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High School Learners' challenges in Solving Circle Geometry Problems

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ABSTRACT

This paper reports on a study that investigated high school learners' difficulties when solving circle geometry problems. This study was conducted on the premise that, if these difficulties are well-known, then the teacher will be guided, to implement appropriate instructional strategies to address them. High school mathematics learners' poor performances in examinations (formal/informal), nationally, necessitated this study. A high school in the Northern Cape Province was purposefully sampled to serve as the research field for the study and circle geometry lessons were conducted, with the study participants. Thereafter, activities, investigation tasks, class work, home-work and standardized tests were administered to collect data. Data was also collected through classroom observations, video recordings and field notes. This was qualitative research hence, qualitative procedures were followed for data collection, analysis and interpretations. Content analysis was carried out on participants' written responses to the standardized tests, utilising Newman's Error Analysis model to inductively, identify participants challenges, which also served as the theoretical framework for the study. The analysed data revealed that learners do not understand circle geometry concepts, hence, their inability to make connections across geometry concepts to solve geometric problems. The researchers concluded this lack of fundamental background prevents learners from applying appropriate techniques in solving circle geometry problems.

Keywords: Problem-solving, Misconceptions, Mathematical errors

INTRODUCTION

Circle Geometry has emerged as a troublesome content area for learners in South Africa. Learners' work is inundated with errors, misconceptions, and misapplications (Seng, 2020; Abakah, 2019; Oladosu, 2014). The mathematics classroom practices that have resulted in

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learners performing so poorly in circle geometry, and geometry in general, have raised some concern among mathematics education researchers and motivated a number of studies. The results indicate that high school learners encounter a lot of difficulties due to the nature of the teaching activities in the circle geometry classrooms (DoBE, 2018). The conclusion was that it is essential to investigate how circle geometry instructions are carried out in the mathematics classroom - the established main cause of learners' poor performance (DoBE, 2018).

Euclidean geometry in general and circle geometry, in particular, make a lot of learners uncomfortable and confused. They complain that the geometry diagrams – which they have termed 'scary' - disturb them causing them to panic. This creates multifaceted challenges in the circle geometry classroom and an indication of the negative attitude learners might have developed towards geometry. This made the researchers to assert that learners' negative attitude tendencies towards geometry, may serve as a psychological barricade in their quest in solving circle geometry problems (Abakah, 2019). This situation can breed lack of confidence and an unwillingness to learn relevant techniques required to answer geometry problems, leading to learners making a lot of mistakes in their solutions.

This study required a tool to identify and analyse the mistakes students made when solving circle geometry problems, thus, the researchers employed the five steps of Newman's Error Analysis model - reading error, comprehension error, transformation error, process skill error, and encoding error **(**Seng, 2020; Siskawati, Zaenuri & Wardono, 2021). The model served as an effective framework for identifying and understanding learners' misconceptions when solving mathematical problems. Appropriateness of the framework was confirmed by the knowledge that it has been effectively implemented to study other content areas of mathematics to achieve the same aim *(*Abdul, 2015; Seng, 2020; Siskawati *et al.*, 2021).

The researchers posit that solving a circle geometry problem, just like mathematical word problems, require the problem-solver to be able to substantially read, understand and interpret the given problems; thereupon, s/he is required to meaningfully identify and apply appropriate procedures, to obtain correct solutions to the given problems; finally, students are expected to justify their solutions to establish whether their solutions logically answer the questions. The researchers maintain that these problem-solving procedures are in accordance with the stages of Newman's error analysis model, hence, the researchers argue for the model's appropriateness for the aims of this study (Seng, 2020; Siskawati *et al.*, 2021).

The researchers conducted this study on the assumption that, if learners' challenges are known, then the teacher will be directed to employ appropriate strategies during teaching and learning to eradicate them or to reduce the possibilities of them occurring (DoBE, 2018; Abakah, 2019). In realising the above aim, the following critical research questions were formulated: (1) *What challenges do Grade 11 learners encounter when solving circle geometry problems?* (1) *How can Grade 11 learners' identified challenges be remedied?*

LITERATURE REVIEW

The inability of teachers to teach relevant and explicit problem-solving instructional approaches and develop in learners' appropriate dispositions, attitudes, habits of mind, and others, may produce misconceptions and misapplication of concepts in geometry in students. This may have devastating effects on developing their mathematical proficiency (DoBE, 2018). According to Ndlovu & Mji, (2012) students' misconceptions in circle geometry classrooms include amongst others the inability to proof geometrically. Students incorrectly either list properties of geometric shapes as proofs and/or rewrite the known variables in a question as proofs. Ndlovu & Mji, (2012) aver that the ineptitude of learners to organise information in a logical chain of reasoning and arguments which develops into a misconception, is another difficulty with geometry which learners face.

In addition, Özerem, (2012); Siskawati, Zaenuri & Wardono, 2021; Seng, 2020) assert that irrelevant vocabulary to describe geometric statements and their relationships, difficulty to assess the validity of geometric arguments and inability of students to know and apply appropriate formulae, theorems, postulates, and axioms might be responsible for students developing a lot of misconceptions in the geometry classroom. Oladosu, (2014), reiterates conclusions of earlier researchers on difficulties learners face when solving geometry problems; these include amongst others, learners' lack of coordination in their views of three-dimensional objects (Battista & Clements, 1996); challenges in learning the appropriate language required for understanding and discussing geometric principles (Swindal, 2000); inability of learners to use theoretical statements in deductive reasoning and to recognise visually-relevant geometrical properties (Laborde, 2005). Some learners experience challenges in how to extract information from objects and form both natural and formal concepts (Battista, 2009). This includes challenges related to measurement and deductive proofs. Learners also experience problems in linking chains of reasoning and understanding definitions in geometry (Chazan, 1993; Groth, 2005; Herbst, Gonzalez, & Macke, 2005). According to Oladosu, (2014), "these difficulties centre around the meanings that students develop in relation to the learning they experience in and out of the geometry classroom".

To buttress the above, learners' ineptitude to design, and to reflect on the content learnt, to test ideas among alternative viewpoints, to evaluate and implement a strategy to achieve a desired goal (solution to a problem) also contribute greatly to their problem-solving difficulties in geometry (AACU, 2009). Furthermore, learners' failure to apply and adapt a variety of appropriate strategies to solve problems by recognizing reasoning and proof as fundamental aspects of mathematics and their inability to make and investigate mathematical conjectures so as to develop and evaluate mathematical arguments and proofs, are also contributory factors to the high failure rate in circle geometry (NCTM, 2000; Cuoco, 2000).

Adding to these challenges in circle geometry are learners' inabilities to select and to use various types of reasoning and methods of proofs, organize and consolidate their mathematical thinking by communicating mathematical thinking coherently and clearly to peers, teachers, and others. Also, majority of mathematics students are unable to analyse and evaluate the mathematical thinking and strategies of others. One of the greatest contributors to a high rate of underperformance in circle geometry is learners' ineptitude to use the appropriate language of mathematics to express mathematical ideas precisely. Learners have difficulties in recognizing and using connections among mathematical ideas, including learners' challenges in understanding how mathematical ideas interconnect and build on one another to form a coherent whole. Another challenge is learners' inability to recognize and to apply mathematics in the contexts of problem-solving situations; this is in combination with their failure to construct mathematical knowledge by shifting between different representations. Learners cannot compare different strategies and connect different concepts and ideas to solve geometric problems (Fennema & Romberg, 1999; NCTM, 2000; Cuoco, 2000).

The above-mentioned challenges are a clear indication that high school mathematics learners experience a lot of difficulties in solving geometric problems based on misconceptions and misapplication of rules and concepts (Fennema & Romberg, 1999; NCTM, 2000; Cuoco, 2000). Non-possession of problem-solving traits would mean that there is a huge gap between the problem-solver and the expected solution to the problem(s). Lack of problem-solving strategies will be like, "an act of chasing after the wind", which produces endless, fruitless efforts in solving a problem; when this goes on incessantly, the problem-solver may develop a negative attitude towards circle geometry, hence, creating mysteries around mathematics. From the above, it can be deduced that the difficulties learners encounter in geometry are enormous, and urgently require attention, however, the researchers aver that learners' challenges are open to remediation (Abakah & Brijlall, 2022).

THEORETICAL FRAMEWORK

Newman's error analysis model

This model is partitioned into five distinct stages showing what is required. Stage 1- reading error: students are expected to read and understand sentences and mathematical symbols from the given questions; Stage 2- comprehension error: students are expected to understand the given questions. Stage 3- transformation error: students are required to choose the appropriate mathematical solution methods, that will be relevant and applicable to solve the given questions; Stage 4- process skill error: students are required to perform mathematics processes correctly and Stage 5- encoding error: students are expected to justify their conjectured solutions. The possible indicators of each of the five stages of Newman's error analysis model and suggested strategies for correcting each of them are comprehensively elaborated in Table 1 below. All these stages are expected to cumulate in improving students' problem-solving competences, which is germane to this study.

The stages of Newman's error analysis model inform the procedures a problem-solver ought to follow, so that s/he will be able to efficaciously solve mathematical word problems; these stages have been extended to other content areas of mathematics. These stages are hierarchical according to the problem-solving difficulty levels (Effandi & Siti Mistima, 2010). These steps are underpinned by three dimensions - development of tasks, validation of tasks, and mandating learners to solve the developed and validated tasks (Newman, 1977). Essential to this study is the fact that this model also guide teachers to ascertain 'how' and 'where' learners' flaws and challenges are experienced. This model also informs teachers on appropriate instructional strategies that may be implemented to address the identified challenges learner's encounter.

Table 1: Possible indicators of each of the five stages of Newman's error analysis model and suggested strategies for eradicating each of them

Type of error	Likely indicators	Suggested strategies for students
Reading/decoding error	•Responses that show little or no engagement with the task • Responses that are consistent with an obvious misreading • Responses consistent with unfamiliarity with technical terms	• Refer to, or create a glossary of new words and their meaning in maths
Comprehension error	• Responses showing only a superficial engagement with the task • Responses consistent with a different (but related) question from the one being asked	• Ask yourself, 'What do I have to find out or show?' • Draw a diagram • Restate the problem in your own words
Transformation error	• Responses consistent with a different (but related) question from the one being asked • Responses consistent with the right numbers being used but with the wrong operations (or in the wrong order)	• Guess and check, • Make a list or table, • Look for a pattern, • Make the numbers simpler, • Be patient: most problems are not solved quickly nor on the first attempt. Try, try and try again until an appropriate solution is reached.
Process skill error	• Arithmetic errors • Procedural errors • Incomplete solutions	• If one approach is not working try a different one.
Encoding error	• Incomplete solutions • Irrelevant responses to questions	• Check if the conjectured solution is correct and meaningful in the context of the question. • Check if the question is completely and well answered.

Adapted from (Australian Council for Educational Research, 2019).

METHODOLOGY

Research design

The researchers realised that the participants (learners) needed to be consistently observed in their natural classroom setting; this will adequately provide unabridged and detailed information so that the research questions can be concisely and empirically answered. To obtain the above aim, the researchers critically perused different research designs. To this end, a qualitative case study research design was found to be the best fit in the context of this study.

This proved an effective medium for data collection as this provided the researchers, the avenue to intensively observe participants as they solved circle geometry problems in their natural habitat - their classroom - over an extended period. The implementation of a case study design for this study, aided the researchers to solicit adequate and relevant information from the participants (McMillan & Schumacher, 2014).

Participants

34 Grade 11 mathematics learners (15 males and 19 females) in a South African high school were selected to serve as participants for this study, irrespective of gender, ethnicity, social and race groups. They were arranged in groups of 3 or 4, in a collaborative classroom setting.

Ethical considerations

Firstly, permission was obtained, in writing from UNISA ethics committee (Ethical clearance letter); the provincial and district departments of education and the school governing body (SGB) at the research site. Concomitantly, the researchers ensured that learners, and their parents/ guardians' consent letters were produced and signed. These processes were undertaken before data collection begun. The identities of all participants were not revealed to any third party; learners were not forced, intimidated, or blackmailed, in any way to serve as participants for this study. The researchers, therefore, reiterated to learners that it was not compulsory for them to participate in this study.

Data Analysis

As said earlier, content analysis was carried out on participants written responses to the standardized test, class exercises, homework, and group activities. These tasks were developed and validated by external experts, thereafter, learners were mandated to solve the tasks. Newman's Error Analysis model was used to inductively, identify participants' errors, difficulties, and misconceptions (Abdullah, Abidin & Ali, 2015). Data analysis was carried out in three stages. Firstly, the overall analysis of participants' written responses to the standardized test was done (see Table 2). After finalizing the first stage, the researchers identified the number of participants who had demonstrated each of the five stages of Newman's error analysis model (see Table 3). Lastly, participants' written responses, on the 'how' and 'why' of each of the five

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categories of errors made by the participants were explained. Each of the three stages of data analysis is elaborated below.

Stage 1- Overall analysis of participants' written responses

At this juncture, content analysis of participants' written responses to the standardized test data was done, by using the stages of Newman's problem-solving approach - reading, comprehension, transformation, process skills, and encoding - as the benchmark for determining participants' problem-solving competence. The number of students who were able to either fulfil (coded as Y) or not fulfil (coded as N), for each of the five stages were noted and summarized (see Table 2).

Table 2: Overall summary of how participants responded to the standardized test items on circle geometry, in accordance with Newman's stages of problem-solving

Participants	APOS categorization of students' mental constructions									
	Reading		Comprehension		Transformation		Process skill		Encoding	
	Y	N	Y	N	Y	N	Y	N	Y	N
Number of participants	34	0	25	9	22	12	20	14	16	18
Percentages	100%	0%	74%	26%	65%	35%	59%	41%	47%	53%

As displayed in Table 2 above, the results after content analysis of participants' written responses were - Reading: 34 (100%) of participants could read; Comprehension: 25 (74%) of participants could comprehend, while 9 participants (26%) could not; Transformation: 22 (65%) of participants were successful, while 12 (35%) were unsuccessful; Process skill: 20 (59%) of participants were successful, while 14 (41%) were unsuccessful; Encoding: 16 (47%) of participants were successful, while 18 (53%) were unsuccessful.

Stage 2- Analysis of participants' responses in view of each stage of Newman's error analysis model

The researchers identified the number of participants who demonstrated each of the five stages of Newman's error analysis model (see Table 3).

Table 3: Overall summary of participants' written responses to the standardized test items on circle geometry in accordance with Newman's error analysis stages

Participants	APOS categorization of students' mental constructions				
	Reading error	Comprehension error	Transformation error	Process skill error	Encoding error
Number of participants	0	9	12	14	18
Percentages	0%	26%	35%	41%	53%

As displayed in Table 3 above, none (0%) of the participants demonstrated reading error; 9(26%) demonstrated comprehension error; 12 (35%) demonstrated transformation error; 14(41%) demonstrated process skill error; 18 (53%) demonstrated encoding error. It can be observed in Table 3 above that the number of participants who demonstrated each of the errors, increased hierarchically. These data-analysis results are in concord with literature; for instance, Effandi & Siti Mistima, (2010), aver that the hierarchical stages of Newman's error analysis is directly proportional to their corresponding problem-solving difficulty levels.

Stage 3- Presentation of participant's written responses

Lastly, participants' written responses, that demonstrated the 'how' and 'why' of each of the five categories of errors were explained. As presented in Table 3 above, none of the participants demonstrated reading error - all could read, identify sentences and mathematical symbols from the given questions, however, other types of errors - comprehension, transformation, process

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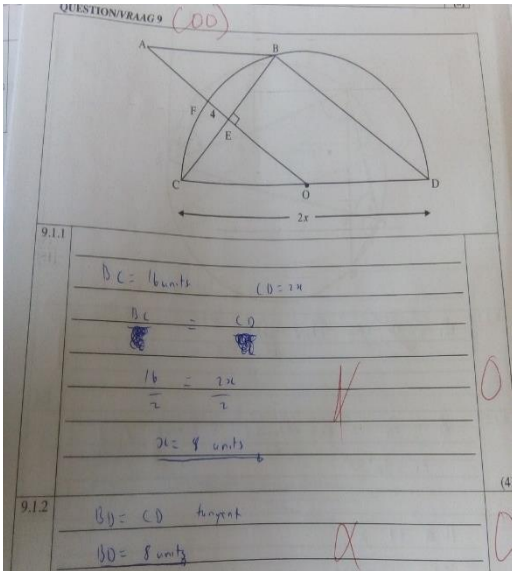
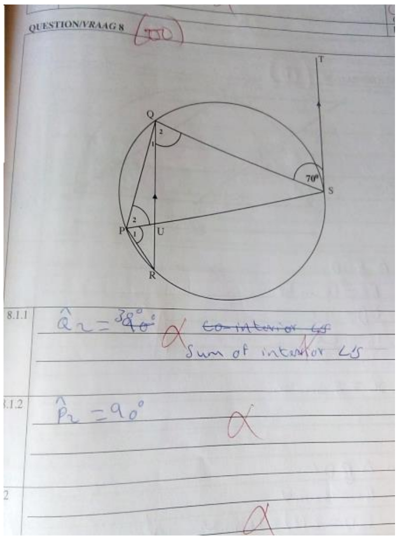
skills and encoding were demonstrated (Abdullah, Abidin & Ali, 2015; Siskawati, Zaenuri & Wardono, 2021; Seng, 2020). They are delineated below.

Comprehension error



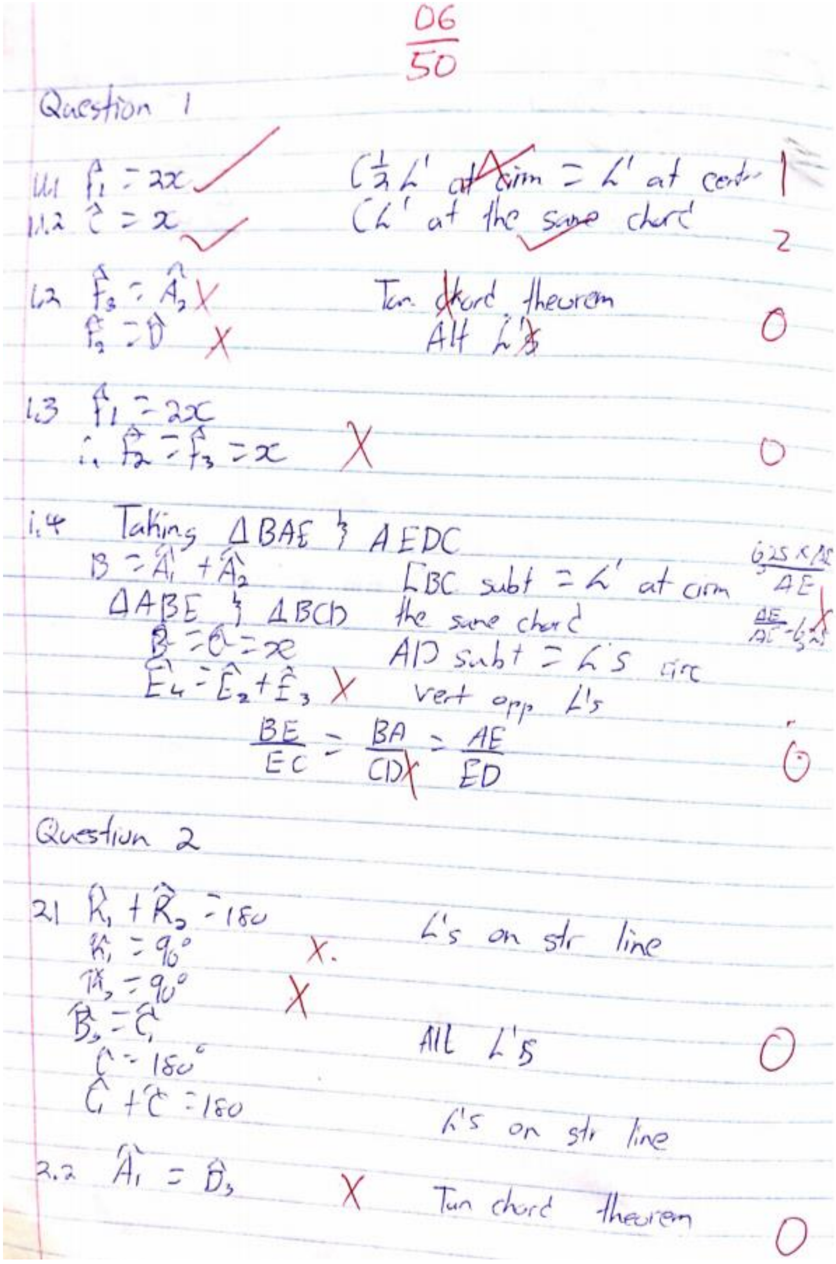


Scan 1 –ST 7

Scan 2- ST 7

The participant's written response (see scans 1&2) indicate that s/he provided irrelevant responses to most of the questions or left them unanswered. S/he could not make any meaningful attempt in solving any of the given questions. This illustrates lack of understanding of circle geometry concepts as well as lack of confidence and technique to approach problems. It is evident in scans 1 &2 that the participant was unable to interpret and understand the given questions and the corresponding geometric diagrams. This, according to Newman's error analysis constitutes comprehension error. This resulted in the participant providing irrelevant or no responses to the given problems.

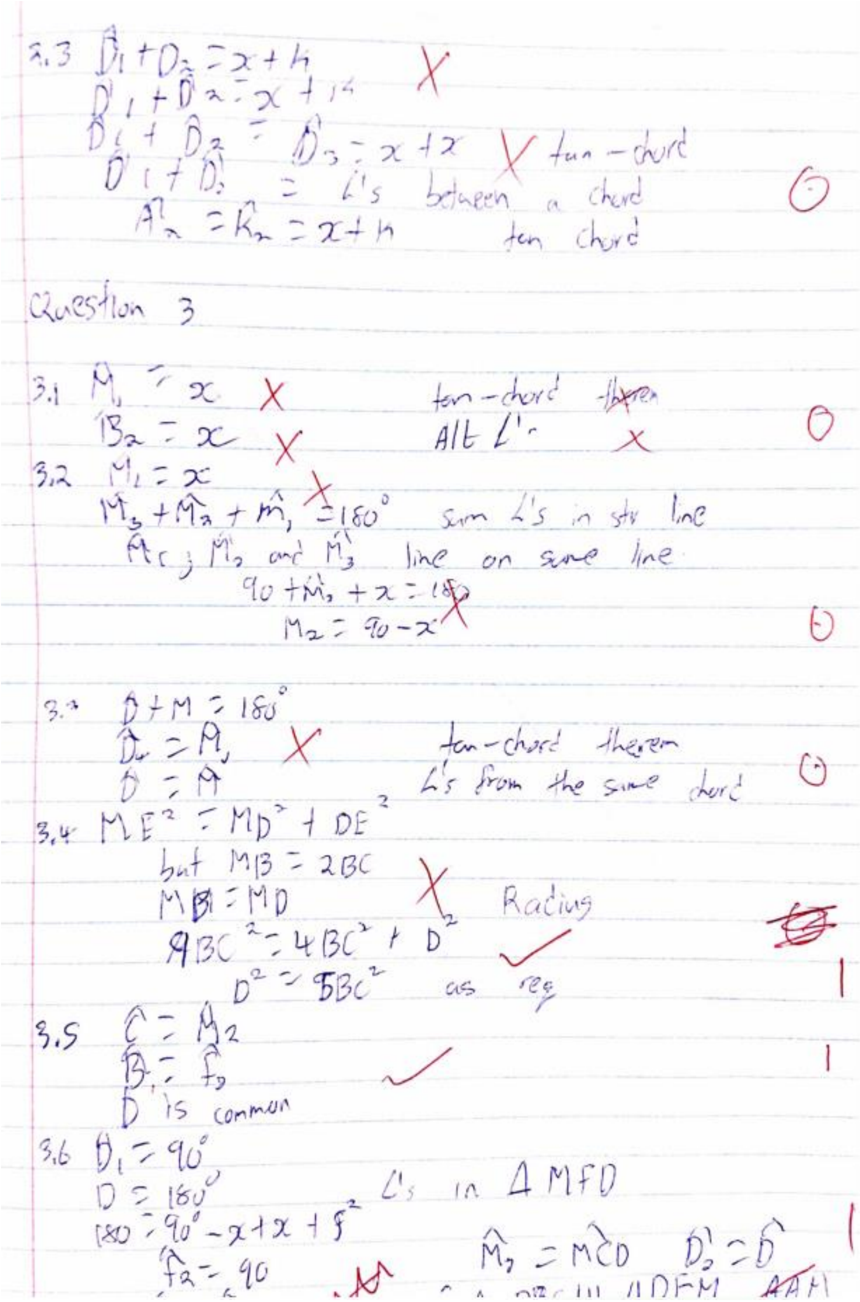
Transformation, process skill and encoding errors



Scan 3: Written response of ST 12

The illustration above (scan 3) reveals the participant's written responses to the question - s/he provided mostly irrelevant responses, hence, scored zero for most sub-questions. This exemplify that s/he was unable to conjecture meaningful and appropriate solutions to the given problems. The next illustration (scan 4) presents the same participant's other written response to the standardized test.

Vol. 78 | No. 12/1 | Dec 2022
DOI: 10.21506/j.ponte.2022.12.9


3.3 $\hat{D}_1 + \hat{D}_2 = x + h$
$\hat{D}_1 + \hat{D}_2 = x + h$
$\hat{D}_1 + \hat{D}_2 = \hat{D}_3 = x + x$ tan – chord
$\hat{D}_1 + \hat{D}_2 = \angle$'s between a chord
$\hat{A}_2 = \hat{K}_2 = x + h$ tan chord

Question 3

3.1 $\hat{M}_1 = x$ tan – chord theorem
$\hat{B}_2 = x$ Alt $\angle$'s
3.2 $\hat{M}_1 = x$
$\hat{M}_3 + \hat{M}_2 + \hat{M}_1 = 180°$ sum $\angle$'s in str line
$\hat{M}_1$, $\hat{M}_2$ and $\hat{M}_3$ line on same line
$90 + \hat{M}_2 + x = 180$
$M_2 = 90 - x$

3.3 $\hat{D} + \hat{M} = 180°$
$\hat{D}_2 = \hat{A}_1$ tan – chord theorem
$\hat{D} = \hat{A}$ $\angle$'s from the same chord
3.4 $ME^2 = MD^2 + DE^2$
but $MB = 2BC$
$MB = MD$ Radius
$4BC^2 = 4BC^2 + D^2$
$D^2 = 5BC^2$ as req
3.5 $\hat{C} = \hat{M}_2$
$\hat{B}_1 = \hat{F}_2$
$\hat{D}$ is common
3.6 $\hat{D}_1 = 90°$
$D = 180°$ $\angle$'s in Δ MFD
$180 = 90° - x + x + f^2$
$\hat{f}_2 = 90$ $\hat{M}_2 = M\hat{C}D$ $\hat{D}_2 = \hat{D}$

Scan 4: Written response of ST 12

The participant's written responses (see scan 4), also display what was observed on scan 3 - that s/he provided mostly irrelevant responses. S/he demonstrated lack of understanding of circle geometry concepts, meaning that s/he could not logically and meaningfully conjecture appropriate responses to the given problems and was unable to identify and apply relevant mathematical methods/procedures, to conjecture correct responses to the given questions. This according to Newman's error analysis, constitute transformation error. This participant demonstrated no evidence of higher order geometric reasoning and creative thinking around circle geometry theorems and concepts. A few of these participants could partly recall and apply the correct statement of theorems and/or converse of theorems, however, they were unable to apply them in their solutions, imaginatively. There were improper and non-meaningful connections in their written responses; this demonstrates transformation error, in accordance with Newman's error analysis model.

It can be observed in scans 3 & 4 that the participant was unable to perform mathematics processes correctly- an indicator of process skill error. In addition, the participant was unable to provide appropriate geometric reasons to justify their conjectured solutions, constituting an encoding error. The written response by ST 12 showed that s/he partly recognised the identified circle geometry theorem(s) required. For instance, in sub-question 1.1.1, the participant stated that $\widehat{F_1} = 2x$, but the corresponding reason was incorrect. S/he stated "$\frac{1}{2} < at\ circumference = < at\ centre''$ as the reason instead of stating that "$\frac{1}{2} < at\ centre = < at\ circumference''$. Also, in sub-question 1.1.2, s/he stated that $\widehat{c_1} = x$, with the corresponding reason, "angles from the same chord", which is correct, however, s/he could not apply the appropriate circle geometry theorems intuitively to obtain meaningfully solutions to the given questions.

FINDINGS AND DISCUSSION

The research findings of this study are presented and discussed below in accordance with each of the research questions.

(1) *What challenges do Grade 11 learners encounter when solving circle geometry problems?*

According to Prakitipong & Nakamura, (2006), students are incapacitated at providing appropriate responses, partly because, they experience problems in language fluency and conceptual understanding (reading and comprehension). The researchers posited that solving a circle geometry problem requires the problem-solver to substantially read, understand and interpret the given problems. The extent to which a problem can be solved, first of all, is centred on what the problem-solver knows about the problem to be solved. As it was emphasized by Posamentier, Smith and Stepelman (2010), problem-solving becomes much effective if - the necessary important contents are covered, useful mathematical techniques are developed and sufficiently practised, and classes of problems are coherent so that the associated concepts and relationships can be constituted at an abstract level.

Learners' knowledge deficiencies in identifying and using correct geometric terminologies, identifying appropriate relevant properties, axioms and theorems, impeded the meaning/understanding they developed in the geometry classroom. This was verified as this study established that 9(26%) of the participants demonstrated comprehension error, hence, the process of acquisition of knowledge can never be underestimated; either it leads to learners understanding and developing the appropriate meaning of the content in context, or, the learners may develop wrong understanding and inappropriate meaning of the content in context, which becomes a misconception. This implies that 9(26%) of the participants had a lot of misconceptions in learning circle geometry concepts which greatly contributed to their poor performance as their conceptions were erroneous (Seng, 2020).

Learners' inability to understand geometric language (an indicator for reading and comprehension errors) was cited by Swindal, (2000), as the starting point of their misconceptions. Learning the appropriate language required for understanding and discussing geometric principles poses a challenge to high school mathematics learners in South Africa; this negatively influences the meaning/understanding learners obtain in the mathematics classroom. As rightly emphasized by Cuoco, (2000), using the language of mathematics to express mathematical ideas precisely is an essential initial step in solving a geometry problem. Chazan, (1993); Groth, (2005); Herbst, Gonzalez, & Macke, (2005) also cited understanding definitions in geometry as key in geometry problem-solving as well as, issues in relation to how students extract information from objects and form both natural and formal concepts is also a contributing factor as to how learners develop meanings of geometry concepts (Battista, 2009).

It can be seen in scans 1 & 2 above, that the participant was unable to provide relevant responses to the questions, which all point to the fact that learners do not understand circle geometry concepts. From the learner's response, it can be observed that, s/he encountered comprehension error, as delineated by Newman's error analysis model. This presupposes that if a learner is unable to comprehend the problem, then his/her ability to formulate the desired problem-solving path to get the expected solution to the problem, would greatly be impeded.

The participant's diagram-interpretation vagueness (inability to interpret a geometric diagram well) is another indicator of comprehension errors. This proved to be a major concern for the study participants and might be one of the reasons they lacked adequate understanding, hence, were unable to develop appropriate meanings of circle geometry concepts. It impeded participant's spatial awareness and geometric thinking which in all impeded the meaning s/he developed of circle geometry. From this study, 9(26%) of the participants who demonstrated comprehension error were unable to interpret the geometry diagrams well. These participants could not make any attempt in solving any of the sub-questions from those supposedly difficult and 'scary diagrams', which presupposes that, participants' inability to interpret a geometric diagram may contribute to their inability to understand the geometric problem and their ineptitude to solve geometric problems well (Oladosu, 2014).

In summary, the factors that contribute to learners not attaching meaning/understanding to circle geometry concepts, as established from this study are - learners' inability to identify and use correct geometric terminology, learners' inability to identify appropriate relevant properties, axioms and theorems, learners' inability to understand geometric language and learners' inability to interpret geometric diagrams well. These are indications that the reasons learners cannot attach meaning/understanding to geometric concepts are multifaceted and the factors raised above can be delineated further, hence, a follow up research on this topic is worthy.

Below are discussions on these identified factors

(i) Learners' inability to make connections across geometry concepts to solve geometric problems

According to Prakitipong & Nakamura, (2006), students are incapacitated from providing appropriate responses, partly because, they encounter problems in mathematical processing - transformation, processing skills and encoding. This implies that, after a learner had read, rightly interpreted and understood the given problem, s/he is required to meaningfully identify and apply appropriate methods to obtain solutions to the given problems; thereafter, students are expected to justify their conjectured solutions - if their solutions adequately and logically answer the given questions.

From this study, although, more than thirty per cent of the participants made no attempt to answer tasks 7&8, it was observed that the rest of the participants who made some attempts to solve the problems, approached the problems by (1) making efforts to interpret the diagram, (2) making efforts to use known information and the language of geometry in their solution path, (3) making efforts to link chains of reasoning together to provide a favourable solution to the circle geometry problem. The disparities in the participants' problem-solving abilities were due to, amongst others - some participants provided irrelevant solutions to the tasks as the approach used did not serve as the desired path to provide solution to the problem, as shown in scans 1-4 above. This constituted transformation, process skills and encoding errors.

From scans 3 & 4 above, it can be observed that learners could not understand the problem to be solved, they could not interpret the geometric diagram, they could not use theoretical statements in deductive reasoning, and they could not recognise visually relevant geometrical properties; these are similar points made by Laborde, (2005). It can be concluded, therefore, that students lacked the desired approach in solving problems in circle geometry. The study participants could, on the average, use their background information in their solution path, however, linking their factual knowledge (identifying and recalling basic concepts, properties, axioms, and theorems) together to promote understanding of the problems to be solved served as the starting point for the problems encountered by the learners. The level of difficulty starts from lack of conceptual knowledge, gradually to procedural knowledge and finally to meta-cognitive knowledge where greater challenges were encountered (Laborde, 2005).

With reference to the taxonomy of the cognitive domain, level two of the cognitive domain (understanding) became the barrier in the thinking process. It was discovered that participants could not attach meaning, understanding, and reasoning relating to circle geometry theorems and concepts, hence, the rest of the thinking process - applying, analysing, evaluating, and creating - were greatly impeded from one level of the cognitive domain to the other. This is an indication that they experienced challenges related to measurement and deductive proof, as well as linking chains of reasoning (Chazan, 1993; Groth, 2005; Herbst, Gonzalez, & Macke, 2005).

From this study, linking chains of reasoning was really a great challenge to the study participants as they were unable to solve non-routine problems, instead they provided irrelevant answers. There was little evidence of logical reasoning and advanced mathematical thinking in their solution path. The participants realised that mathematics is about seeking solutions, not just memorizing procedures (since the procedures they memorized could not help). Learning mathematics is about exploring patterns, not just memorizing formulas; it is about formulating conjectures and not just doing exercises (Schoenfeld, 2016). A mathematical problem can be viewed as requiring critical thinking; it entails the use of mathematical methods to make representations and analysis, to obtain solutions to problems. The study participants experienced that solving circle geometry problems demand strategic reasoning, insightfulness, personal persistence, choosing an effective strategy and the ability to apply these strategies to solving the problems. The situation also demanded linking mathematical ideas, comprehension of the language of mathematics to enable the expressing of mathematical ideas precisely; recognizing and using connections among mathematical ideas; understanding how mathematical ideas interconnect and build on one another to produce a coherent whole (Cuoco, 2000). These attributes, were lacking in the written responses (see scans 1-4) as more than 30% of participants (see Table 3) as noted earlier, demonstrated transformation, process skills and encoding errors.

(2) How can Grade 11 learners' identified challenges when solving circle geometry problems be remedied?

As said earlier, this study was conducted on the premise that, if learners' challenges are well-known, then the teacher will be guided, to implement appropriate strategies to address them (Ndlovu & Mji, 2012; Abakah & Brijlall, 2022). As asserted earlier, teaching students' relevant instructional approaches and guiding them to develop the right dispositions, attitudes and habits of mind, in the geometry classroom, is key in managing students' misconceptions and difficulties (Swartz, 2012). For instance, Schwieger, (2003), suggested ways for dealing with students' attitudes and misconceptions about problem-solving and these included - asking interesting and real-life problems which students can relate to and demonstrating to students the 8 problem-solving skills - to classify, deduce, estimate, generate patterns, hypothesize, translate, try, modify, and verify. In addition, the teacher is expected - to give problem-solving examples, illustrate the application of these skills; give practice that results in students sharpening these skills; demonstrate the necessity of implementing these problem-solving strategies; demonstrate that since multiple strategies are available, problem-solving is not necessarily impeded because some particular mathematical tool is unavailable; show that trials which do not lead to solution usually provide useful information to guide re-trials, and that trial information should not be destroyed until after solutions are reached; remind students that reaching solutions often takes time and that experimentation is to be expected; remind students that there are no algorithms for true problems so they should not waste effort in trying to remember ‘how we did this one the last time’; teach students that careful reading and comprehending the problem statement or situation are necessary and the search for ‘key words’ is likely to be counterproductive as well as give students practice with ‘multiple’ or ‘conditional’ solution problems.

Özerem, (2012) also suggested solutions to manage students’ misconceptions. He established that it is necessary for teachers to (1) use relevant vocabulary to describe geometric statements and their relationships; (2) apply logic to assess the validity of geometric arguments, and (3) help students to memorize formulae easily. Additionally, according to Ndlovu & Mji, (2012), students’ difficulties and misconceptions have pedagogical implications. This involves teachers adjusting their pedagogical strategies to deal with the difficulty/misconception identified. The implication is that teachers need to adopt and implement relevant instructional approaches in mathematics classrooms (Abakah & Brijlall, 2022). This can help eradicate or reduce to the barest minimum, students’ difficulties, and misconceptions.

Most learners learn circle geometry theorems/concepts as a set of rules or principles to be followed, but circle geometry problems go beyond merely being able to memorize rules or principles. It requires learners to link ideas from circle geometry theorems to form a productive thought and to link the chains of their reasoning to form a meaningful solution path, which will be helpful in solving a problem. Learners’ mere ability to memorize circle geometry theorems or concepts impedes their ability to be competent in geometric proofs and solving high-order geometric problems. They do not learn circle geometry as a content which requires rigorous mathematical thinking, analysing, being creative, conjecturing and linking chains of reasoning together to provide a meaningful solution path. They also do not realise that circle geometry is a content which demands patience and a ‘never give up’ attitude which will motivate them to try, try and try again until an appropriate solution is reached (Cuoco, 2000; DBE, 2018).

CONCLUSION

As we asserted earlier in this paper, the difficulties experienced by mathematics learners in the circle geometry classrooms are open to remediation by a different approach to instruction

(Abakah & Brijlall, 2022). Learners' acceptable level of performance in examinations is the aim of every mathematics teacher, hence, a learning environment which will give mathematics learners the opportunity to study mathematics as an exploratory, dynamic, and evolving discipline rather than as a rigid, absolute, closed body of rules to be memorized, must be promoted (Schoenfeld, 2016). This is because, some participants who could identify and use correct geometric terminology, as well as identify appropriate relevant properties, axioms, and theorems, lacked appropriate techniques and skills in solving problems. Most notably, they could not appropriately, make connections across the circle geometry concepts to solve more complex problems. Participants' progression from knowledge dimension of circle geometry concepts to the application dimension, demand these concepts to be taught in an innovative, exploratory, and experimental manner; such an approach could evoke learners' creativity, thinking skills and spatial awareness, which enhance learners' problem-solving skills. It is imperative, therefore, for teachers to endeavour to make mathematics an interesting subject to drive away the fear of mathematics in learners (DoBE, 2018).

RECOMMENDATIONS

The researchers highly recommended that learning and teaching materials, particularly, technological learning and teaching aids must be inculcated in the learning and teaching of geometry in South African high schools. The researchers posited that this would help in eradicating students' errors, difficulties, and misconceptions. The unavailability of these materials forces geometry instructions to be carried out in an abstract manner which impedes learners' understanding and problem-solving skills. As described by Rose & Arline (2009); Roble, (2016) students demonstrate conceptual understanding in mathematics problem-solving when they use diagrams and instructional materials to demonstrate solution path. Also, as spelt out clearly in the NCTM's standards, teaching and learning materials are essential tools for instruction in mathematics; these can support investigations carried out by students in every area of mathematics, including geometry. Appropriate tools furnish visual images of mathematical ideas and enables the learners to think divergently and intuitively. According to the NCTM, when technological tools are available and used efficiently in problem-solving situations, students can focus more on decision-making, reflection, reasoning, and problem-solving, all of which have a tremendous positive impact on their performance (NCTM, 2000).

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