Innovative Pedagogy: Implications of Genetic Decompositions for Problem Solving in Management Courses

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Abstract

Mathematics is a subject on which many business and management courses rely. In this paper the researcher reports on the processes involved in the design of itemised genetic decompositions for certain mathematics tasks. The manner by which this is done may be extended to teaching and learning in management courses in higher education. The researcher used two tasks, one in a high school context from a previously published study and another from a qualitative case study which explored the development of mental constructions of a group of fourth-year pre-service teachers during definition making of certain mathematical concepts in real analysis, at a South African higher education institution. Questionnaires based on two items involving the concept of infinity were administered to those participating pre-service teachers. The pre-service teachers were allowed to respond to these items without lecturer involvement so that a learner-centred learning environment was created. Their mental constructions of these concepts were analysed by using an APOS (Action-Process-Object-Schema) framework. Forty six pre-service teachers, specialising in the teaching of mathematics in the FET school curriculum, participated in the project. It was found that the APOS designed questionnaire generated thick data that led to a modification of the initial genetic decomposition formulated. This modified genetic decomposition has didactical implication for business and management courses taught in higher education. This is illustrated in the recommendations section of this paper.

Keywords: APOS, Injective Functions, Itemised Genetic Decomposition, Optimisation

Introduction

Mathematics student performance has been for decades recognised as a problem in society. This is the case not only in schools but also at universities, especially at the undergraduate level. It is thought that if one understands how students think when engaging in mathematics activities then one might be able to improve on ways of making the learning of mathematics more meaningful. Hence exploring student mathematical thinking is important not only to mathematics education research but the country and a global society as a whole. All of the research work the applicant engages with involves investigations targeting aspects that address the improvement in mathematics learning In the Revised National Curriculum Statement grades R-12 (Schools)[12] it is emphasized that curriculum and teacher development theories in recent times should focus on the role of teachers and specialists in the development and implementation of effective teaching. There is a need for teachers to know the ideas with which students often have difficulty and ways to help bridge common misunderstandings. For example, one important facet of mathematical knowledge is the ability to move flexibly among different representations of a mathematical concept. Teachers need to be able to make flexible use of representations before they are able to create an environment that allows learners the freedom to use developing mathematical solutions[2]. In the mathematics community of practice there is consensus that students' ways of thinking should be taken into account when planning instruction and that teachers should choose or design sequences of lessons for use and discussion in

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class[18]. We thought that this should also be applicable to university lecturers in the higher education arena. In this study we aimed to explore what mental constructions are evoked when the pre-service teachers express their thoughts in writing when answering particular mathematical questions. The study aimed to explore learners' mental construction of knowledge in mathematics in answering questions based on infinite sets. This was done by exploring the research question: *How do classroom practitioners and mathematics educationists design itemized genetic decompositions for mathematics tasks?* To unpack this question we asked the following sub-questions.

- How can we design a genetic decomposition for an optimization problem?
- What impact does this decomposition have on student's responses?
- How can we design a genetic decomposition to cater for pre-service teachers' mental constructions of action, process, and object derived from their written responses?

Constructivism forms basis of how learners learn and this study was underpinned by APOS theory which is fundamentally a constructivist one. The focus of the study was not only to understand how learners construct knowledge but also to explore the cognitive structures involved in the construction of knowledge. This theory clearly describes the cognitive structures used by learners to construct knowledge, through action, process, object and schema; so the acronym APOS. Based on APOS Theory, the construction of knowledge when dealing with problems involving infinite sets were explored through identifying the relevant initial itemised genetic decomposition.

Related Literature Review

The notion of the infinite always presents difficulty to mathematics students. Tsamir[18] studied the different types of representations which arise when comparing two infinite sets. She worked with prospective teachers. Those teachers were given the set of odd numbers and the set of natural numbers and asked whether the set of natural numbers was larger than/smaller than or equal to the other set. She found that many students explained that the set of odd numbers was smaller since it was a subset of the set of natural numbers. That was an Israeli study and so we decided to ask a similar question in the South African context. In a Korean study, Choi & Do (2005) studied the equality of $0,\bar{9}$ and 1, where $0,\bar{9}$ represents the recurring 9 in the decimal expansion. They found that the fundamental problem about two numbers being equal was related to the mathematical ideas inherent in extensions of number systems. Equality cannot be understood properly unless it was interpreted in terms of underlying mathematical structures. We hence have decided to extend these findings in this study by exploring the mental structures pre-service teachers develop when engaging with the equality of $0,\bar{9}$ and 1. We hoped that understanding such mental structures would be linked with the relevant mathematical structures mentioned in the study by Choi & Do (2005). The equality of $0,\bar{9}$ and 1 was studied by Brijlall, Maharaj, Bansilal, Mkhwanazi & Dubinsky[5]. In that South African study the researchers explored the intervention of APOS designed activity sheets to aid the understanding of the equality of $0,\bar{9}$ and 1. They found that most students (even after engaging with the activities) did not believe that the quantities $0,\bar{9}$ and 1 are equal. The researchers in that study did not focus on the concept of infinite sets which we do in this study.

Conceptual Framework

In recent developments the focus on mathematics is on the mental processes that an individual employs to understand a learnt concept[6]. A number of learning theories such as Piaget's theory on constructivism, Vygotsky theory on scaffolding, Skinner theory on behavioural learning focus on learning. This study mainly engages with APOS theory which is a framework for the process of learning mathematics that pertains specifically to learning more complex mathematical concepts[20].

APOS theory is premised on the hypothesis that mathematical knowledge consists of an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemas* to make sense of the situations and solve the problems[14]. This theory builds on Piaget's notion of reflective abstraction. According to Dubinsky[13] reflective abstraction refers to the construction of logico- mathematical structures by an individual during the course of cognitive development. Piaget[17] distinguished three types of abstraction which are empirical abstraction, pseudo empirical and reflective abstraction. Cetin[7] stated that in empirical abstraction the focus is on general characteristics of objects and in reflective abstraction the focus is on the actions or operations done by a subject on mental objects. So he further elaborated that action, process, object, schema are the mental structures that an individual builds by the mental mechanism of reflective abstraction. Therefore APOS allows for the development of ways of thinking about how abstract mathematics can be assimilated and learned[10]. In looking at mathematics this theory is very much applicable in understanding learners' learning of different concepts in calculus such as derivatives, optimisation problems, etc.

According to Dubinsky[13] there are five kinds of reflective abstractions; interiorization, encapsulation, coordination, reversal and generalisation. These can be linked to the four stages of APOS. The following are the definitions as cited in Brijlall & Maharaj[3, 4] which we adopt: Action is a repeatable physical or mental manipulation that transforms objects. Action is based on rules and algorithms where a rule is practised repeatedly until it becomes routine, this takes place without adequate thinking. In the action stage, the manipulation of entities is thought of as external and the learner only knows how to perform an operation from memory or a clearly given instruction[19]. If an individual interiorises such an action conception such an individual we say has a process conception. Once an individual becomes aware that actions or operations can be performed on a concept then he/she might encapsulate a process to an object. Object is a static entity one transforms[7]. According to Weyer[19] encapsulation refers to the mental construction of process into a cognitive object that can be seen as a total entity which is referred to as a schema. In the schema stage the learner has collection of actions, processes, objects and other schemas that the learners understand in relation to calculus. These notions helped Brijlall & Ndlovu[6] to devise a linear model of an *itemised genetic decomposition*.

Itemised Genetic Decomposition

The genetic epistemology of Jean Piaget is found to be useful for this study. At the centre of Piaget's work is a fundamental cognitive process which he termed "equilibration"[17]. Through an application of the model of equilibration to a series of written tasks we are able to generate an account of the arrangements of component concepts and cognitive connections prerequisite to the acquisitions of the concept of the maxima and minima. These arrangements which are called "genetic decomposition" do not necessarily represent how trained mathematicians understand the concepts. Brijlall & Ndlovu[6] introduced the notion of itemised genetic decomposition. An Itemised Genetic Decomposition (IGD) they defined as a genetic decomposition specific to a mathematics task an individual is confronted with. For example they spoke of an IGD for rectangle area which dealt with a specific problem requiring learners to find the maxima/minima area of a given rectangle. We extend this notion of IGD in this study for two tasks which we highlight in the methodology section of this paper.

Methodology

For this study we used a qualitative methodological paradigm. We present this section in five sub-sections which are: 1) the case study, 2) background of participants, 3) ethical issues, 4) validity and reliability and 5) tasks design and their IGD.

The Case Study

In any type of dialogue it is effective when one uses a particular instance to illustrate something that is more general. It is easier to engage with your audience when you talk about real people and events instead of discussing theories and ideas that are abstract[16]. People generally understand an idea better if an example is used to illustrate the idea. We are all familiar with specific details and that a single instance assists us to see how the abstract principles fit together[16]. "A case study is a specific instance that is frequently designed to illustrate a more general principle" (Nisbet & Watt as cited in Cohen et al, 2007, p.253). Our research was specific in that we invited twenty eight pre-service teachers from a South African university.

Participants

The participants in our research were twenty eight pre-service teachers. The teachers were chosen for two reasons: firstly the teachers were students from the fourth year class which one of the researchers was teaching, and secondly they were chosen purposively. Access is a key issue in research and it is a factor that must be considered early in the research procedure[8]. The participants were specialising in high school mathematics teaching. They were generally the better qualified (better matric entry mathematics mark) than the other students who were specialising to teach from grade1 to grade 9. The pre-service teachers who participated in this study had already passed five courses which included differential and integral calculus, linear algebra, complex numbers and differential equations.

Ethical Issues

"A major ethical dilemma is that which requires researchers to strike a balance between the demands placed on them as professional scientists in pursuit of truth, and their subjects' rights and values potentially threatened by the research"[8]. In terms of ethical considerations we followed the procedures as stipulated by the university research office. Participation by those teachers was totally voluntary and their confidentiality, privacy and anonymity were assured. The consent letters included details of the study and data collection procedures. The participants were also assured that if they chose to be part of the study they could withdraw at any time without being prejudiced in any way.

Validity and Reliability

In qualitative research "validity might be addressed through the honesty, depth, richness and scope of the data achieved, the participants, the extent of the triangulation and the disinterestedness or objectivity of the researcher"[8]. Validity can be improved through careful sampling, using the appropriate instruments and data analysis techniques. Validity is not something that can be achieved absolutely but it can be maximized. According to Cohen et al[8] reliability can be seen as the correlation between the researcher's recorded data and what actually happens in the natural setting of the research. This we achieved by analysing the pre-service teachers' written responses against the itemised genetic decompositions formulated.

Tasks Design and IGD

The task one (see Frame 1) is a task given to pre-service teachers to work with. We also provide an IGD for each task.

Task one

Frame 1: The Task Given to University Pre-service Teachers.

You are given the two sets: $N = \{1;2;3;4;5;......\}$ and $T \{1;3;5;7;9;....\}$.

Is the number of elements in set N greater than/ equal to/ smaller than the number of elements in set T ?

We thought that in order to correctly answer this question an individual required to possess a schema for set theory which incorporates the various APOS shown in Figure 1.

In an IGD for task 1(which we shall refer to as IGD1), we thought that the pre-service teachers were required to possess a schema for set theory. Within this schema, he/she should realise that a function is a set (of ordered pairs). The individual should also have an object conception of a function. Within his/her set theory schema, the individual would be expected to further classify a function according to further characteristics. In this way

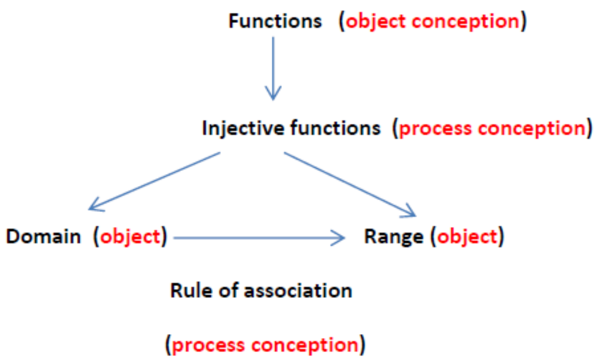

Figure 1: An IGD for task 1.

the individual would be expected to have at least a process conception of an injective function. Such a process should entail an association between two sets, namely the domain and the range. For this task then if the individual could identify this domain as N, say and the set T as the range and deduce an association (like $n \rightarrow 2n-1$) then it can be proved that the elements of set can be arranged in a one-to-one correspondence with the elements of the set. Hence the individual could conclude that these sets have the "same" number of elements.

Task 2 we use from a study reported in Brijlall & Ndlovu[6]. That study was conducted in a school consisting of only black African learners as well as teachers. It is a small school with an enrolment of about 600 learners with 19 teachers. Most learners reside in a nearby township and some are farm dwellers. The learners belong to a low socio-economic background as their parents are farm workers and farm dwellers. In the school there are only ten learners in grade twelve who do pure mathematics and are the participants in the study. During the study learners worked collaboratively. Since there were only ten learners who do pure mathematics we decided to have three groups. Two groups had three members and one had four members. The groups were formed using their academic achievement as a guide to create groups consisting of learners of mixed abilities. The following codes were used: a) R for the researcher and b) L1, L2 and L3 for the respective group leaders.

To gather data we used designed activity sheets and interviews. The responses for the tasks were categorised as: a) correct, b) partially correct and c) wrong. Even though marks were not allocated the responses were judged based on their correctness, judging whether the necessary understanding to solve that particular problem was applied appropriately with thoughtful reasoning. The category correct was for those learners who executed the correct steps in solving a task. The category partially was for those learners whose answers contain some steps or explanations that were correct but not entirely solved. The last category was for those learners who provided the entire incorrect solutions to a task. This analysis was followed by the researchers' interpretations and the data from semi structured interviews.

Task 2

The following task (see Frame 2) we extract from Brijlall & Ndlovu[6]. We make use of the IGD constructed in that paper as an illustration on how one can design IGD for school mathematics tasks.

Frame 2: The Problem Statement for Task Two (Adapted from Brijlall & Ndlovu[6]).

A rectangular box has the following dimensions:

- Length 5x units
- Breath (9-2x) units
- Height x units

1. Write down the formula for finding the volume of the box.
2. Hence find the volume in terms of x.
3. Find the value of x for which the box will have a minimum value.
4. Explain all the steps you followed when finding the minimum value.

According to Weyer[19] encapsulation refers to the mental construction of process into a cognitive object that can be seen as a total entity which is referred to as a schema. In the schema stage the learner has collection of actions, processes, objects and other schemas that the learners understand in relation to calculus. These notions helped us to devise a linear model shown in Figure 2 to depict an optimisation schema[2].

Discussion of Data for Task Three

In exploring how learners constructed their knowledge in dealing with the concept of minimisation of the volume of a cube we analysed data from the written response of Group two. The part of the activity sheets that show the problem statement for task three is shown in Extract 3. The itemised genetic decomposition that was used to analyse this task is presented

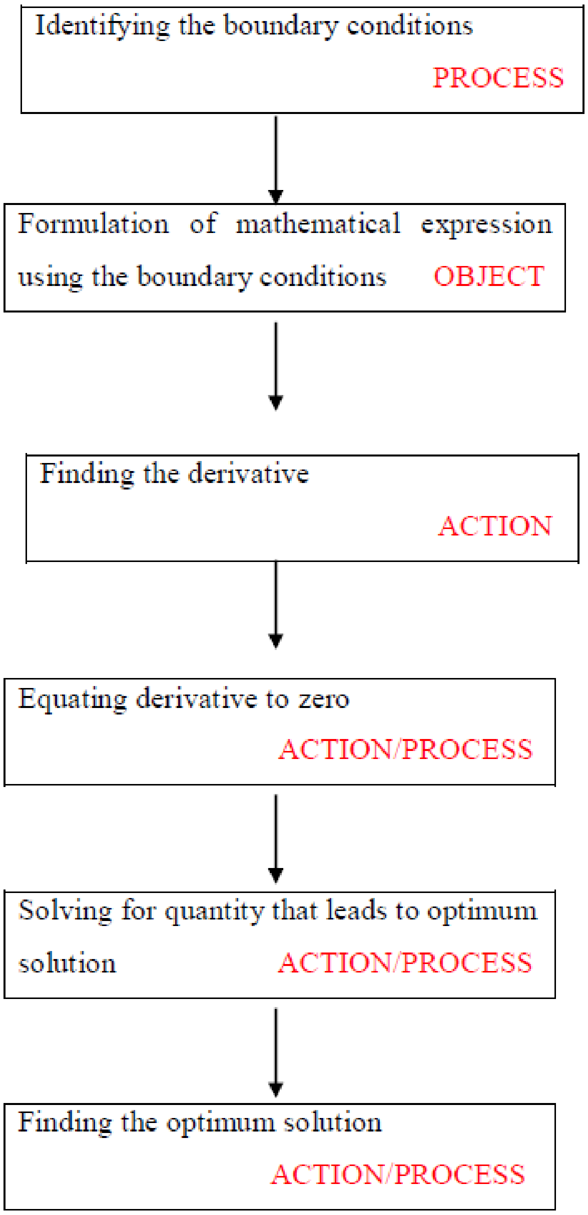

Figure 2: A schema to solve optimisation tasks in calculus[6].

For task two Group two learners showed that they knew the formula for the volume of a box. Using length and breadth in their formula showed they figured out that the base of this figure was a rectangle. In question two as they had already knew the formula they effectively substituted into the formula and worked out the volume of the box. In question three they had shown that they understood to find the minimum value they needed to first find the derivative of a function. They did however confuse this with a quadratic function where they could have used $x = -\frac{b}{2a}$ Also they knew that for a minimum to occur the derivative should be zero as they had equated their derivative to zero.

They were now at the process stage in terms of APOS theory because without actually writing all the steps they displayed that the derivative was the same as the gradient at the point and where the gradient was zero the minimum value occurred. This showed that the gradient schema and derivative schema had developed which now led to the construction of minima/maxima schema. We coded their attempt as partially correct because after finding the two *x* values they did not state which one among the two values gave the minimum value. It was possible that they that they had not known how to determine which *x* value provided the minimum value. Also it was possible that they considered the values of *x* as both the minimum values. By leaving their answer as they did they implied that it could either be both of them which was not true and which proved that they did not possess a complete volume schema. In order to verify what had been observed in Group two learners' written responses to questions in task three and to explore their mental constructions related to the concept of minima/maxima an extract of the interview with the Group two leader was conducted;

R: If we look at this figure would you say it is a 2D or a 3D shape?

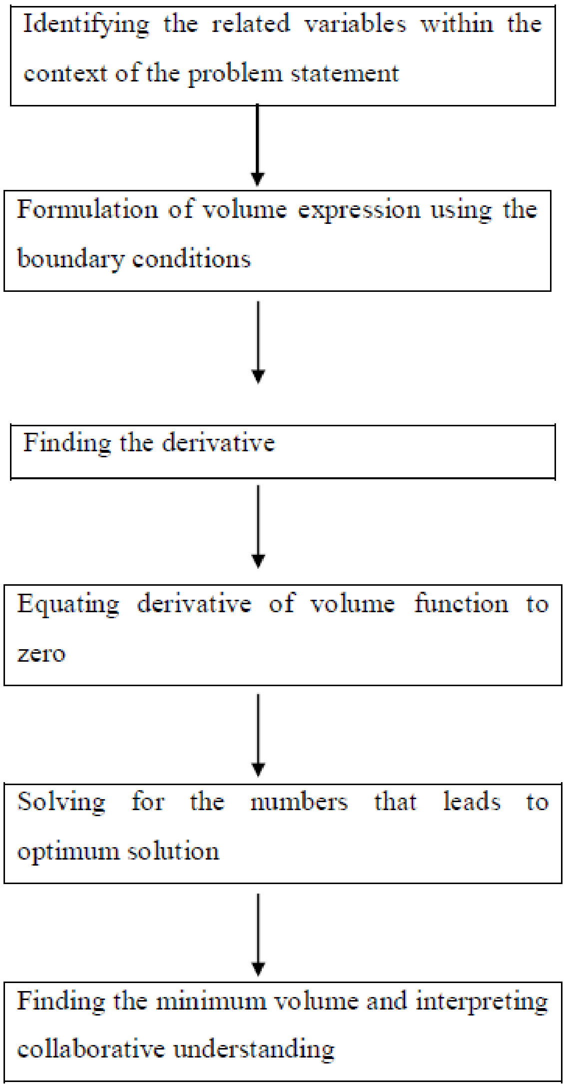

Figure 3: An IGD for volume of cube (Adapted from Brijlall & Ndlovu)

L2: It's a 3D shape.

R: What is a 3D shape?

L2: It has eh.... the dimensions miss it has like this, this side that are facing each other (pointing to the figure) *it has three sides miss.*

He knew that this figure was a 3D even though he couldn't describe it properly. He attempted to describe it as he saw without using mathematical terminology. The reasons these questions were asked during the interview even though they were not in the activity sheet was that for learners to work out the volume correctly they should identify the shapes they confronted. When it came to finding the formula he clearly knew his formula and understood the meaning of the variables used in the formula. Since he knew the formula he could extract the required information from the given figure and substituted into the formula to find the volume in terms of *x*. We decided to assess whether the members of Group two had an object level in their mental construction of the optimisation of a volume of a cube and so proceeded with the following interview:

R: if you were to sketch this graph or any other graph at which point will you find the minima/ maxima values?

L2: Maximum miss you will find it where it turns at the positive like this (sketching) at the top miss and at the bottom you will find the minimum.

R: But what do we call those points where will find the maxima and minima value?

L2: Turning points.

R: What is happening at those points?

L2: The gradient is.....what happened miss? We know that the gradient is equal to zero.

R: Is the gradient and derivative the same?

L2: Yes miss because if you derive you equate it to zero, the gradient miss

He had encapsulated the concept of a function into an object as he could, by looking at the equation given, tell if the function will first have the maxima then minima or vice versa. The gradient and derivative schema had developed as he understood that the derivative is the gradient at a point and for maxima and minima that point was equal to zero. Since the gradient schema has developed he could now assimilate the maxima/minima schema into his cognitive structures and that was why he could explain his thinking and constructed the required knowledge about the concepts and showed knowledge of the procedures followed in calculating and not memorising it. Even though he has constructed the knowledge he still used the incorrect terminology when referring to the derivative as he kept saying we derive. The term derive according to him meant the same as differentiate (finding the derivative).

Conclusion

Although both studies were small scale studies and the findings cannot be generalised, the formulation of IGD and the theory of APOS provided a valuable way of exploring the learning of maximising/minimising in calculus in a grade twelve mathematics class. The study in Brijlall & Ndlovu[6] revealed that some aspects of APOS theory were not fully operational because most of the learners were successful with the questions where they could use formulas and substitute when necessary. These manipulations took place externally as they could follow the given instruction and applied the rules that they had memorised. Some of them had acquired the process stage of APOS theory like group three who interiorised the volume formula into a process where they showed understanding that for maxima/minima to occur the gradient at that point is zero.

This conference presentation was based on itemised genetic decomposition formulation and the task one forms part of a study which will be reported in a possible journal publication and will require an analysis of data and a report on the findings thereon.

Recommendations

Research in APOS theory and its applications have been done in mathematics education extensively. It has potential for application in business and management courses taught at higher education institutions. It is recommended that explorations should occur in business and management courses where problem solving appears. To highlight this possibility the researcher provides a task entrenched in a business and management context. The task statement is provided in Frame 3.

Frame 3: A Typical Task that Students in Economic Studies are Faced With.

A toy company has fixed costs of 1000 rupees. In addition, each toy costs 4.5 rupees to manufacture. The weekly revenue is given by $R(x) = 100 - 0.01x^2$ for $0 \leq x \leq 8000$, where x is the number of toys produced and sold each week. Determine the maximum profits the toy company makes in a week.

The schema for solving optimisation problems illustrated in Figure 2 can be applied to this problem. Following the stages in that figure we notice that for the first stage, in the task in Frame 3, the boundary conditions are explicit where the number of toys is constrained to be between 1 and 800. The formulation of the mathematical expression in this case is done by recognising that profit will be obtained from the difference of total weekly cost of manufacture and total weekly revenue. Here we deduce that the expression for profit will be $P(x) = 100x - 0.01x^2 - 4.5x - 1000$ or $P(x) = -0.01x^2 + 95.5x - 1000$. Stage 3 in Figure 2 suggests we find the derivative of the profit function which leads to $P'(x) = -0.02x + 95.5$. Applying stages 4 and 5 we arrive at $x = 4775$ and stage 6 allows us to get a maximum profit of $P(4775) = 227006.25$ rupees.

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