpha) = u_{\eta_2}(t_1,\alpha)$ and $u_{\eta_1}(t,\alpha) < u_{\eta_2}(t,\alpha)$ for $t \in [t_0, t_1)$.
By lemma (3.2), we have that

$$ {}^{C}D_{+}^{q}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha)) > 0. $$

However,

$$ \begin{aligned} {}^{C}D_{+}^{q}(u_{\eta_1}(t_1,\alpha) - u_{\eta_2}(t_1,\alpha_)) &= {}^{C}D_{+}^{q}u_{\eta_1}(t_1,\alpha) - {}^{C}D_{+}^{q}u_{\eta_2}(t_1,\alpha) \\ &= g(t_1, u(t_1,\alpha)+\eta_1)) - [g(t_1, u(t_1,\alpha)+\eta_2))] \\ &= \eta_1 - \eta_2 < 0, \end{aligned} $$

which is a contradiction and so (21) is true. Thus, (11) and (21) guarantee that the family of solutions $\{u_\eta(t,\alpha)\}$, $t \in [t_0, T]$ of (9) is uniformly bounded, i.e. there exists $P > 0$ with $|u_\eta(t,\alpha)| \leq P$, with bound $P$ on $[t_0, T]$.
We now show that the family $\{u_\eta(t,\alpha)\}$ is equicontinuous on $[t_0, T]$. Assume $K = sup\{g(t,x) : (t,x) \in [t_0, T] \times [-P, P]\}$. Also, fix a decreasing sequence $\{\eta_i\}_{i=1}^{\infty}(t)$, such that $\lim_{i\to\infty} \eta_i = 0$ and consider a sequence of functions $u_{\eta_i}(t,\alpha)$. Again let $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$, then we have the following estimate

$$ \begin{aligned} \|u_{\eta_i}(t_2,\alpha) - u_{\eta_i}(t_1,\alpha)\| &= \|u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i) \\ &- (u_0 + \eta_i + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)) + \eta_i))\| \\ &= \frac{1}{\Gamma(q)}\|\int_{t_0}^{t_2}(t_2-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds \end{aligned} $$

$$
\begin{aligned}
&- \int_{t_0}^{t_1}(t_1-s)^{q-1}(g(s,u_{\eta_i}(s,\alpha)))ds\| \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_2}(t_2-s)^{q-1}-\int_{t_0}^{t_1}(t_1-s)^{q-1}\right|ds \\
&= \frac{k}{\Gamma(q)}\left|-(\int_{t_0}^{t_1}(t_1-s)^{q-1}-\int_{t_0}^{t_2}(t_2-s)^{q-1})ds\right| \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}-(\int_{t_0}^{t_1}(t_2-s)^{q-1}+\int_{t_1}^{t_2}(t_2-s)^{q-1})ds\right| \\
&= \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}-\int_{t_0}^{t_1}(t_2-s)^{q-1}-\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right| \\
&\leq \frac{k}{\Gamma(q)}\left|\int_{t_0}^{t_1}(t_1-s)^{q-1}-\int_{t_0}^{t_1}(t_2-s)^{q-1}\right|ds+\left|\int_{t_1}^{t_2}(t_2-s)^{q-1}ds\right| \\
&= \frac{k}{\Gamma(q)}\left|\frac{(t_1-t_0)^q}{q}+\frac{(t_2-t_1)^q}{q}-\frac{(t_2-t_0)^q}{q}\right|+\frac{(t_2-t_1)^q}{q} \\
&\leq \frac{k}{\Gamma(q+1)}(t_1-t_0)^q+(t_2-t_1)^q-(t_2-t_0)^q+(t_2-t_1)^q \\
&= \frac{k}{\Gamma(q+1)}(t_1-t_0)^q-(t_2-t_0)^q+2(t_2-t_1)^q \\
&\leq \frac{2k}{\Gamma(q+1)}(t_2-t_1)^q<\epsilon,
\end{aligned}
$$

provided $\|t_2-t_1\|<\delta=(\frac{\epsilon\Gamma(q+1)}{2k})^{\frac{1}{q}}$, proving that the family of solutions $\{u_{\eta_i}(t;\alpha)\}$ is equi-continuous. By the Arzela-Ascoli theorem, $\{u_{\eta_i}(t;\alpha)\}$has a subsequence $\{u_{\eta_{i_j}}(t;\alpha)\}$ which converges uniformly to a function $r(t)$ on $[t_0,T]$. We then show that $r(t)$ is a solution of (10). Now, (10) becomes

$$
u_{\eta_{i_j}}(t,\alpha)\ =u_{0_{i_j}}+\eta_{i_j}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g_{i_j}(s,u_{i_j}(s,\eta_{i_j}))+\eta_{i_j})ds, \tag{22}
$$

taking the limit as $i_j\to\infty$ in (22), yields

$$
r(t)=u_0+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(g(s,r(t)))ds. \tag{23}
$$

Thus, $r(t)$ is a solution of (6) on $[t_0,T]$. We claim that $r(t)$ is the maximal solution of (6). Then from (11), we have that $m(t,\alpha)<u_\eta(t,\alpha)\leq r(t)$ on $[t_0,T]$.

## 5. Main Results

In this section, we will obtain sufficient conditions for the uniform eventual stability of the system (4).

**Theorem 5.1.** *Assume the following*

(i) $g\in PC[\mathbb{R}_+\times\mathbb{R}^n_+,\mathbb{R}^n]$ *satisfies* $(A_0)(ii)$ *and* $g(t,u)$ *is quasi-monotone non-decreasing in* $u$ *with* $g(t,0)\equiv 0$.

(ii) *$V : \mathbb{R}_+ \times S_\psi \to \mathbb{R}^N_+$, and $V \in \mathcal{L}$ with $V(t,0) \equiv 0$ such that*

$$^{C}D^q_+ V(t,x) \le g(t, V(t,x)), t \ne t_k, (t,x) \in \mathbb{R}_+ \times S_\psi, \tag{24}$$

*holds for all $(t,x) \in \mathbb{R}_+ \times S_\psi$.*

(iii) *there exists a $\psi_0 > 0$ such that $x_0 \in S_\psi$ implies that*

$$x + I_k(x(t_k)) \in S_\psi \text{ and } V(t_k^+, x + I_k(x(t_k))) \le \psi_k(V(t,x)), t = t_k, x \in S_\psi,$$

*and the function $\psi_k : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ is nondecreasing for k=1,2,...*

(iv) *$b(\|x\|) \le V_0(t,x) \le a(\|x\|)$, where $a, b \in \mathcal{K}$ and $V_0(t,x) = \sum_{i=1}^N V_i(t,x)$.*

*Then the uniform eventual stability of the trivial solution $u = 0$ of the IFrDE (6) implies the uniform eventual stability of the trivial solution $x = 0$ of (4).*

*Proof.* Let $0 < \epsilon < \psi$ and $t_0 \in \mathbb{R}_+$ be given.

Assume that the solution $u = 0$ of (6) is uniformly eventually stable. Then given each $b(\epsilon) > 0$, and $t_0 \in \mathbb{R}_+$, there exist a positive function $\delta_1 = \delta_1(\epsilon) > 0$ such that whenever

$$u_0 = \sum_{i=1}^n u_{i0} \le \delta, \text{ we have } \sum_{i=1}^n u_i(t; t_0, u_0) < b(\epsilon), \ t \ge t_0, \tag{25}$$

where $u(t; t_0, u_0)$ is any solution of (6).

let us choose $V(t_0^+, x_0) \le u_0$ and

$$\sum_{i=1}^n u_{i0} = a(t_0, \|x_0\|).$$

Since $a(t, \mathcal{K})$ and $a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ we can find a positive function $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \|x_0\|) < \delta_1, \text{ and } \|x_0\| < \delta, \tag{26}$$

hold simultaneously. We claim that if

$$\|x_0\| \le \delta, \text{ then } \|x(t, t_0, x_0)\| \le \epsilon, \ t \ge t_0.$$

Suppose that this claim is not true. Then there would exists a point $t_1 > t_0$ and a solution $x(t)$ with $\|x_0\| < \delta$ such that

$$\|x(t_1)\| = \epsilon \text{ and } \|x(t)\| < \epsilon, \text{ for } t \in [t_0, t_1). \tag{27}$$

This implies that $x(t) + I_k(x(t_k)) \in S_\psi$ for $t \in [t_0, t_1)$.

From (7) we have that

$$V_0(t, x(t)) \le r_0(t, t_0, u_0) \ for \ t \in [t_0, t_1). \tag{28}$$

Combining condition $(iv)$ and (28) we have

$$b(\epsilon) \le \sum_{i=1}^n V_i(t_1, x(t_1)) \le \sum_{i=1}^n r_i(t; t_0, u_0). \tag{29}$$

Using (25), (27) and (29) we have,

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, x(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < b(\epsilon),$$

which leads to an absurdity that $b(\epsilon) < b(\epsilon)$.

Hence, the uniform eventual stability of the trivial solution $u(t) = 0$ of (6) implies the uniform eventual stability of the set $x(t) = 0$ of (4). □

## 6. Application

Let the points $t_k, t_k < t_{k+1}, lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive Caputo fractional differential equations

$$\begin{aligned} {}^C D^q x_1(t) &= -15x_1 - \frac{x_2^2 cosecx_1}{2x_1} + x_1 sinx_2 + \frac{3x_2^2 sinx_1}{x_1} \\ {}^C D^q x_2(t) &= \frac{3x_1^2}{x_2} - x_2 sinx_1 - 5x_2 cosecx_1 - x_1^2 cosx_2 \\ \Delta x_1 &= s_k(x(t_k)), \Delta x_2 = n_k(x(t_k)), k = 1, 2, .. \end{aligned} \tag{30}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_{10} \quad \text{and} \quad x_2(t_0^+) = x_{20}.$$

Consider a vector $V = (V_1, V_2)^T$, where

$V_1(t, x_1, x_2) = x_1^2$ and $V_2(t, x_1, x_2) = x_2^2$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$.

Now,

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = x_1^2 + x_2^2,$$

and so $b(\|x\|) \leq V_0(t, x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, implying that $a, b \in \mathcal{K}$. From (5), we compute the Caputo fractional Dini derivative for $V_1(t, x_1, x_2) = x_1^2$ for $t > 0, t \neq t_k$ as follows:

$$\begin{aligned} &{}^C D_+^q V_1(t; x_1, x_2) \\ =\ & \limsup_{h\to 0^+} \frac{1}{h^q} \left\{ V(t, x) - V(t_0, x_0) + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^q C_r)[V(t - rh, x - h^q f(t, x)) - V(t_0, x_0)] \right\} \\ =\ & \limsup_{h\to 0^+} \frac{1}{h^q} \left\{ x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^q C_r)[(x_1 - h^q f_1(t; x_1, x_2)^2 - x_{10}^2] \right\} \\ \leq\ & \limsup_{h\to 0^+} \frac{1}{h^q} \{ x_1^2 - x_{10}^2| + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^r ({}^q C_r)[x_1^2 - 2x_1 h^q f_1(t; x_1, x_2) + h^{2q} f_1(t, x_1) - x_{10}^2] \} \end{aligned}$$

$$= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_1^2 - x_{10}^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_1^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)2x_1h^qf_1(t;x_1,x_2)$$

$$- \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^{2q}f_1(t;x_1,x_2)\}$$

$$= \limsup_{h\to 0^+} \frac{1}{h^q}\{x_1^2 + \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_1^2 - x_{10}^2 - \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_{10}^2$$

$$-2x_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^qf_1(t;x_1,x_2)|\}$$

$$= \limsup_{h\to 0^+} \frac{1}{h^q}\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_1^2 - \sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_{10}^2 - 2x_1\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^qf_1(t;x_1,x_2)\}.$$

Recall that,

$$\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r) = \frac{x_1^2}{t^q\Gamma(1-q)}, \text{ and } \quad \lim_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r) = -1.$$

Applying (3.7) and (3.8) in [4]. we obtain,

$$\begin{aligned}
{}^CD_+^qV_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)} - \frac{x_{10}^2}{t^q\Gamma(1-q)} + 2x_1f_1(t;x_1,x_2)\\
&\leq \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1f_1(t;x_1,x_2)\\
{}^CD_+^qV_1(t;x_1,x_2) &\leq \frac{x_1^2}{t^q\Gamma(1-q)} + 2x_1(-15x_1 - \frac{x_2^2cosecx_1}{2x_1} + x_1sinx_2 + \frac{3x_2^2sinx_1}{x_1})\\
&= \frac{x_1^2}{t^q\Gamma(1-q)} - 30x_1^2 - x_2^2cosecx_1 + 2x_1^2sinx_2 + 6x_2^2sinx_1.
\end{aligned}$$

As $t\to\infty$, $\frac{x_1^2}{t^q\Gamma(1-q)} \to 0$, so that we have

$$\begin{aligned}
{}^CD_+^qV_1(t;x_1,x_2) &\leq -30x_1^2 - x_2^2cosecx_1 + 2x_1^2sinx_2 + 6x_2^2sinx_1\\
&= 2x_1^2(-15 + sinx_2) + x_2^2(6sinx_1 - cosecx_1)\\
&\leq 2x_1^2(-15 + |sinx_2|) + x_2^2(6|sinx_1| - \frac{1}{|sinx_1|})\\
&\leq 2x_1^2(-15+1) + x_2^2(6-1)\\
&\leq 2x_1^2(-14) + 5x_2^2\\
&\leq x_1^2(-14) + 5x_2^2.
\end{aligned}$$

Therefore,

$$^{C}D_{+}^{q}V_1(t;x_1,x_2) \quad \leq \quad -14V_1+5V_2. \tag{31}$$

Also, for $x_0 \in S_\psi$, for $t=t_k, k=1,2,..,$ we have

$$V(t,x(t)+c_k)=|c_k+x(t)|\leq V(t,x(t)).$$

Similarly, using (5), we compute the Caputo fractional Dini derivative for $V_2(t,x_1,x_2)=x_2^2$ as follows:

$$\begin{aligned}
^{C}D_{+}^{q}V_2(t;x_1,x_2) &= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{V(t,x)-V(t_0,x_0)\\
&\quad+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)[V(t-rh,x-h^qf(t,x))-V(t_0,x_0)]\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_2^2-x_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)[(x_2-h^qf_2(t;x_1,x_2)^2-x_{20}^2]\big\}\\
&\leq \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_2^2-x_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)[x_2^2-2x_2h^qf_2(t;x_1,x_2)\\
&\quad+h^{2q}f_2(t,x_2)-x_{20}^2]\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_2^2-x_{20}^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_2^2\\
&\quad-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)2x_2h^qf_2(t;x_1,x_2)-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^{2q}f_2(t;x_1,x_2)\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{x_2^2+\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_2^2-x_{20}^2-\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_{20}^2\\
&\quad-2x_2\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^qf_2(t;x_1,x_2)|\big\}\\
&= \limsup_{h\to 0^+}\frac{1}{h^q}\big\{\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_2^2-\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)x_{20}^2\\
&\quad-2x_2\sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^r({}^qC_r)h^qf_2(t;x_1,x_2)\big\}\\
^{C}D_{+}^{q}V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)}-\frac{x_{20}^2}{t^q\Gamma(1-q)}+2x_2f_2(t;x_1,x_2)
\end{aligned}$$

$$^{C}D_{+}^{q}V_2(t;x_1,x_2) \quad \leq \quad \frac{x_2^2}{t^q\Gamma(1-q)}+2x_2f_2(t;x_1,x_2) \tag{32}$$

$$\begin{aligned}
{}^{C}D_{+}^{q}V_2(t;x_1,x_2) &\leq \frac{x_2^2}{t^q\Gamma(1-q)} + 2x_2\left(\frac{3x_1^2}{x_2} - x_2 sinx_1 - 5x_2 cosecx_1 - \frac{x_1^2 cosx_2}{x_2}\right) \\
&= \frac{x_1^2}{t^q\Gamma(1-q)} - 2x_2^2 sinx_1 + 6x_1^2 - 10x_2^2 cosecx_1 - 2x_1^2 cosx_2 \\
&\leq -2x_2^2 sinx_1 - 10x_2^2 cosecx_1 + 6x_1^2 - 2x_1^2 cosx_2 \\
&= -2x_2^2(sinx_1 + 5cosecx_1) + 2x_1^2(3 - cosx_2) \\
&\leq -2x_2^2(|sinx_1| + \frac{5}{|sinx_1|}) + 2x_1^2(3 - |cosx_2|) \\
&\leq -2x_2^2(1+5) + 2x_1^2(3-1) \\
&= -2x_2^2(6) + 2x_1^2(2) \\
&\leq -12V_2 + 4V_1.
\end{aligned}$$

Therefore,

$$ {}^{C}D_{+}^{q}V_1(t;x_1,x_2) \leq 4V_1 - 12V_2. \tag{33}$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$V(t, x(t) + d_k) = |d_k + x(t)| \leq V(t, x(t))$$

Combining (31) and (33), we have

$$ {}^{C}D_{+}V \leq \begin{pmatrix} -13 & 5 \\ 4 & -12 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V). \tag{34}$$

Now, consider the comparison system

$$ {}^{C}D^{q}u = g(t, u) = Au. \tag{35}$$

where $A = \begin{pmatrix} -13 & 5 \\ 4 & -12 \end{pmatrix}$.
Thus, the vectorial inequality (34) and all other conditions of Theorem (5.1) are satisfied since the eigenvalues of $A$ are all negative real parts. Hence, the system (30) is uniformly eventually stable. Therefore, the set $x(t) = 0$ for the system of IFrDE (30) is uniformly eventually stable.

## 7. CONCLUSION

In this paper, the uniform eventual stability of impulsive Caputo fractional differential equations using a class of piecewise continuous functions which generalizes the vector Lyapunov functions is examined. Together with the comparison results, sufficient conditions for the uniform eventual stability of the set $x(t) = 0$ is established with an illustrative example.

## Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.


## Funding

This research was partly funded by Topfaith University, Mkpatak, Nigeria.


## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References


[1] J.E. Ante, O.O. Itam, J.U. Atsu, S.O. Essang, E.E. Abraham, M.P. Ineh, On the novel auxiliary Lyapunov function and uniform asymptotic practical stability of nonlinear impulsive caputo fractional differential equations via new modelled generalized dini derivative, Afr. J. Math. Stat. Stud. 7 (2024), 11–33. https://doi.org/10.52589/AJMSS-VUNAIOBC.

[2] J.E. Ante, J.U. Atsu, E.E. Abraham, O.O. Itam, E.J. Oduobuk, A.B. Inyang, On a class of piecewise continuous Lyapunov functions and asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized dini derivative, IEEE-SEM, 12 (2024), 1–21.

[3] J.O. Achuobi, E.P. Akpan, R. George, A.E. Ofem, Stability analysis of Caputo fractional time-dependent systems with delay using vector Lyapunov functions, AIMS Math. 9 (2024), 28079–28099. https://doi.org/10.3934/math.20241362.

[4] R. Agarwal, D. O'Regan, S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions, Appl. Math. 60 (2015), 653–676. https://doi.org/10.1007/s10492-015-0116-4.

[5] R. Agarwal, S. Hristova, D. O'Regan, Stability of solutions of impulsive Caputo fractional differential equations, Elec. J. Diff. Equ. 2016 (2016), 58.

[6] E.P. Akpan, O. Akinyele, On the $\phi_0$-stability of comparison differential systems, J. Math. Anal. Appl. 164 (1992), 307–324. https://doi.org/10.1016/0022-247X(92)90116-U.

[7] L. Arnold, B. Schmalfuss, Lyapunov's second method for random dynamical systems, J. Diff. Equ. 177 (2001), 235–265. https://doi.org/10.1006/jdeq.2000.3991.

[8] D. Baleanu, O.G. Mustafa, On the global existence of solutions to a class of fractional differential equations, Comp. Math. Appl. 59 (2010), 1835–1841. https://doi.org/10.1016/j.camwa.2009.08.028.

[9] S.E. Ekoro, A.E. Ofem, F.A. Adie, J. Oboyi, G.I. Ogban, M.P. Ineh, On a faster iterative method for solving nonlinear fractional integro-differential equations with impulsive and integral conditions. Palestine J. Math. 12 (2023), 477–484.

[10] T.A. Burton, Fractional differential equations and lyapunov functionals, Nonlinear Anal.: Theory Meth. Appl. 74 (2011), 5648–5662. https://doi.org/10.1016/j.na.2011.05.050.

[11] J. Vasundhara Devi, F.A. Mc Rae, Z. Drici, Variational lyapunov method for fractional differential equations, Comp. Math. Appl. 64 (2012), 2982–2989. https://doi.org/10.1016/j.camwa.2012.01.070.

[12] K. Diethelm, The analysis of fractional differential equations: an application-oriented exposition using differential operators of Caputo type, Springer, Berlin, 2010. https://doi.org/10.1007/978-3-642-14574-2.

[13] J. R. Haddock, Some new results on stability and convergence of solutions of ordinary and functional differential equations, Funkcial. Ekvac. 19 (1976), 247–269.

[14] J.K. Hale, Theory of functional differential equations, Springer, New York, 1977. https://doi.org/10.1007/978-1-4612-9892-2.

[15] D.K. Igobi, E. Ndiyo, M.P. Ineh, Variational stability results of dynamic equations on time-scales using generalized ordinary differential equations, World J. Appl. Sci. Technol. 15 (2024), 245–254. https://doi.org/10.4314/wojast.v15i2.14.

[16] I.D. Kanu, M.P. Ihen, Results on existence and uniqueness of solutions of dynamic equations on time scale via generalized ordinary differential equations, Int. J. Appl. Math. 37 (2024), 1–20. https://doi.org/10.12732/ijam.v37i1.1.

[17] I. Dodi K, A. Jackson E, Results on existence and uniqueness of solution of impulsive neutral integro-differential system, J. Math. Res. 10 (2018), 165–174. https://doi.org/10.5539/jmr.v10n4p165.

[18] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional integrals and derivatives: Theory and applications, Gordon and Breach, 1993.

[19] G.K. Kulev, Uniform asymptotic stability in impulsive perturbed systems of differential equations, J. Comp. Appl. Math. 41 (1992), 49–55. https://doi.org/10.1016/0377-0427(92)90237-R.

[20] M.P. Ineh, E.P. Akpan, H. Nabwey, On lyapunov stability of caputo fractional dynamic equations on time scale using a new generalized derivative, preprints (2024), 2024062042. https://doi.org/10.20944/preprints202406.2042.v1.

[21] V. Lakshmikantham, On the method of vector Lyapunov functions, Mathematics Technical Papers 51, (1974). https://mavmatrix.uta.edu/math_technicalpapers/51.

[22] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of impulsive differential equations, World Scientific, Singapore, (1989).

[23] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal.: Theory Meth. Appl. 69 (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042.

[24] V. Lakshmikantham, S. Leela, M. Sambandham, Lyapunov theory for fractional differential equations, Comm. Appl. Anal. 12 (2008), 365–376.

[25] V. Lakshmikantham, S. Leela, Differential and integral inequalities: Theory and application, Academic Press, 1969.

[26] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of fractional dynamic systems, Cambridge Scientific Publ, Cottenham, 2009. .

[27] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett. 21 (2008), 828–834. https://doi.org/10.1016/j.aml.2007.09.006.

[28] V. Lakshmikantham, S. Leela, A.S. Martynyuk, Practical stability analysis of nonlinear systems, World Scientific, 1990.

[29] V. Lakshmikantham, S. Sivasundaram, On vector Lyapunov functions and stability analysis of non-linear systems, Springer, 1991.

[30] Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: lyapunov direct method and generalized mittag–leffler stability, Comp. Math. Appl. 59 (2010), 1810–1821. https://doi.org/10.1016/j.camwa.2009.08.019.

[31] C. Li, D. Qian, Y. Chen, On Riemann-liouville and Caputo derivatives, Discr. Dyn. Nat. Soc. 2011 (2011), 562494. https://doi.org/10.1155/2011/562494.

[32] Z. Lipcsey, J.A. Ugboh, I.M. Esuabana, I.O. Isaac, Existence theorem for impulsive differential equations with measurable right side for handling delay problems, J. Math. 2020 (2020), 7089313. https://doi.org/10.1155/2020/7089313.

[33] A. Ludwig, B. Pustal, D.M. Herlach, General concept for a stability analysis of a planar interface under rapid solidification conditions in multi-component alloy systems, Mater. Sci. Eng.: A 304–306 (2001), 277–280. https://doi.org/10.1016/S0921-5093(00)01451-9.

[34] V.D. Milman, A.D. Myshkis, On motion stability with shocks, Sibirsk. Mat. Zh. 1 (1960), 233–237.

[35] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, San Diego, 1998.

[36] Z. Qunli, A class of vector Lyapunov functions for stability analysis of nonlinear impulsive differential systems, Math. Probl. Eng. 2014 (2014), 649012. https://doi.org/10.1155/2014/649012.

[37] A. Sood, S.K. Srivastava, On retention of eventual stability of perturbed impulsive differential systems, J. Math. fund. Sci. 48 (2016), 1–11.

[38] A.A. Soliman, Stability criteria of impulsive differential systems, Appl. Math. Comp. 134 (2003), 445–457. https://doi.org/10.1016/S0096-3003(01)00293-4.

[39] S.K. Srivastava, K. Amanpreet, A new approach to stability of impulsive differential equations, Int. J. Math. Anal. 3 (2009), 179–185.

[40] I. Stamova, Eventual stability and eventual boundedness for impulsive differential equations with "supremum", Math. Model. Anal. 16 (2011), 304–314. https://doi.org/10.3846/13926292.2011.580470.

[41] J.A. Ugboh, I.M. Esuabana, Existence and uniqueness result for a class of impulsive delay differential equations, Int. J. Chem. Math. Phys. 2 (2018), 27–32.

[42] T. Yoshizawa, Stability theory by Lyapunov's second method, The Mathematical Society of Japan, 1966.

[43] C. Wu, A general comparison principle for Caputo fractional-order ordinary differential equations, Fractals 28 (2020), 2050070. https://doi.org/10.1142/S0218348X2050070X.