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# Algebraic Structures of Generalised Symmetries of $n$ th-order Scalar Ordinary Differential Equations of Maximal Lie Point Symmetry

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**Abstract:** We compute for the representative scalar ordinary differential equation of maximal point symmetry the generalised symmetries of order-one and two. We examine the Lie Brackets for the generalised symmetries and see that closure does not occur for generalised symmetries of order-two. Consequently all generalised symmetries up to the maximum order possible must be admitted.

**Keywords:** Generalised symmetries; $n$th-order scalar ODEs; algebraic structures; Lie Brackets
**MSC 2010 Numbers:** 34A30; 34C14

## 1 Introduction

Any $n$th-order scalar ordinary differential equation of maximal Lie point symmetry is equivalent under a point transformation to the equation

$$y^{(n)} = 0 \qquad (1)$$

in which the independent variable is $x$. The Lie point symmetries of ordinary differential equations have been the subject of many studies beginning with those of Lie in which he showed that the maximum number of point symmetries of a scalar second-order equation is eight [9][p 405]. Subsequently it was shown that the algebra of the symmetries found in the operation of taking the Lie Bracket was $sl(3,R)$. The internal structure of the algebra is not without intrinsic interest and has relevance to studies of higher-order equations and so the subject of this paper. The subalgebras are

$$\begin{array}{ll} \left.\begin{array}{l} \Gamma_1 = \partial_y \\ \Gamma_2 = x\partial_y \end{array}\right\} & \begin{array}{l}\text{Solution Symmetries}\\ 2A_1\end{array} \\ \left.\begin{array}{l} \Gamma_3 = y\partial_x \\ \Gamma_4 = xy\partial_x + y^2\partial_y \end{array}\right\} & \begin{array}{l}\text{Noncartan Symmetries}\\ 2A_1\end{array} \\ \left.\begin{array}{l} \Gamma_5 = \partial_x \\ \Gamma_6 = 2x\partial_x + y\partial_y \\ \Gamma_7 = x^2\partial_x + xy\partial_y \end{array}\right\} & \begin{array}{l}\text{Special Linear Group}\\ sl(2,R)\end{array} \\ \left.\begin{array}{l} \Gamma_8 = y\partial_y \end{array}\right\} & \begin{array}{l}\text{Homogeneity}\\ 1A_1.\end{array} \end{array} \qquad (2)$$

Even in the earliest days of the study of symmetries of differential equations it was natural to consider transformations generated by symmetries which were not point symmetries [10] and one finds in the work of Noether [14] an assumption that the symmetries could be generalised. As Noether was studying the invariance of the Action Integral and the Lagrangian is directly related to the Hamiltonian the theory of which essentially contains canonical transformations, this assumption is simply a reflection of the fact that the canonical transformations in general correspond to generalised symmetries.

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Lie had investigated the symmetries of equations of order greater than the second, but it was not until more than a century after his time that there were systematic studies of the point symmetries of equations of higher-order. In particular these studies were concerned with those of linear - equally linearisible by a point transformation - scalar ordinary differential equations. Initial studies by Krause and Michel [7,8] were rendered more complete in [11,12]. A second-order scalar ordinary differential equation of maximal point symmetry always had eight Lie point symmetries and the algebra $sl(3,R)$. By way of contrast for higher-order equations there were three possibilities for an equation which was linearisible by means of a point transformation. There could be $n+4$, $n+2$ or $n+1$ Lie point symmetries. Evidently second-order equations were exceptional since they did not obey these rules. A second-order equation of maximal symmetry did not have these three possibilities but just the one, eight, which was more than expected by the general result. The additional two symmetries with the subalgebra $2A_1$ indicated above as Noncartan - also called fibre-preserving in the literature [6] - are not reflected in the point symmetries of equations of higher-order. However, there was some hint of a persistence of this property in third-order equations in that they possessed contact symmetries - not necessarily equations of maximal point symmetry; the Kummer-Schwarz equation is an example of an exception - making ten symmetries in all with the algebra $sp(5)$ [2] and in the case of a linear equation, with seven plus three symmetries being a natural generalisation of the six plus two of the second-order equation of maximal symmetry. Further properties were noted in [13]. Equations of higher-order than the third did not possess contact symmetries.

In this paper we explore the generalised symmetries and associated algebras of linear equations of higher-order of maximal symmetry. The complexities of the calculations limit our attention to equations of the form (1), but it has to be remembered that the considerations apply to all equations which can be linearised to (1). The linearising transformation need not be point since we now consider generalised symmetries. It is our intention to establish general results for the number of generalised symmetries of order-two and three and their associated algebras. There is a practical need for the establishment of general results as the calculations become impossible even for equations of not excessive order. This is despite using one of the better symmetry-determining packages currently available, Sym [3,4,5,1]. From the results which we can obtain we infer general rules and then prove that they are in fact general being quite aware of the dictum of Popper [15]. Because we are dealing with generalised symmetries, we write all symmetries in vertical form.

## 2 Some Computational Results

We present the generalised symmetries of order-two for several equations of the class (1).

1. $y^{(4)}=0$

$$\Gamma_s = x^i\partial_y, \quad i=0,3$$
$$\Gamma_h = y\partial_y$$
$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-3xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-6xy\right)\partial_y;\ \left(x^4y''-4x^3y'+6x^2y\right)\partial_y$$

where the subscripts refer to solution, homogeneity, Jetspace order-one and Jetspace order-two symmetries. The subalgebra of $sl(2,R)$ is obtained from the Jetspace order-one symmetries and the homogeneity symmetry. The standard form for $y^{(n)}=0$ is [12]

$$\Gamma_1 = \partial_x,\ \Gamma_2 = x\partial_x+(n-1)y\partial_y \text{ and } \Gamma_3 = x^2\partial_x+(n-1)xy\partial_y$$

and the connection with the Jetspace order-one symmetries and heterogeneity symmetry is manifest.

2. $y^{(5)}=0$

We do not repeat the expressions for $\Gamma_s$ and $\Gamma_h$ as the only difference is that the number of solution symmetries increases by one with the order of the equation.

$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-4xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-12xy\right)\partial_y;\ \left(x^4y''-6x^3y'+12x^2y\right)\partial_y.$$

3. $y^{(6)}=0$

$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-5xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-20xy\right)\partial_y;\ \left(x^4y''-8x^3y'+20x^2y\right)\partial_y.$$

4. $y^{(7)}=0$

$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-6xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-30xy\right)\partial_y;\ \left(x^4y''-10x^3y'+30x^2y\right)\partial_y.$$

5. $y^{(8)}=0$

$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-7xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-42xy\right)\partial_y;\ \left(x^4y''-12x^3y'+42x^2y\right)\partial_y.$$

6. $y^{(9)}=0$

$$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y'-8xy\right)\partial_y$$
$$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y''-56xy\right)\partial_y;\ \left(x^4y''-14x^3y'+56x^2y\right)\partial_y.$$

7. $y^{(10)} = 0$

$\Gamma_{J1} = y'\partial_y;\ xy'\partial_y;\ \left(x^2y' - 9xy\right)\partial_y$

$\Gamma_{J2} = y''\partial_y;\ xy''\partial_y;\ x^2y''\partial_y;\ \left(x^3y'' - 72xy\right)\partial_y;$
$\left(x^4y'' - 16x^3y' + 72x^2y\right)\partial_y.$

We note that the Jetspace order-one symmetries are always three in number (at least for the number of equations considered, but a detailed proof in terms of the equivalent point symmetries is found in [12]) and that the number of Jetspace order-two symmetries is five for each of the equations considered here. One can find similar results for equations of order 2 and 3. Naturally in the case of the second-order equation the Jetspace order-two symmetries are completely trivial.

Before we proceed to establish a general formula for the Jetspace order-two symmetries from the evidence before us and then to verify the formula in general it is appropriate to consider the algebraic structure of the symmetries which we have before us. The first task is to establish the subalgebra of the Jetspace order-one and order-two symmetries, respectively, and then to consider the relationships between the subalgebras.

For the fifth-order equation the Lie Brackets for the symmetries of order-one are

$$
\begin{aligned}
&[\Gamma_{J11}, \Gamma_{J12}]_{LB} = -\Gamma_{J11}\\
&[\Gamma_{J11}, \Gamma_{J13}]_{LB} = -2\Gamma_{J12} + 4\Gamma_h\\
&[\Gamma_{J12}, \Gamma_{J13}]_{LB} = -\Gamma_{J13}.
\end{aligned}
$$

This is reconciled with $sl(2,R)$ if we define $\tilde{\Gamma}_{J12} = \Gamma_{J12} - 2\Gamma_h$. With this redefinition the subalgebra is closed as $sl(2,R)$.

The Lie Brackets of the symmetries of order-two are

$$
\begin{aligned}
&[\Gamma_{J21}, \Gamma_{J22}]_{LB} = -2\Gamma_{J31}\\
&[\Gamma_{J21}, \Gamma_{J23}]_{LB} = -4\Gamma_{J32} - 2\Gamma_{J21}\\
&[\Gamma_{J21}, \Gamma_{J24}]_{LB} = -6\left(\Gamma_{J33} + \Gamma_{J22} - 4\Gamma_{J11}\right)\\
&[\Gamma_{J21}, \Gamma_{J25}]_{LB} = -4\left(2\Gamma_{J34} - 6\Gamma_{J23} + 3\Gamma_{J12} + 6\Gamma_h\right)\\
&[\Gamma_{J22}, \Gamma_{J23}]_{LB} = -2\left(\Gamma_{J33} + \Gamma_{J22}\right)\\
&[\Gamma_{J22}, \Gamma_{J24}]_{LB} = -2\left(2\Gamma_{J34} + 3\Gamma_{J23} - 12\Gamma_{J12}\right)\\
&[\Gamma_{J22}, \Gamma_{J25}]_{LB} = -6\left(x^4y''' - 3x^3y'' + 2x^2y' + 4xy\right)\partial_y\\
&[\Gamma_{J23}, \Gamma_{J24}]_{LB} = -2\left(x^4y''' + 2x^3y'' - 12x^2y'\right)\partial_y\\
&[\Gamma_{J23}, \Gamma_{J25}]_{LB} = -2\left(2x^5y''' - 7x^4y'' + 6x^3y' + 12x^2y\right)\partial_y\\
&[\Gamma_{J24}, \Gamma_{J25}]_{LB} = -2\left(x^6y''' - 6x^5y'' + 18x^4y' - 24x^3y\right)\partial_y.
\end{aligned}
$$

Obviously there is no closure. One infers that this be the case for all symmetries of order higher than the second. From the results for the last four brackets one can easily believe that the algebra closes only with the addition of higher-order symmetries. The order of the ultimate set of symmetries must necessarily be limited by the order of the equation. From the evidence one can only conjecture the number of symmetries if all possible generalised symmetries are admitted. In the case of the tenth-order equation this may well be in excess of 100 which could make the calculation of the Lie Brackets a somewhat daunting task.

## 3 Conjectures

That the subalgebra of the solution symmetries is $nA_1$ follows from construction of the solution set from the differential equation itself. The homogeneity symmetry follows from the linearity of the equation in the dependent variable, $y$. The existence of the three-dimensional subalgebra, $sl(2,R)$, has already been established in [12]. It remains to consider both the structures and algebras of the Jetspace order-one and order-two symmetries and from the evidence before us to conjecture what the general case can be.

If we observe the initial symmetry in the cases of $\Gamma_h$, $\Gamma_{J1}$ and $\Gamma_{J2}$, we see that we have $y\partial_y$, $y'\partial_y$ and $y''\partial_y$, respectively. In the case of the homogeneity there is just the one symmetry. In the case of the Jetspace order-one there is also $xy'\partial_y$ whereas for Jetspace order-two one finds $xy''\partial_y$ and $x^2y''\partial_y$. This leads us to the first conjecture.

**Conjecture I:** The generalised symmetries for the equation $y^{(n)} = 0$ include symmetries of the form

$$\Gamma_{\mathrm{jet}ij} = x^i y^{(j)}\partial_y, \tag{3}$$

where $i = 0, j$ and $j = 0, J$ and $J < n$ is the order of the Jetspace under consideration. We note that the notation here is more precise than that used in Section 2 in that we relate the indices to the order of the derivative of $y$ and the exponents of the power of $x$.

In addition to the generalised symmetries of the simple structure proposed in Conjecture I there exist generalised symmetries of more complex structure determined by the order of the equation and the order of the Jetspace. Thus we have

**Conjecture II:** The generalised symmetries for the equation $y^{(n)} = 0$ include symmetries of the forms

$\Gamma_{J11} = \left(x^2y' - (n-1)xy\right)\partial_y$ for a Jetspace of order-one,
$\Gamma_{J21} = \left(x^3y'' - (n-1)(n-2)xy\right)\partial_y$ and
$\Gamma_{J22} = \left(x^4y'' - 2(n-2)x^3y' + (n-1)(n-2)x^2y\right)\partial_y$

for a Jetspace of order-two.

In general, if the Jetspace is of order-$j$, we assume that there are $j$ such symmetries of the general form

$$\Gamma_{\mathrm{jet}ji} = \sum_{m=0}^{j} a_m x^{j+i-m} y^{(j-m)}\partial_y,$$

where $j$ is the order of the Jetspace and $i+j$ the degree of the polynomial expression.

## 4 Proofs of the Conjectures

**Proof of Conjecture I:**
The differential operator

$$\Gamma_{\mathrm{jet}ij} = x^i y^{(j)} \partial_y, \tag{4}$$

where $i = 0, j$ and $j = 0, J$ and $J < n$ is the order of the Jetspace under consideration, is a symmetry of the equation $y^{(n)} = 0$ if

$$\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}\left(x^i y^{(j)}\right) = 0. \tag{5}$$

A general term in the Leibniz expansion of the derivative above is

$$\binom{n}{k} \frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\left(x^i\right) \frac{\mathrm{d}^{\mathrm{n-k}}}{\mathrm{dx}^{\mathrm{n-k}}}\left(y^{(j)}\right).$$

When the derivatives are expanded, this becomes

$$\binom{n}{k} \frac{i!}{(i-k)!} x^{i-k} y^{(j+n-k)}$$

which is nonzero only if $k \leq i$ and $k \geq j$. As $i \leq j$, this is a contradiction and Conjecture I is proven.

**Proof of Conjecture II:**
**Part (a)**
We assume that the equation $y^{(n)} = 0$ has a generalised symmetry of the form

$$\Gamma_{J11} = \left(x^2 y' + c_1 xy\right) \partial_y.$$

The $n$th extension of $\Gamma_{J11}$, which is given by the form

$$\binom{n}{k} \frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}\left(x^2\right) \frac{\mathrm{d}^{\mathrm{n-k}}}{\mathrm{dx}^{\mathrm{n-k}}}\left(y'\right) + c_1 \binom{n}{k} \frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{dx}^{\mathrm{k}}}(x) \frac{\mathrm{d}^{\mathrm{n-k}}}{\mathrm{dx}^{\mathrm{n-k}}}(y),$$

must be zero when it acts upon the differential equation. When we perform the calculation of the $n$th extension, take into account the differential equation and the eventual annihilation of the derivative of the exponents of the independent variable $x$, we obtain

$$\binom{n}{2} 2! y^{(n-1)} + c_1 \binom{n}{1} 1! y^{(n-1)} = 0$$

the solution of which is

$$c_1 = -\frac{(n-1)!}{(n-2)!} = -(n-1). \tag{6}$$

**Part (b)**
By similar argument, if we have the generalised symmetry of the form

$$\Gamma_{J21} = \left(x^3 y'' + c_2 xy\right) \partial_y,$$

we obtain that

$$\frac{n!}{(n-3)!3!}.6 + nc_2 = 0$$

whence

$$c_2 = -\frac{(n-1)!}{(n-3)!} = -(n-1)(n-2). \tag{7}$$

**Part (c)**
In this case we take the generalised symmetry to be of the form

$$\Gamma_{J22} = \left(x^4 y'' + ax^3 y' + bx^2 y\right) \partial_y.$$

When we gather those terms in the $n$th extension of $\Gamma_{J22}$ which are not automatically zero and make some useful cancellations of terms, we obtain the two equations

$$\begin{aligned} &4n(n-1)(n-2) + 3an(n-1) + 2bn = 0, \\ &n(n-1)(n-2)(n-3) + an(n-1)(n-2) + bn(n-1) = 0. \end{aligned}$$

The solution of this system of equations gives

$$a = -2(n-2) \quad \text{and} \quad b = (n-1)(n-2). \tag{8}$$

**Part (d)**
From the indications given by the results for Jetspaces 0, 1 and 2 we make the Ansatz that there are symmetries of the form

$$\Gamma_{\mathrm{jet}ji} = \sum_{m=0}^{j} a_m x^{j+i-m} y^{(j-m)} \partial_y, \tag{9}$$

where $j \leq n$ is the order of the Jetspace, $i+j$ the degree of the polynomial expression in $x$ with $i \leq j$ and $a_0$ is set at one. We require that

$$\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{dx}^{\mathrm{n}}}\left(\sum_{m=0}^{j} a_m x^{j+i-m} y^{(j-m)}\right) = 0 \tag{10}$$

for the equation $y^{(n)} = 0$.

## 5 Discussion

In the computational results presented here there are several points to be noted. The first is that the capacity to calculate generalised symmetries in a reasonable time is limited to Jetspaces 1 and 2. Whether this limitation be due to machine or program is not obvious, but attempts made with Jetspace 3 at various orders of equation were uniformly unsuccessful. That generalised symmetries of higher-order exist is easily demonstrated. For example the equation $y^{(10)} = 0$ possesses the symmetry $\Gamma = y^{(9)}\partial_y$, corresponding to Jetspace 9, and doubtless with a little more effort one could devise other generalised symmetries for this equation. However, there is no way that one could claim completeness by means of educated guessing. The second point is that the type of generalised symmetry at any particular order varies with the order of the equation. The third-order equation has generalised symmetries of both the first degree and the second degree in $y'$. The former are the vertical forms of the three symmetries constituting the $sl(2,R)$ subalgebra. This property is repeated in all higher-order equations. The latter are the vertical forms of the contact symmetries which are peculiar to third-order equations. At the fourth-order generalised symmetries of order-two and degree one are introduced, whereas at the fifth and higher-orders there are also symmetries of degree two. An interesting feature of the coefficient functions of these symmetries is that they factor and we illustrate this property in the case of $y^{(6)} = 0$. The symmetries as far as order two are

$$
\begin{aligned}
&\Gamma_{s1} = \partial_y, \quad \Gamma_{s2} = x\partial_y, \quad \Gamma_{s3} = x^2\partial_y,\\
&\Gamma_{s4} = x^3\partial_y, \quad \Gamma_{s5} = x^4\partial_y, \quad \Gamma_{s6} = x^5\partial_y\\
&\Gamma_h = y\partial_y\\
&\Gamma_{js11} = y'\partial_y, \quad \Gamma_{js12} = xy'\partial_y, \quad \Gamma_{js13} = \left(x^2y' - 5xy\right)\partial_y\\
&\Gamma_{js21} = y''\partial_y, \quad \Gamma_{js22} = xy''\partial_y, \quad \Gamma_{js23} = x^2y''\partial_y,\\
&\Gamma_{js24} = \left(x^3y'' - 20xy\right)\partial_y,\\
&\Gamma_{js25} = \left(x^4y'' - 8x^3y' + 20x^2y\right)\partial_y.
\end{aligned}
$$

The factors of the symmetries of order two are

$$
\begin{aligned}
&\Gamma_{js21} = \mathrm{D(D)y}\partial_y, \quad \Gamma_{\mathrm{js22}} = \mathrm{xD(D)y}\partial_{\mathrm{y}},\\
&\Gamma_{js23} = x\mathrm{D(xD-1)y}\partial_y, \quad \Gamma_{\mathrm{js24}} = (\mathrm{xD}-6)\left(\mathrm{x^2D}+4\mathrm{x}\right)\mathrm{y}\partial_{\mathrm{y}}\\
&\text{and } \Gamma_{js25} = \left(x^2\mathrm{D} - 6\mathrm{x}\right)\left(x^2\mathrm{D} - 4\mathrm{x}\right)y\partial_y,
\end{aligned}
$$

where $\mathrm{D} = \mathrm{d}/\mathrm{dx}$. Note that the factors are not necessarily unique.

Although currently there does not appear to be any particular application of generalised symmetries in the case of ordinary differential equations – naturally we exclude the contact symmetries of the third-order equation – to the resolution of problems, we have increased our theoretical knowledge of the properties of this class of equations.

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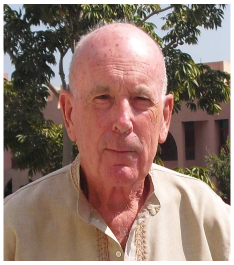

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