PTEP
Prog. Theor. Exp. Phys. **2021**, 103E01 (20 pages)
DOI: 10.1093/ptep/ptab116

Higher-dimensional inhomogeneous composite fluids: energy conditions

Byron P. Brassel*, Sunil D. Maharaj*, and Rituparno Goswami*

Astrophysics Research Centre, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
E-mail: drbrasselint@gmail.com; maharaj@ukzn.ac.za; goswami@ukzn.ac.za

Received June 7, 2021; Revised August 27, 2021; Accepted August 28, 2021; Published September 4, 2021

The energy conditions are studied, in the relativistic astrophysical setting, for higher-dimensional Hawking–Ellis Type I and Type II matter fields. The null, weak, dominant and strong energy conditions are investigated for a higher-dimensional inhomogeneous, composite fluid distribution consisting of anisotropy, shear stresses, non-vanishing viscosity as well as a null dust and null string energy density. These conditions are expressed as a system of six equations in the matter variables where the presence of the higher dimension N is explicit. The form and structure of the energy conditions is influenced by the geometry of the $(N-2)$-sphere. The energy conditions for the higher-dimensional Type II fluid are also generated, and it is shown that under certain restrictions the conditions for a Type I fluid are regained. All previous treatments for four dimensions are contained in our work.

Subject Index E00, E01

1. Introduction

Higher-dimensional gravity has come to the forefront in recent times with the advent of modified gravitation theories such as the $f(R)$ models [1], the Lovelock gravity models [2], supergravity [3], Jackiw–Teitelboim gravity [4], de Rham–Gabadadze–Tolley (dRGT) massive gravity [5] and others. In this regard, the energy conditions for realistic matter fields pertaining to any relevant theory of gravity become important, as does their connection to the spacetime dimension. In the context of general relativity, the energy conditions have been described comprehensively for Type I, II, III and IV matter distributions by Hawking and Ellis [6]. Of these, the Type III and Type IV matter tensors are not physically viable matter distributions, especially for astrophysical models such as radiating stars, since they violate the null energy condition [7]. The physical model we have in mind is a radiating star in general relativity. The interior matter distribution is a Type I fluid and the exterior matter distribution can be taken to be a Type II fluid. Therefore, for the purposes of this paper, only the former Type I and Type II fluids will be addressed.

The energy conditions have been studied in the context of both cosmology and astrophysical applications. Santos *et al.* [8] used type Ia supernovae observations for attractive gravity and cosmic acceleration to study the energy conditions. Xiong and Xhu [9] showed that the appearance of a quantum geometry potential strengthened the violation of the strong energy condition in small volume regions, in loop quantum cosmology. In modified gravity, Nashed [10] studied the validity of the energy conditions of built-in inflation models in $f(T)$ gravity, formulating the system of equations by assuming a Friedmann–Robertson–Walker (FRW) universe for both the flat and non-flat cases. Various features of energy conditions were investigated by Martín-Moruno and Visser [11–15]. With regards to relativistic astrophysics, the energy conditions are paramount in describing physically reasonable static stars in general relativity [16–19] as well as radiating stars [20–28]. In some of these treatments, the systems of equations generated for the energy conditions contain errors. Kolassis *et al.* [29] first considered the energy conditions for a Type I inhomogeneous fluid distribution with a radial heat flow and undergoing dissipation. The matching conditions for a radiating star in general relativity were found by Santos [30] for a shear-free matter distribution; an interior matter distribution which was barotropic in nature and containing a radial heat flow was matched across a surface to the Vaidya atmosphere. It was found that the pressure at the boundary was balanced by the heat flux emanating from the core of the star.

It is important to highlight the role of the spacetime dimension in the generation of the energy conditions. For example, in the framework of higher dimensions Paul [31] showed that the upper bound for the mass–radius limit in a constant density star varies as the dimension changes. As the number of spacetime dimensions are vital in the modeling of higher-dimensional radiating spheres and stars, the discovery of the higher-dimensional version of the Santos boundary condition became the prelude point for further development of relativistic stellar theories in higher dimensions. Bhui *et al.* [32], Shah *et al.* [33] and Banerjee and Chaterjee [34] analogously considered the interior matter to be shear-free which matches the higher-dimensional Vaidya exterior. Chaterjee and Banerjee [35] extended the C-field cosmology of Hoyle and Narlikar [36] to higher dimensions and obtained closed form solutions for a five-dimensional homogeneous spacetime. The energy conditions were then tested for these solutions. More recently, Maeda and Martínez [7] studied the energy conditions for Type I, II, III and IV Hawking–Ellis energy momentum tensors in higher-dimensional general relativity. The most general canonical forms of the four types of matter tensors were derived in higher dimensions and it was shown that the Type I and Type II matter tensors were the most physically relevant. These were then applied to various matter fields.

Our objective is to show that composite matter fields that arise in the modeling of radiating stars yield energy conditions in a simple form for the matter variables involving Type I fields in the interior and Type II fields across the boundary of the star. We aim to generate the energy conditions for a higher-dimensional composite Type I matter distribution. This matter field contains viscous matter, anisotropy, a null dust and a null string energy density. The junction conditions for the matching of an internal composite fluid distribution to the exterior generalised Vaidya spacetime in four dimensions were given by Maharaj and Brassel [37]. It was found that the pressure at the surface of the radiating star was proportional to the heat flux, the internal null string energy density and the anisotropy. This was further extended to include an electromagnetic field in Ref. [38] where the interior charge distribution became an added component in the aforementioned boundary condition. This boundary condition, containing the heat flux, internal null string, anisotropy and the charge, is the most generally known. That work was extended to higher dimensions in Ref. [39]. It was shown that the role of dimension in the dynamics is critical. The energy conditions for a composite matter distribution in four dimensions were completed in Ref. [40]. The present treatment is an extension of that work to arbitrary dimensions in order to unravel further the interplay between dimension and gravitational dynamics.

The basis of this paper is to find the energy conditions for a generalised higher-dimensional composite fluid, and a null dust and null string fluid. This result will be applicable to various cosmological and astrophysical models in both general relativity as well as any other modified gravity theory. We assume that the spacetime has the general spherically symmetric metric. The fluid distribution is a composite, containing a barotropic fluid, null dust and a null string energy density. The energy conditions are found for the composite matter distribution as a general system of six equations; this composite fluid can be Type I, Type II or Type IV. We determine the effect of the higher dimensions on the resulting energy conditions. A higher-dimensional null dust and null string matter field is considered and the energy conditions are presented.

2. Preliminaries

In higher dimensions, it is important to consider units carefully. In this treatment we assume $G = c = 1$ and the N-dimensional spacetime manifold has local Lorentzian signature $(-, +, +, +, \ldots, +)$. The Einstein coupling constant in higher dimensions is derived from Poisson's equation, and is given by

$$\kappa_N = \frac{2(N-2)\pi^{(N-1)/2}}{(N-3)\,\{[(N-1)/2]-1\}!}, \tag{1}$$

in terms of the factorial function. In four dimensions $\kappa_4 = 8\pi$.

The shear tensor σ_{ab} is defined in N dimensions as

$$\sigma_{ab} = u_{(a;b)} + \mathcal{A}_{(a}u_{b)} - \frac{1}{N-1}\Theta(g_{ab} + u_a u_b), \tag{2}$$

where $\mathbf{u}$ represents the N-velocity vector, $\mathcal{A}_a$ is the acceleration vector, and Θ is the expansion scalar. We can write

$$\mathcal{A}_a = u_{a;b}u^b, \qquad \Theta = u^a{}_{;a},$$

in N dimensions. The semicolon indicates covariant differentiation and the round brackets represent symmetrisation about the indices. For a viscous fluid, the anisotropic stress tensor π_{ab} has the form

$$\pi_{ab} = \eta\sigma_{ab}, \tag{3}$$

and $\eta \geq 0$ is the shear viscosity.

The N-dimensional Einstein field equations are given by

$$G_{ab} = \kappa_N T_{ab}, \tag{4}$$

where $G_{ab} = R_{ab} - (1/2)Rg_{ab}$ is the Einstein tensor and T_{ab} is the energy momentum tensor. Observe that the field equations depend critically on the dimension N of spacetime due to Einstein's gravitational constant κ_N. When $N = 4$, then $\kappa_4 = 8\pi$ and

$$G_{ab} = 8\pi T_{ab}, \tag{5}$$

which is the usual form in four dimensions.

2.1. *Composite fluids*

The interior of a stellar body can be described by a generalised inhomogeneous composite matter distribution. Such a distribution takes the form

$$\begin{aligned} T_{ab} = {} & (\mu + p_\perp)u_a u_b + p_\perp g_{ab} + (p_{||} - p_\perp)X_a X_b + q_a u_b + q_b u_a - 2\eta\sigma_{ab} \\ & + \epsilon l_a l_b + (\rho + \mathcal{P})(l_a n_b + l_b n_a) + \mathcal{P}g_{ab}, \end{aligned} \tag{6}$$

where μ is the energy density, $p_{||}$ is the radial pressure, $p_\perp$ is the tangential pressure, $\mathbf{X}$ is a N-vector along the radial direction and $\mathbf{q}$ is the heat flux vector measure relative to the fluid velocity $\mathbf{u}$. Also, ϵ is the energy density of the null dust, ρ is the null string energy density and $\mathcal{P}$ is the pressure of the null string fluid. The two quantities $\mathbf{l}$ and $\mathbf{n}$ are null vectors. The vectors $\mathbf{u}$, $\mathbf{q}$, $\mathbf{X}$, $\mathbf{l}$ and $\mathbf{n}$ obey the following restrictions:

$$u^a q_a = 0, \quad X_a X^a = 1, \quad u^a u_a = -1,$$
$$l^a l_a = n^a n_a = 0, \quad l_a n^a = -1, \quad l^a u_a = -1.$$

When $\rho = \mathcal{P} = 0$, the matter tensor (6) reduces to the viscous matter field with a null dust, as studied by Di Prisco *et al.* [41]. For the case where $\epsilon = \rho = \mathcal{P} = 0$, Eq. (6) reduces to the purely viscous/barotropic matter field, studied by [20–23] and others. We note that the above energy momentum tensor (6) is a mixture of a Type I fluid, a Type II fluid and a Type IV fluid in the Hawking–Ellis classification. Note that Eq. (6) can be Type IV depending on the values of the heat flow vector $\mathbf{q}$ or the shear viscosity η; if these quantities are large enough, the tensor (6) can be Type IV. Our interest is Type I and Type II fluids, on physical grounds. Note that the composite distribution is a Type I fluid *if and only if* the matrix of eigenvalues can be diagonalised. If one eigenvector is timelike and the comoving eigenvectors are spacelike, then the energy momentum tensor $T_{(ij)}$ in the tetrad basis is diagonalisable (see Appendix A.1). When $\mu = p_{||} = p_\perp = q = \eta = 0$, we then have a Type II fluid [7].

2.2. *Null and string fluids*

The energy momentum tensor for the Hawking–Ellis Type II matter distribution admits one eigenvector which is doubly null; thus two of the eigenvalues will be identical. The matter tensor for a Type II fluid is given by

$$T_{ab} = \nu \mathcal{L}_a \mathcal{L}_b + (\tilde{\rho} + P)(\mathcal{L}_a \mathcal{N}_b + \mathcal{L}_b \mathcal{N}_a) + P g_{ab}, \tag{7}$$

where we note that

$$T^{(I)}_{ab} = \nu \mathcal{L}_a \mathcal{L}_b,$$
$$T^{(II)}_{ab} = (\tilde{\rho} + P)(\mathcal{L}_a \mathcal{N}_b + \mathcal{L}_b \mathcal{N}_a) + P g_{ab}$$

are the null dust fluid and the additional null string fluid, respectively. Here, ν is the energy density of the null dust, $\tilde{\rho}$ is the energy density of the null string and P is the pressure of the null string. In the above, the vectors $\mathcal{L}$ and $\mathcal{N}$ are null. The null vector $\mathcal{L}$ is a double null eigenvector of the energy momentum tensor (7). The vectors $\mathcal{L}$ and $\mathcal{N}$ satisfy

$$\mathcal{L}_c \mathcal{L}^c = \mathcal{N}_c{}^c = 0, \quad \mathcal{L}_c \mathcal{N}^c = -1. \tag{8}$$

We note that when $\bar{\rho} = P = 0$, the matter tensor (7) reduces to null dust.

3. Energy conditions

In order for any relativistic Hawking–Ellis energy momentum tensor to be deemed physically reasonable, it should obey the null, weak, dominant and strong energy conditions [6,29,42]. There is in fact a weaker, nontrivial form of the above conditions that is worth noting. The resulting condition is called the flux energy condition. If $\mathbf{v}$ is a timelike velocity vector, then $F^a = T^a{}_b v^b$ can be interpreted as the energy momentum flux as measured by an observer moving with velocity $\mathbf{v}$ [43]. If it is assumed that this flux is timelike or null (that is, it is causal in its very nature), then this corresponds to mass–energy flowing at speeds less than the speed of light, but without any assumption that the energy densities be positive. This *freedom* makes the flux energy condition significantly weaker than the dominant energy condition [43,44]. When considering an N-dimensional spacetime manifold, obtaining the energy conditions involves solving an Nth degree polynomial, a difficult endeavour in general. If the matrix of eigenvalues is diagonalisable, then the energy momentum tensor is that of a Hawking–Ellis Type I fluid. If the eigenvalue matrix is not diagonalisable, the matter tensor is Hawking–Ellis Type II, III or IV. In this paper, we are concerned only with Types I and II, as these correspond to realistic matter, specifically for describing astrophysical objects.

The general definitions for the energy conditions are given by:

(i) The null energy condition: For any future-pointing null vector $\mathbf{k}$, the total energy density must obey $T_{ab}k^a k^b \geq 0$.
(ii) The weak energy condition: For any future-pointing timelike vector $\mathbf{w}$, the total energy density, at each event in the manifold, obeys $T_{ab}w^a w^b \geq 0$. The the weak energy condition contains the null energy condition.
(iii) The strong energy condition: For any future-pointing timelike unit vector $\mathbf{w}$, the stresses of the matter distribution are constrained by the condition $(N-2)T_{ab}w^a w^b + T \geq 0$, at each event in the manifold, where T represents the trace of the energy momentum tensor $\mathbf{T}$.
(iv) The dominant energy condition: For any future-pointing timelike or null vector $\mathbf{w}$, the overall energy density must obey $T_{ab}w^a w^b \geq 0$ (this is the weak energy condition), and the four-momentum density vector $T_{ab}w^b$ must be null, or future-pointing and timelike, at each event in the spacetime manifold (this is the flux energy condition). In other words, the mass–energy flow is positive and less than the speed of light, according to any observer.

All four of these conditions should be obeyed in general if an astrophysical or cosmological model is to be deemed physically relevant. Over and above this, the conditions of causality should be adhered to, i.e. in any matter field, the speed of sound c_s^2 is constrained by the condition $0 < c_s^2 < 1$ in our units.

4. Higher-dimensional model

Any relativistic fluid moving along a patch of a spacetime manifold will incur a shear stress on the manifold. Composite matter distributions containing shear stresses were studied in detail by Maharaj and Brassel [39] in higher dimensions. We make use of the N-dimensional general spherically symmetric metric

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\Omega^2_{N-2}, \tag{9}$$

where $A = A(r,t)$, $B = B(r,t)$, $C = C(r,t)$, and the $(N-2)$-sphere is given by

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta_j)\right](d\theta_i)^2. \tag{10}$$

The vectors **u**, **X**, **q**, **l** and **n** take the form

$$u^a = \left(\frac{1}{A}, 0, 0, ..., 0\right), \quad q^a = (0, q, 0, ..., 0),$$
$$X^a = \left(0, \frac{1}{B}, 0, ..., 0\right), \quad l^a = \left(\frac{1}{A}, \frac{1}{B}, ..., 0\right),$$
$$n^a = \left(\frac{1}{2A}, \frac{1}{2B}, 0, ..., 0\right), \tag{11}$$

in N dimensions.

The expansion scalar evaluates to

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + (N-2)\frac{\dot{C}}{C}\right), \tag{12}$$

and the shear tensor (2) has the N non-zero components

$$\sigma_{11} = \frac{N-2}{N-1}\left\{\frac{B^2}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\right\}, \tag{13a}$$
$$\sigma_{22} = -\frac{1}{N-1}\left\{\frac{C^2}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)\right\}, \tag{13b}$$
$$\sigma_{33} = \sin^2\theta_1\sigma_{22}, \tag{13c}$$
$$\vdots$$
$$\sigma_{N-1N-1} = \prod_{i=1}^{N-3}\sin^2\theta_i\sigma_{22}, \tag{13d}$$

where $^{\cdot} = \partial/\partial t$ and $^{\prime} = \partial/\partial r$. When $N = 4$, Eqs. (12) and (13) reduce to the corresponding four-dimensional expressions. We define the following scalar σ as

$$|\sigma| = \pm\frac{1}{N-1}\frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right), \tag{14}$$

where $\sigma^2 = 1/2\sigma^{ab}\sigma_{ab}$. We can then write

$$\sigma^1{}_1 = \frac{1}{B^2}\sigma_{11} = (N-2)|\sigma|, \tag{15a}$$
$$\sigma^2{}_2 = \frac{1}{C^2}\sigma_{22} = -|\sigma|, \tag{15b}$$
$$\sigma^3{}_3 = \frac{1}{C^2\sin^2\theta_1}\sigma_{22} = -|\sigma|, \tag{15c}$$
$$\vdots$$
$$\sigma^{N-1}{}_{N-1} = \frac{1}{C^2}\prod_{i=1}^{N-3}\frac{1}{\sin^2\theta_i}\sigma_{22} = -|\sigma|, \tag{15d}$$

so that $\sigma^a{}_a = 0$ and the shear tensor is trace-free.

The N non-vanishing components of the energy momentum tensor (6) become

$$T_{00} = A^2(\mu + \epsilon + \rho), \tag{16a}$$

$$T_{01} = -AB^2\left(q + \frac{1}{B}\epsilon\right), \tag{16b}$$

$$T_{11} = B^2\left(p_{||} + \epsilon - \rho - 2\eta\sigma^1{}_1\right), \tag{16c}$$

$$T_{22} = C^2\left(p_\perp + \mathcal{P} - 2\eta\sigma^2{}_2\right), \tag{16d}$$

$$T_{33} = \sin^2\theta_1 T_{22}, \tag{16e}$$

$$\vdots$$

$$T_{N-1N-1} = \prod_{i=1}^{N-3}\sin^2\theta_i T_{22}, \tag{16f}$$

for the spacetime (9).

The non-vanishing components of the Einstein tensor are

$$G_{00} = (N-2)\frac{\dot{B}\dot{C}}{BC} + \frac{(N-2)(N-3)}{2}\left(\frac{A^2}{C^2} + \frac{\dot{C}^2}{C^2}\right) - \frac{A^2}{B^2}\left[(N-2)\frac{C''}{C} + \left(\frac{(N-2)(N-3)}{2}\right)\frac{C'^2}{C^2} - (N-2)\frac{B'C'}{BC}\right], \tag{17a}$$

$$G_{01} = (N-2)\left[-\frac{\dot{C}'}{C} + \frac{\dot{B}C'}{BC} + \frac{A'\dot{C}}{AC}\right], \tag{17b}$$

$$G_{11} = \frac{B^2}{A^2}\left[-(N-2)\frac{\ddot{C}}{C} - \left(\frac{(N-2)(N-3)}{2}\right)\frac{\dot{C}^2}{C^2} + (N-2)\frac{\dot{A}\dot{C}}{AC}\right] + \left(\frac{(N-2)(N-3)}{2}\right)\frac{C'^2}{C^2} + (N-2)\frac{A'C'}{AC} - \left(\frac{(N-2)(N-3)}{2}\right)\frac{B^2}{C^2}, \tag{17c}$$

$$G_{22} = -\frac{C^2}{A^2}\left[\frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + (N-3)\frac{\dot{B}\dot{C}}{BC} - (N-3)\frac{\dot{A}\dot{C}}{AC} + (N-3)\frac{\ddot{C}}{C}\right] + \frac{C^2}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + (N-3)\frac{A'C'}{AC} - (N-3)\frac{B'C'}{BC} + (N-3)\frac{C''}{C}\right] - \left(\frac{(N-3)(N-4)}{2}\right)\left(\frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} + 1\right), \tag{17d}$$

$$G_{33} = \sin^2\theta_1 G_{22}, \tag{17e}$$

$$\vdots$$

$$G_{N-1N-1} = \prod_{i=1}^{N-3}\sin^2\theta_i G_{22}. \tag{17f}$$

Using Eqs. (16) and (17) we can write the N-dimensional Einstein field equations as

$$\kappa_N\left(\mu + \epsilon + \rho\right) = \frac{(N-2)}{A^2}\frac{\dot{B}\dot{C}}{BC} + \left(\frac{(N-2)(N-3)}{2}\right)\left(\frac{1}{C^2} + \frac{\dot{C}^2}{A^2C^2}\right) - \frac{(N-2)}{B^2}\left[\frac{C''}{C} + \left(\frac{(N-3)}{2}\right)\frac{C'^2}{C^2} - \frac{B'C'}{BC}\right], \tag{18a}$$

$$\kappa_N \left(p_{||} + \epsilon - \rho - 2(N-2)\eta|\sigma|\right) = \frac{(N-2)}{A^2}\left[-\frac{\ddot{C}}{C} - \left(\frac{(N-3)}{2}\right)\frac{\dot{C}^2}{C^2} + \frac{\dot{A}\dot{C}}{AC}\right] + \frac{(N-2)}{B^2}\left[\left(\frac{(N-3)}{2}\right)\frac{C'^2}{C^2} + \frac{A'C'}{AC}\right] - \left(\frac{(N-2)(N-3)}{2}\right)\frac{1}{C^2}, \tag{18b}$$

$$\kappa_N \left(p_\perp + \mathcal{P} + 2\eta|\sigma|\right) = -\frac{1}{A^2}\left[\frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} + (N-3)\frac{\dot{B}\dot{C}}{BC} - (N-3)\frac{\dot{A}\dot{C}}{AC} + (N-3)\frac{\ddot{C}}{C}\right] + \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + (N-3)\frac{A'C'}{AC} - (N-3)\frac{B'C'}{BC} + (N-3)\frac{C''}{C}\right] - \left(\frac{(N-3)(N-4)}{2}\right)\left(\frac{1}{C^2}\right)\left(\frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} + 1\right), \tag{18c}$$

$$\kappa_N \left(q + \frac{1}{B}\epsilon\right) = -\frac{(N-2)}{AB^2}\left[-\frac{\dot{C}'}{C} + \frac{\dot{B}C'}{BC} + \frac{A'\dot{C}}{AC}\right], \tag{18d}$$

for the N-dimensional spherically symmetric metric (9) and the composite matter distribution (6). If we set $C = rB$ then the shear vanishes and we regain the corresponding shear-free field equations [32,37].

The four energy conditions for a matter field of Hawking–Ellis Type I in N dimensions are

(1) Null energy condition:

$$-\lambda_0 + \lambda_i \geq 0, \tag{19}$$

(2) Weak energy condition:

$$-\lambda_0 \geq 0, \quad -\lambda_0 + \lambda_i \geq 0. \tag{20}$$

(3) Dominant energy condition:

$$-\lambda_0 \geq 0, \quad \lambda_0 \leq \lambda_i \leq -\lambda_0. \tag{21}$$

(4) Strong energy condition:

$$-(N-3)\lambda_0 + \sum_{i=1}^{N-1}\lambda_i \geq 0, \quad -\lambda_0 + \lambda_i \geq 0. \tag{22}$$

In the above, $i \in \{1, 2, 3, \ldots, N-1\}$. The energy momentum tensor has eigenvalues λ which are the roots of the equation

$$\left|T_{ab} - \lambda g_{ab}\right| = 0, \tag{23}$$

which has to be satisfied in a general higher-dimensional spacetime.

We can now generate the energy conditions for the higher-dimensional metric (9). Using Eqs. (6) and (23) we acquire an $N \times N$ matrix $T_{ab} - \lambda g_{ab}$ leading to the determinant equation

$$\begin{vmatrix} A^2(\mu+\epsilon+\rho+\lambda) & -\tilde{q}AB & 0 & 0 & \dots & 0 \\ -\tilde{q}AB & B^2\left(p_{||}+\epsilon-\rho-\lambda-2\eta\sigma^1{}_1\right) & 0 & 0 & \dots & 0 \\ 0 & 0 & C^2\left(p_\perp+\mathcal{P}-\lambda-2\eta\sigma^2{}_2\right) & 0 & \dots & 0 \\ 0 & 0 & 0 & & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & C^2\prod_{i=1}^{N-3}\sin^2\theta_i\left(p_\perp+\mathcal{P}-\lambda-2\eta\sigma^2{}_2\right) \end{vmatrix} = 0,$$

with $\tilde{q} = qB + \epsilon$. Evaluating the determinant yields the following equation in λ:

$$\left[\lambda^2 + (\mu - p_{||} + 2\rho)\lambda + \tilde{q}^2 - (\mu+\epsilon+\rho)(p_{||}+\epsilon-\rho) + 2(\mu+\epsilon+\rho+\lambda)\eta\sigma^1{}_1\right] \times \left(p_\perp + \mathcal{P} - \lambda - 2\eta\sigma^2{}_2\right)^{N-2}\left(-A^2B^2C^{2N-4}\prod_{i=1}^{N-3}\sin^2\theta_i\right) = 0, \quad (24)$$

which is an Nth degree polynomial in λ. In the above,

$$-A^2B^2C^{2N-4}\prod_{i=1}^{N-3}\sin^2\theta_i \neq 0.$$

Thus, one solution of Eq. (24) is given by

$$\lambda^2 + (\mu - p_{||} + 2\rho)\lambda + \tilde{q}^2 - (\mu+\epsilon+\rho)(p_{||}+\epsilon-\rho) + 2(\mu+\epsilon+\rho+\lambda)\eta\sigma^1{}_1 = 0, \quad (25)$$

which is a second-order polynomial in λ and yields the two roots

$$\lambda_0 = -\frac{1}{2}\left[\mu - p_{||} + 2\rho + 2\eta\sigma^1{}_1 + \Delta\right], \quad (26a)$$

$$\lambda_1 = -\frac{1}{2}\left[\mu - p_{||} + 2\rho + 2\eta\sigma^1{}_1 - \Delta\right], \quad (26b)$$

with

$$\Delta^2 = \left(\mu + p_{||} + 2\epsilon - 2\eta\sigma^1{}_1\right)^2 - 4\tilde{q}^2. \quad (27)$$

The energy momentum tensor T_{ab} is Type I in the region $\Delta^2 > 0$. When $\Delta^2 = 0$, we have that T_{ab} is a Type II fluid. When $\Delta^2 < 0$, the energy momentum tensor T_{ab} becomes a Type IV fluid since there will be two complex eigenvalues. The other solution of Eq. (24) is given by

$$\left(p_\perp + \mathcal{P} - \lambda - 2\eta\sigma^2{}_2\right)^{N-2} = 0,$$

which gives the $N-2$ repeated roots

$$\lambda_{2,3,\ldots,N} = p_\perp + \mathcal{P} - 2\eta\sigma^2{}_2. \tag{28}$$

In the system (26), we must have $\Delta \geq 0$ in order for these roots to be real. This implies

$$\left|\mu + p_{||} + 2\epsilon - 2\eta\sigma^1{}_1\right| - 2|\tilde{q}| \geq 0. \tag{29}$$

The above expression (29) appears in several treatments, for example models of dissipative radiating stars in Refs. [25,29]. The difference in our analysis is the explicit appearance of the null dust term ϵ arising from the more general form of the energy momentum tensor (6). We make the observation that for very large $\tilde{q} = qB + \epsilon$, the second term in Eq. (29) will dominate and the matter field (6) will then become Type IV, and all energy conditions will be violated. We also note that if the shear viscosity η is large enough, the first term within the modulus sign may become negative, in which case the matter field may also become Type IV.

We make the observation that classical matter fields do not lead to Type IV fluids. However, such fluids may be related to renormalised expectation values of evaporating black holes [45], the Unruh vacuum far from the horizon [46] and massless conformal scalars in the Unruh state [47].

It is important to emphasise the physical importance of equation (29). It relates the physical quantities of energy density μ, the radial pressure $p_{||}$, the null dust energy density ϵ, the shear viscosity η and the heat flux q. We have demonstrated that the result holds in four and in N dimensions. It is of particular interest in the modeling of radiating stars in general relativity, and in dissipative relativistic fluids in general.

In four dimensions, a 4th-degree polynomial arises for a Type I fluid in the energy conditions, yielding two repeated roots and two distinct roots. We observe that the main difference for the energy conditions in N dimensions, when compared with the composite fluid in four dimensions, is that an Nth degree polynomial equation in λ arises. The resulting equation in λ has N real roots: there are two distinct roots, λ_0 and λ_1, and $N-2$ repeated roots, $\lambda_{2,3,\ldots,N}$.

4.1. *Null energy conditions*

Making use of Eqs. (15) and (19) with the solutions (26) and (28), the null energy conditions (NEC) become

$$\mu + p_{||} + 2\epsilon - 2(N-2)\eta|\sigma| + \Delta \geq 0 \tag{30a}$$

$$\mu - p_{||} + 2\rho + 2N\eta|\sigma| + 2\left(p_\perp + \mathcal{P}\right) + \Delta \geq 0, \tag{30b}$$

$$\Delta > 0. \tag{30c}$$

These are similar to the conditions in four dimensions [40]. However, the components now depend on N.

4.2. *Weak energy conditions*

Using Eqs. (15) and (20) along with Eqs. (26) and (28), the weak energy conditions (WEC) become

$$\mu - p_{||} + 2\rho + 2(N-2)\eta|\sigma| + \Delta \geq 0, \tag{31a}$$

$$\mu - p_{||} + 2\rho + 2N\eta|\sigma| + 2\left(p_\perp + \mathcal{P}\right) + \Delta \geq 0, \tag{31b}$$

$$\Delta > 0. \tag{31c}$$

These conditions are similar in form to the four-dimensional case, but note the presence of N. These weak conditions imply the null energy conditions (30).

4.3. *Dominant energy conditions*

Using Eq. (15) together with the solutions (26) and (28), along with the dominant energy conditions (DEC; Eq. 21), we have the following expressions:

$$\mu - p_{||} + 2\rho + 2(N-2)\eta|\sigma| \geq 0, \tag{32a}$$

$$\mu - p_{||} + 2\rho + 2N\eta|\sigma| + 2\left(p_{\perp} + \mathcal{P}\right) + \Delta \geq 0, \tag{32b}$$

$$\mu - p_{||} + 2\rho + 2(N-4)\eta|\sigma| - 2\left(p_{\perp} + \mathcal{P}\right) + \Delta \geq 0, \tag{32c}$$

$$\Delta > 0. \tag{32d}$$

Again, the conditions are similar in form to their four-dimensional counterparts. The comment about the dependence on dimensions for the components, made above, also applies. We also note the presence of the $2(N-4)\eta|\sigma|$ term in Eq. (32c), which is absent in four dimensions.

4.4. *Strong energy conditions*

Making use of Eq. (15) and the roots, Eqs. (26) and (28), and evaluating the sum with $i \in \{1, 2, ..., N-1\}$ in Eq. (22), the strong energy conditions (SEC) can be written as

$$\begin{aligned}&(N-4)\left(\mu - p_{||} + 2\rho + 2(N-2)\eta|\sigma|\right)\\&+2(N-2)\left(p_{\perp} + \mathcal{P} + 2\eta|\sigma|\right) + (N-2)\Delta \geq 0,\end{aligned} \tag{33a}$$

$$\mu - p_{||} + 2\rho + 2N\eta|\sigma| + 2\left(p_{\perp} + \mathcal{P}\right) + \Delta \geq 0, \tag{33b}$$

$$\Delta > 0. \tag{33c}$$

In this case the parameter N for dimensions appears explicitly in Eq. (33a). We also note the first term containing $N-4$, which is not present in four dimensions. For all the energy conditions, we have

$$\Delta = \sqrt{\left(\mu + p_{||} + 2\epsilon - 2(N-2)\eta|\sigma|\right)^2 - 4\tilde{q}^2}, \tag{34}$$

with $\tilde{q} = qB + \epsilon$. In the above conditions we have replaced the components $\sigma^1{}_1$ and $\sigma^2{}_2$ with the shear scalar $|\sigma|$.

We note that when $\Delta = 0$, the energy momentum tensor (6) will become a Hawking–Ellis Type II fluid [7]. This is true because, with a local Lorentz boost, the energy momentum tensor $T_{(ij)}$ in Appendix A.1 with $\Delta = 0$ may be transformed into the canonical form (A.6). The above energy conditions will then change accordingly; in summary, they will become the following:

- NEC:

$$\mu + p_{||} + 2\epsilon - 2(N-2)\eta|\sigma| \geq 0, \tag{35a}$$

$$\mu - p_{||} + 2\rho + 2N\eta|\sigma| + 2\left(p_{\perp} + \mathcal{P}\right) \geq 0. \tag{35b}$$

- WEC:

$$\mu - p_{||} + 2\rho + 2(N-2)\eta|\sigma| \geq 0, \tag{36}$$

in addition to system (35).

- DEC:

$$\mu - p_{||} + 2\rho + 2(N-4)\eta|\sigma| - 2\left(p_{\perp} + \mathcal{P}\right) \geq 0, \tag{37}$$

in addition to system (35).

- SEC:

$$(N-4)\left(\mu - p_{||} + 2\rho + 2(N-2)\eta|\sigma|\right) + 2(N-2)\left(p_{\perp} + \mathcal{P} + 2\eta|\sigma|\right) \geq 0, \tag{38}$$

in addition to system (35).

Equations (30)–(33) and (35)–(38) highlight the role of the dimension N in a transparent fashion. Brassel *et al.* [40] found the energy conditions for a composite fluid in four-dimensional general relativity. These energy conditions are a generalisation to N dimensions for a composite fluid with viscous barotropic matter, null dust and a null string. When $N = 4$, we regain the energy conditions of Ref. [40]. We have established in this paper that the dimension N is a critical component in the energy conditions. We can now state the following theorems:

THEOREM 1 *Consider an N-dimensional time-oriented Lorentzian manifold $\mathcal{M}$ with the general spherically symmetric metric (9) and a Type I matter field (since $\Delta > 0$) comprising a combination of an anisotropic viscous barotropic fluid, null dust and a null string fluid. The null, weak, dominant and strong energy conditions are satisfied when the composite fluid distribution satisfies the energy conditions in Eqs. (30), (31), (32) and (33).*

THEOREM 2 *Consider an N-dimensional time-oriented Lorentzian manifold $\mathcal{M}$ with the general spherically symmetric line element (9) and a Type II matter field (since $\Delta = 0$) comprising a combination of an anisotropic viscous barotropic fluid, null dust and a null string fluid. The null, weak, dominant and strong energy conditions will be satisfied when the composite fluid distribution satisfies the conditions in Eqs. (35), (36), (37) and (38).*

Note the appearance of the terms $(N-4)\eta|\sigma|$, $2(N-4)\eta|\sigma|$, $2(N-2)\eta|\sigma|$ and $2N\eta|\sigma|$ in the energy conditions (30)–(33) and (35)–(38), which are all related to the shear scalar and viscosity. The particular terms $2(N-4)\eta|\sigma|$ in the dominant energy conditions and $(N-4)\left(\mu - p_{||} + 2\rho + 2\eta|\sigma|\right)$ in the strong energy conditions are *new* components that do not appear in four dimensions. It is a consequence of the geometry of the $(N-2)$-sphere (10) playing a role in all of the energy conditions. With regards to the spherically symmetric metric (9), for $N > 4$, these $(N-4)$-terms appear as a result of the higher-dimensional geometry of the $(N-2)$-sphere, hence their appearances in the $G_{22}, G_{33}, \ldots, G_{N-1N-1}$ terms (17d)–(17f). Thus, the energy conditions have a new structure in higher dimensions which is not evident in four dimensions. *We have shown that there is a deep connection between the spacetime geometry, represented by the $(N-2)$-sphere, and the energy conditions in N dimensions.* This connection becomes transparent when $N > 4$ and is not as apparent in four dimensions.

5. Energy conditions for a null dust and null string fluid in higher dimensions

For any massive astrophysical body undergoing gravitational contraction, the radiative processes and fluid pressure must contribute to this dynamical process. In this regard, the spacetime that closely mirrors this scenario of collapse is the generalised Vaidya spacetime, the Hawking–Ellis energy momentum tensor of which is a combination of a Type I fluid and a Type II matter field (which has double null eigenvectors). Such matter distributions are important as they describe the exterior of a radiating star in general relativity. The aim of this section is to generate the energy conditions for this Type II fluid in higher dimensions and to show that, under certain restrictions, these reduce to the energy conditions pertaining to the Type I fluid discussed earlier.

The generalised Vaidya metric in higher dimensions is given by

$$ds^2 = -\left(1 - \frac{2m(v,\mathsf{r})}{(N-3)\mathsf{r}^{N-3}}\right) dv^2 + 2\varepsilon dv d\mathsf{r} + \mathsf{r}^2 d\Omega^2_{N-2}. \tag{39}$$

In the above, $d\Omega^2_{N-2}$ is the $(N-2)$-sphere given by Eq. (10), and we have

$$\varepsilon = \begin{cases} 1, & \text{Collapsing radiation,} \\ -1, & \text{Outgoing radiation.} \end{cases} \tag{40}$$

The function $m(v,\mathsf{r})$ is regarded as the higher-dimensional Misner–Sharp mass [48,49] containing the gravitational energy within the given radius r.

The Einstein tensor components become

$$G^0{}_0 = G^1{}_1 = -\frac{(N-2)m_{\mathsf{r}}}{(N-3)\mathsf{r}^{N-2}}, \tag{41a}$$

$$G^1{}_0 = \frac{(N-2)m_v}{(N-3)\mathsf{r}^{N-2}}, \tag{41b}$$

$$G^2{}_2 = G^3{}_3 = \ldots = G^{N-1}{}_{N-1} = -\frac{m_{\mathsf{rr}}}{(N-3)\mathsf{r}^{N-3}}, \tag{41c}$$

for the N-dimensional generalised Vaidya metric (39). The null vectors $\mathcal{L}$ and $\mathcal{N}$ are given by

$$\mathcal{L}_a = (1, 0, 0, \ldots, 0),$$
$$\mathcal{N}_a = \left(\frac{1}{2}\left[1 - \frac{2m(v,\mathsf{r})}{(N-3)\mathsf{r}^{N-3}}\right], -\varepsilon, 0, \ldots, 0\right),$$

in higher dimensions. Therefore the non-vanishing energy momentum tensor components (7) representing the Type II fluid in higher dimensions are

$$T^0{}_0 = T^1{}_1 = -\bar{\rho}, \tag{42a}$$

$$T^1{}_0 = \frac{1}{\varepsilon}\nu, \tag{42b}$$

$$T^2{}_2 = T^3{}_3 = \ldots = T^{N-1}{}_{N-1} = P, \tag{42c}$$

for the metric (39).

The Einstein field equations $G^a{}_b = \kappa_N T^a{}_b$ are then written as

$$\kappa_N \nu = \frac{(N-2)m_v}{\varepsilon(N-3)\mathsf{r}^{N-2}}, \tag{43a}$$

$$\kappa_N \bar{\rho} = \frac{(N-2)m_{\mathsf{r}}}{(N-3)\mathsf{r}^{N-2}}, \tag{43b}$$

$$\kappa_N P = -\frac{m_{\mathsf{rr}}}{(N-3)\mathsf{r}^{N-3}}, \tag{43c}$$

for the metric spacetime (39). This system of equations (43) describes gravitational dynamics of the combination of null radiation and the additional Type II null string fluid. The energy momentum tensor (7) is a generalisation of the Vaidya solution in higher dimensions.

We introduce the vectors

$$\mathbb{E}^a_{(0)} = \frac{\mathcal{L}^a + \mathcal{N}^a}{\sqrt{2}}, \tag{44a}$$

$$\mathbb{E}^a_{(1)} = \frac{\mathcal{L}^a - \mathcal{N}^a}{\sqrt{2}}, \tag{44b}$$

$$\mathbb{E}^a_{(2)} = \frac{1}{\mathsf{r}}\delta^a{}_2, \tag{44c}$$

$$\mathbb{E}^a_{(3)} = \frac{1}{\mathsf{r}\sin\theta}\delta^a{}_3, \tag{44d}$$

$$\vdots$$

$$\mathbb{E}^a_{(N-1)} = \frac{1}{\mathsf{r}}\prod_{i=1}^{N-3}\frac{1}{\sin\theta_i}\delta^a{}_{N-1}, \tag{44e}$$

to form a tetrad basis $\{\mathbb{E}^a_{(0)}, \mathbb{E}^a_{(1)}, \ldots, \mathbb{E}^a_{(N-1)}\}$. The derivation of the tetrad basis for Type I and Type II fluids is given in the Appendix. Projecting the energy momentum tensor (7) to the orthonormal basis using the above null vectors gives

$$T_{(ij)} = \begin{pmatrix} \frac{\nu}{2}+\tilde{\rho} & \frac{\nu}{2} & 0 & 0 & \ldots & 0 \\ \frac{\nu}{2} & \frac{\nu}{2}-\tilde{\rho} & 0 & 0 & \ldots & 0 \\ 0 & 0 & P & 0 & \ldots & 0 \\ 0 & 0 & 0 & P & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & P \end{pmatrix}, \tag{45}$$

where it is evident that two eigenvalues are identical; the above matrix cannot be diagonalised. This follows because one vector is doubly null; details are given in Hawking and Ellis [6]. The energy conditions for a Type II fluid in higher dimensions are then given by

(1) NEC:

$$\nu \geq 0, \qquad \tilde{\rho} + P \geq 0. \tag{46}$$

(2) WEC:

$$\nu \geq 0, \qquad \tilde{\rho} \geq 0, \qquad \tilde{\rho} + P \geq 0. \tag{47}$$

(3) SEC:

$$\nu \geq 0, \qquad \tilde{\rho} + P \geq 0, \qquad (N-4)\tilde{\rho} + (N-2)P \geq 0. \tag{48}$$

(4) DEC:

$$\nu \geq 0, \qquad \tilde{\rho} \geq |P|(\geq 0). \tag{49}$$

Note that these conditions will hold for any reasonable choice of the mass function $m(v, \mathsf{r})$ in any dimension, and $\nu \neq 0$.

When the mass is a function only of the retarded time coordinate, $m = m(v)$, the model reduces to the pure Vaidya metric and the energy conditions reduce to the single relation

$$\nu \geq 0. \tag{50}$$

If $m = m(r)$, we have that $\nu = 0$ and the energy momentum tensor becomes a diagonal matrix of the eigenvalues, which is a Type I fluid. The energy conditions then have the form

(1) NEC:

$$\tilde{\rho} + P \geq 0. \tag{51}$$

(2) WEC:

$$\tilde{\rho} \geq 0, \qquad \tilde{\rho} + P \geq 0. \tag{52}$$

(3) SEC:

$$\tilde{\rho} + P \geq 0, \qquad (N-4)\tilde{\rho} + (N-2)P \geq 0. \tag{53}$$

(4) DEC:

$$\tilde{\rho} \geq 0, \qquad -\tilde{\rho} \leq P \leq \tilde{\rho}. \tag{54}$$

Note the explicit presence of the higher dimensions N in the SEC as this case corresponds to a Type I fluid. These conditions are analogous with Eqs. (19), (20), (21) and (22).

6. Discussion

We generated the energy conditions pertaining to a higher-dimensional generalised composite Hawking–Ellis matter tensor which we showed to be of Type I. This field consisted of viscous barotropic matter, null dust, null string energy density, anisotropy and the shear. The metric utilised was the general higher-dimensional spherically symmetric geometry. Obtaining the energy conditions involved solving an Nth degree polynomial for the eigenvalues λ, which is a difficult task in general. The resulting equation in λ had N real roots, two of which were distinct and the other $N-2$ roots were repeated. The energy conditions were then found as a system of six equations which would need to hold for any physically reasonable study in cosmology or relativistic astrophysics. It is important to highlight the effect of the higher dimensions on these energy conditions. We note that the energy conditions also possess a different analytic structure and form in N dimensions which is not present when $N = 4$. This is evident particularly in the energy conditions, with the appearance of the additional term $(N-4)\eta|\sigma|$ (in addition to $(N-4)\left(\mu - p_{||} + 2\rho + 2\eta|\sigma|\right)$, $2(N-4)\eta|\sigma|$, $2(N-2)\eta|\sigma|$ and $2N\eta|\sigma|$) which arises because of the geometry of the $(N-2)$-sphere. All previous treatments in four and higher dimensions are contained in our analysis for both shearing and shear-free spacetime geometries. The higher-dimensional Type II fluid distribution with the general mass function $m(v, \mathsf{r})$ was then addressed, and it was observed that the energy conditions for this type of matter distribution are affected by an increase in dimension. However, under certain conditions, namely when $m = m(\mathsf{r})$, these reduce to the Type I matter field. This work can be applied to stellar astrophysics in the general relativistic setting. For example, the junction conditions for a higher-dimensional radiating composite star were found in Refs. [37,39], where a Type I composite matter interior was matched across a stellar boundary to the higher-dimensional exterior atmosphere which consisted of a Hawking–Ellis Type II fluid. This provides a realistic physical scenario for application of higher-dimensional energy conditions. There are many other situations in relativistic astrophysics and cosmology where the energy conditions play an important role.

Acknowledgements

BPB, SDM and RG thank the University of KwaZulu-Natal for its continued support. SDM acknowledges that this work is based upon research supported by the South African Research Chair Initiative of the Department of Science and Technology and the National Research Foundation. The authors are also indebted to the referee for important insights, suggestions, and the outline for the Appendix; these have greatly improved this paper.

A. Appendix

A.1. Orthonormal basis: composite fluid

In an orthornormal basis all the vectors are unit and orthogonal to each other. The N-tetrad (or verbien) is given by $\{\mathbb{E}^0, \mathbb{E}^1, \mathbb{E}^2, \mathbb{E}^3, \dots, \mathbb{E}^{N-1}\}$ with the components $(\mathbb{E}_a^{(0)}, \mathbb{E}_a^{(1)}, \mathbb{E}_a^{(2)}, \mathbb{E}_a^{(3)}, \dots, \mathbb{E}_a^{(N-1)})$ so that

$$\eta_{ij} = g_{ab}\mathbb{E}^a_{(i)}\mathbb{E}^b_{(j)}. \tag{A.1}$$

Equation (A.1) relates the vector components of the tetrad and the metric g_{ab} to Minkowski space η_{ij}. Projection of the energy momentum tensor T_{ab} into the tetrad basis can be affected by using the expression

$$T_{(ij)} = \mathbb{E}^a_{(i)}\mathbb{E}^b_{(j)}T_{ab}, \tag{A.2}$$

which relates the null vectors of the tetrad to the energy momentum tensor.

Using the expression (A.1) together with the metric (9) generates the following vectors

$$\mathbb{E}^a_{(0)} = \frac{1}{A}\delta^a{}_0,$$

$$\mathbb{E}^a_{(1)} = \frac{1}{B}\delta^a{}_1,$$

$$\mathbb{E}^a_{(2)} = \frac{1}{C}\delta^a{}_2,$$

$$\mathbb{E}^a_{(3)} = \frac{1}{C\sin\theta}\delta^a{}_3,$$

$$\vdots$$

$$\mathbb{E}^a_{(N-1)} = \frac{1}{C}\prod_{i=1}^{N-3}\frac{1}{\sin\theta_i}\delta^a{}_{N-1}.$$

Projection on to the energy momentum tensor (6), with the help of Eq. (A.2), gives

$$T_{(ij)} = \begin{pmatrix} \mu+\epsilon+\rho & -B\left(q+\frac{1}{B}\epsilon\right) & 0 & \dots & 0 \\ -B\left(q+\frac{1}{B}\epsilon\right) & \begin{matrix} p_{||}+\epsilon-\rho \\ -2\eta\sigma^1{}_1 \end{matrix} & 0 & \dots & 0 \\ 0 & 0 & \begin{matrix} p_\perp+\mathcal{P} \\ -2\eta\sigma^2{}_2 \end{matrix} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \begin{matrix} p_\perp+\mathcal{P} \\ -2\eta\sigma^2{}_2 \end{matrix} \end{pmatrix}$$

in the N-dimensional orthonormal basis. This above matrix has off-diagonal components; however, it can be diagonalised with certain restrictions. To prove this we perform a local Lorentz boost in the (01)-plane such that we have the following:

$$\bar{\mathbb{E}}^a_{(0)}\frac{\partial}{\partial x^a} := \left(\cosh\alpha\mathbb{E}^a_{(0)} - \sinh\alpha\mathbb{E}^a_{(1)}\right)\frac{\partial}{\partial x^a} = \cosh\alpha\frac{1}{A}\frac{\partial}{\partial t} - \sinh\alpha\frac{1}{B}\frac{\partial}{\partial r},$$
$$\bar{\mathbb{E}}^a_{(1)}\frac{\partial}{\partial x^a} := \left(\cosh\alpha\mathbb{E}^a_{(1)} - \sinh\alpha\mathbb{E}^a_{(0)}\right)\frac{\partial}{\partial x^a} = \cosh\alpha\frac{1}{B}\frac{\partial}{\partial t} - \sinh\alpha\frac{1}{A}\frac{\partial}{\partial r},$$

where we have the parametrisation

$$\cosh\alpha = \frac{1}{\sqrt{1-v^2}}, \qquad \sinh\alpha = \frac{v}{\sqrt{1-v^2}}.$$

We then arrive, with the new basis vectors, at

$$T_{(00)} = -\frac{1}{v^2-1}\left(\frac{1}{A^2}T_{00} - 2v\frac{1}{A}\frac{1}{B}T_{01} + v^2\frac{1}{B^2}T_{11}\right), \tag{A.3a}$$

$$T_{(01)} = \frac{1}{v^2-1}\left(v\frac{1}{A^2}T_{00} - (1+v^2)\frac{1}{A}\frac{1}{B}T_{01} + v\frac{1}{B^2}T_{11}\right), \tag{A.3b}$$

$$T_{(11)} = -\frac{1}{v^2-1}\left(v^2\frac{1}{A^2}T_{00} - 2v\frac{1}{A}\frac{1}{B}T_{01} + \frac{1}{B^2}T_{11}\right). \tag{A.3c}$$

When $T_{(01)} = 0$ we have the following condition for v:

$$v = \frac{1}{2T_{01}}AB\left(\frac{1}{A^2}T_{00} + \frac{1}{B^2}T_{11} \pm \Delta\right). \tag{A.4}$$

In the above,

$$\Delta^2 = \left(\frac{1}{A^2}T_{00} + \frac{1}{B^2}T_{11}\right)^2 - 4\left(\frac{1}{A^2}\frac{1}{B^2}\right)T_{01}^2$$
$$= \left(\mu + p_{||} + 2\epsilon - 2\eta\sigma^1{}_1\right)^2 - 4\tilde{q}^2,$$

where $\tilde{q} = qB + \epsilon$. We note that $T_{(ij)}$ is Type I only when $-1 < \nu < 1$. We have that $T_{(ij)}$ is Type II in the region $\nu = \pm 1$. If $|\nu| > 1$, then two eigenvalues will become complex and $T_{(ij)}$ is then a Type IV fluid. Substituting Eq. (A.4) into Eqs. (A.3a) and (A.3c) gives

$$\begin{aligned} T_{(00)} &= \frac{1}{2}\left(\frac{1}{A^2}T_{00} - \frac{1}{B^2}T_{11} \mp \Delta\right) \\ &= \frac{1}{2}\left(\mu - p_{||} + 2\rho + 2\eta\sigma^1{}_1 \mp \Delta\right), \\ T_{(11)} &= -\frac{1}{2}\left(\frac{1}{A^2}T_{00} - \frac{1}{B^2}T_{11} \pm \Delta\right) \\ &= -\frac{1}{2}\left(\mu - p_{||} + 2\rho + 2\eta\sigma^1{}_1 \pm \Delta\right). \end{aligned} \tag{A.5}$$

We now have that

$$T_{(ij)} = \mathrm{diag}\left(T_{(00)}, T_{(11)}, T_{(22)}, \ldots, T_{(N-1N-1)}\right),$$

and so $T_{(ij)}$ has been diagonalised. Therefore the inequalities (19)–(21) for the Type I energy conditions may be used.

A.2. Orthonormal basis: null dust and null string fluid

Transforming to an orthonormal basis for a Type II fluid is more complicated than for a Type I fluid. Projection of the energy momentum tensor (7) into the orthonormal basis can be achieved using Eq. (A.2). We obtain

$$T_{(ij)} = \begin{pmatrix} \frac{\nu}{2} + \tilde{\rho} & \frac{\nu}{2} & 0 & 0 & \ldots & 0 \\ \frac{\nu}{2} & \frac{\nu}{2} - \tilde{\rho} & 0 & 0 & \ldots & 0 \\ 0 & 0 & P & 0 & \ldots & 0 \\ 0 & 0 & 0 & P & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & P \end{pmatrix}, \tag{A.6}$$

as in Eq. (45). This projection requires the N null vectors

$$\mathbb{E}^a_{(0)} = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\left(1 - \frac{1}{2}\left(1 - \frac{2m}{(N-3)\mathrm{r}^{N-3}}\right)\right), 0, \ldots, 0\right),$$
$$\mathbb{E}^a_{(1)} = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\left(1 + \frac{1}{2}\left(1 - \frac{2m}{(N-3)\mathrm{r}^{N-3}}\right)\right), 0, \ldots, 0\right),$$

$$\mathbb{E}^a_{(2)} = \left(0, 0, \frac{1}{\mathsf{r}}, 0, \ldots, 0\right),$$

$$\mathbb{E}^a_{(3)} = \left(0, 0, 0, \frac{1}{\mathsf{r}\sin\theta}, 0, \ldots, 0\right),$$

$$\vdots$$

$$\mathbb{E}^a_{(N-1)} = \left(0, 0, 0, \ldots, \frac{1}{\mathsf{r}} \prod_{i=1}^{N-3} \frac{1}{\sin\theta_i}\right).$$

The above system can then be simplified using Eq. (7) together with the null vectors $\mathcal{L}^a$ and $\mathcal{N}^a$, to find the basis vectors

$$\mathbb{E}^a_{(0)} = \frac{\mathcal{L}^a + \mathcal{N}^a}{\sqrt{2}}, \tag{A.7a}$$

$$\mathbb{E}^a_{(1)} = \frac{\mathcal{L}^a - \mathcal{N}^a}{\sqrt{2}}, \tag{A.7b}$$

$$\mathbb{E}^a_{(2)} = \frac{1}{\mathsf{r}}\delta^a{}_2, \tag{A.7c}$$

$$\mathbb{E}^a_{(3)} = \frac{1}{\mathsf{r}\sin\theta}\delta^a{}_3, \tag{A.7d}$$

$$\vdots$$

$$\mathbb{E}^a_{(N-1)} = \frac{1}{\mathsf{r}} \prod_{i=1}^{N-3} \frac{1}{\sin\theta_i}\delta^a{}_{N-1}, \tag{A.7e}$$

in the tetrad basis. The Hawking–Ellis Type II energy momentum tensor in the orthonormal basis is *not* diagonalisable.

References

[1] H. A. Buchdahl, Mon. Not. Roy. Astron. Soc. **150**, 1 (1970).
[2] D. Lovelock, J. Math. Phys. **12**, 498 (1971).
[3] P. Nath and R. Arnowitt, Phys. Lett. B **56**, 177 (1975).
[4] R. B. Mann, A. Shiekh, and L. Tarasov, Nuc. Phys. B **341**, 134 (1990).
[5] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010).
[6] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
[7] H. Maeda and C. Martínez, Prog. Theor. Exp. Phys. **2020** 043E02 (2020).
[8] J. Santos, J. S. Alcaniz, N. Pires, and M. J. Rebouços, Phys. Rev. D **75**, 083523 (2007).
[9] H.-H. Xiong and J.-Y. Zhu, Int. J. Mod. Phys. A. **22**, 3137 (2007).
[10] G. G. L. Nashed, Adv. High Energy Phys., **2015**, 687940 (2015).
[11] P. Martín-Moruno and M. Visser, Fund. Theor. Phys. **189**, 193 (2017).
[12] P. Martín-Moruno and M. Visser, Class. Quantum Grav. **34**, 225014 (2017).
[13] P. Martín-Moruno and M. Visser, Class. Quantum Grav. **35**, 125003 (2018).
[14] P. Martín-Moruno and M. Visser, Class. Quantum Grav. **35**, 185004 (2018).
[15] P. Martín-Moruno and M. Visser, Class. Quantum Grav. **37**, 015013 (2020).
[16] S. K. Maurya, Y. K. Gupta, S. Ray, and D. Deb, Eur. Phys. J. C **77**, 45 (2017).
[17] S. K. Maurya, A. Banerjee, and S. Hansraj, Phys. Rev. D **97**, 044022 (2018).
[18] M. K. Jasim, D. Deb, S. Ray, Y. K. Gupta, and S. R. Chowdhury, Eur. Phys. J. C **78**, 603 (2018).
[19] M. F. Shamir, Phys. Lett. B **811**, 135927 (2020).
[20] R. Chan, Mon. Not. Roy. Astron. Soc. **288**, 589 (1997).

[21] R. Chan, Mon. Not. Roy. Astron. Soc. **316**, 588 (2000).
[22] R. Chan, Astron. Astrophys. **368**, 325 (2001).
[23] R. Chan, Int. J. Mod. Phys. D **12**, 1131 (2003).
[24] C. F. C. Brandt, M. F. A. da Silva, J. F. V. da Rocha, and R. Chan, Int. J. Mod. Phys. D **12**, 1315 (2003).
[25] G. Pinheiro and R. Chan, Gen. Relativ. Gravit. **40**, 2149 (2008).
[26] G. Pinheiro and R. Chan, Int. J. Mod. Phys. D **19**, 1797 (2010).
[27] J. M. Z. Pretel and M. F. A. Silva, Gen. Relativ. Gravit. **51**, 3 (2019).
[28] K. Charan, O. P. Yadav and B. C. Tewari, Eur. Phys. J. C **81**, 60 (2021).
[29] C. A. Kolassis, N. O. Santos and D. Tsoubelis, Class. Quantum Grav. **5**, 1329 (1988).
[30] N. O. Santos, Mon. Not. Roy. Astron. Soc. **216**, 403 (1985).
[31] B. C. Paul, Class. Quantum Grav. **18**, 2637 (2001).
[32] B. Bhui, S. Chatterjee, and A. Banerjee, Astrophys. Space Sci. **226** 7 (1995).
[33] H. Shah, Z. Ahmad, and S. Khan, Can. J. Phys. **96**, 1201 (2018).
[34] A. Banerjee and S. Chatterjee, Astrophys. Space Sci. **299**, 219 (2005).
[35] S. Chatterjee and A. Banerjee, Gen. Relativ. Gravit. **36**, 303 (2004).
[36] F. Hoyle, and J. V. Narlikar, Proc. Roy. Soc. A **290**, 162 (1966).
[37] S. D. Maharaj and B. P. Brassel, Eur. Phys. J. C **81**, 366 (2021).
[38] S. D. Maharaj and B. P. Brassel, Eur. Phys. J. C **81**, 783 (2021).
[39] S. D. Maharaj and B. P. Brassel, Class. Quantum Grav. **38**, 195006 (2021).
[40] B. P. Brassel, S. D. Maharaj, and R. Goswami, Phys. Rev. D submitted for publication.
[41] A. Di Prisco, L. Herrera, G. Le Denmat, M. A. H. MacCallum, and N. O. Santos, Phys. Rev. D **76**, 064017 (2007).
[42] R. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[43] G. Abreu, C. Barceló, and M. Visser, J. High Energy Phys. **1112**, 092 (2011).
[44] P. Martín-Moruno and M. Visser, Phys. Rev. D **88**, 061701(R) (2013).
[45] T. A. Roman, Phys. Rev. D **33**, 3526 (1986).
[46] P. Martín-Moruno and M. Visser, J. High Energy Phys. **1309**, 050 (2013).
[47] P. Martín-Moruno and M. Visser, Phys Rev. D **103**, 124003 (2021).
[48] C. W. Misner and D. H. Sharp, Phys. Rev. **136**, B571 (1964).
[49] M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971).